and

(2) $$|f_n(z)| > 2^n \quad \text{on} \quad B_n.$$

We assert that the sequence $\{f_n(z)\}$, $n = 1, 2, \ldots$, fulfills all the conditions (i), (ii), (iii).

(i) It follows from the construction of A_n that, for every $z \in C$, there exists a n_z such that $z \in A_n$ for $n \geqslant n_z$ and, consequently, $|f_n(z)| < 1/2^n$ from (1) for $n \geqslant n_z$.

(ii) For $m < n$ clearly $\frac{1}{2^n} < \frac{1}{2^m} - \frac{1}{2^{m+2}}$, for $m > n$ clearly $\frac{1}{2^n} > \frac{1}{2^m} + \frac{1}{2^{m+2}}$, hence $\frac{1}{2^n} \in A_m$ for $m \neq n$ and (ii) follows from (1).

(iii) $1/2^n \in B_n$ for $n = 1, 2, \ldots$, hence (iii) follows from (2).

2. Define

$$f(z, w) = \sum_{n=1}^{\infty} f_n(z) w^n.$$

From (i) it follows that the sequence $\{f_n(z_0)\}$, $n = 1, 2, \ldots$, is bounded for every $z_0 \in C$. Hence the series $f(z, w)$ converges for every $(z, w) \in C \times D_2$ and the function $f(z_0, w)$ is holomorphic in D_2 for every $z_0 \in C$. Therefore all conditions required by Pták are fulfilled for $f(z, w)$.

3. Now we show that for every fixed $w_0 \in D_2$, $w_0 \neq 0$, the function $f(z, w_0)$ is not bounded and so not holomorphic in any neighborhood of $z = 0$. Thus take such a w_0 and denote $|w_0| = r_0$, $0 < r_0 < 1$. Choose n_0 so that $1/2^{n_0} < r_0$ and estimate $|f(1/2^{nn_0}, w_0)|$ for $n = 1, 2, \ldots$ From (ii), (iii) it follows

$$\left|f\left(\frac{1}{2^{nn_0}}, w_0\right)\right| \geqslant \left|f_{nn_0}\left(\frac{1}{2^{nn_0}}\right) w_0^n\right| - \sum_{m \neq nn_0}^{\infty} \left|f_m\left(\frac{1}{2^{nn_0}}\right) w_0^m\right|$$

$$> 2^{nn_0} r_0^n - \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} > (2^{n_0} r_0)^n - 1 \to \infty \quad \text{for} \quad n \to \infty.$$

4. From the assertion in 3 it follows that $f(z, w)$ is not holomorphic in any neighborhood of the point $(0, 0) \in D_1 \times D_2$.

Reference

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THE COEFFICIENT PROBLEM FOR FUNCTIONS WITH POSITIVE REAL PART IN A FINITELY CONNECTED DOMAIN*

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We consider the following problem: Let D denote a domain of finite order n of connectivity; set

$$\partial D = \sum_{\nu=1}^{n} K_\nu$$

where the components K_ν are supposed to be proper continua. Without restriction of generality we suppose that $0 \in D$, $\infty \notin \bar{D}$ (closure of D), and that each K_ν is an analytic curve. Let $\mathfrak{P}$ denote the following family of functions:

(1) $f \in \mathfrak{P}$ if and only if (a) f is holomorphic in D; (b) $\operatorname{Re} f(z) > 0$ for $z \in D$; (c) $f(0) = 1$.

If

(2) $$f(z) = 1 + \sum_{\mu=1}^{\infty} a_\mu z^\mu$$

is the power series development of $f \in \mathfrak{P}$ near 0, the problem is to characterize the set

(3) $$\mathfrak{C}_m = \{a_1, \ldots, a_m\}_{f \in \mathfrak{P}} \subset C^m$$

for any m and, in particular, the functions $P \in \mathfrak{P}$ for which

$$a := (a_1, \ldots, a_m) \in \partial \mathfrak{C}_m$$

(extremal functions).

We call $\mathfrak{C}_m$ the mth Carathéodory-body of $\mathfrak{P}$, for it was Carathéodory who, for the special case $D = U$, the unit disc, solved the problem in 1907, [1]. The solution was carried on to a very elegant algebraic characterization of $\partial \mathfrak{C}_m$ by Toeplitz, Carathéodory and E. Fischer in 1911, see [8], [2], [3]. We present here a sol-

* A two hours lecture with this title was given at the Banach Center by the author on April 28, 1979. This article gives a modified (§§ (e), (f), (i)) and extended (§ (k)) version.

ution of the general problem which mainly corresponds to that of Carathéodory of 1907 (of course in a less explicit form). Considering the intimate connection of function theoretic problems in a finitely connected domain with the theory of algebraic functions, which rests on the idea of the Schottky double of D, it might well be possible to find a characterization of the extremal functions corresponding to the one given in the papers of 1911, mentioned above.

The method we use was initiated by Z. Nehari in 1948 and 1951 (see [6] and [7]), but he did not exploit it to its full implications.

Our results may be summed up in the following

THEOREM. 1) *$\mathfrak{C}_m$ is convex and compact and $0 \in \mathfrak{C}_m$ (open core of $\mathfrak{C}_m$).*

2) *If H is a supporting hyperplane of $\mathfrak{C}_m$ in C^m then $H \cap \mathfrak{C}_m =: \Pi^r$ is a convex polyhedron of dimension r with $0 \leqslant r \leqslant m-1$.*

3) *Any $P \in \mathfrak{P}$ belonging to a point $a \in \Pi^r$ adopts each value w with $\operatorname{Re} w > 0$ equally often, $n+r \leqslant n+m-1$ times. If s is the smallest number such that $a \in \Pi^s$, where Π^s is a side of dimension s of Π^r, $0 \leqslant s \leqslant r$, then P adopts each value $n+s$ times; in particular ($s = 0$) there exist functions $P \in \mathfrak{P}$ such that $P(D)$ convers $\operatorname{Re} w > 0$ exactly n times.*

4) *To each $a = (a_1, \ldots, a_m) \in \partial\mathfrak{C}_m$ there exists exactly one $P \in \mathfrak{P}$ whose power series at 0 is $P(z) = 1 + \sum_{\mu=1}^{m} a_\mu z^\mu + \ldots$*

5) *Each point $a \in \partial\mathfrak{C}_m$ is a point of some Π^r* (cf. 3)) *with $r \geqslant m-(n+1)/2$.*

6) *For the number e of vertices of Π^r we have the estimates* $\max(1, m-(n-1)/2) \leqslant e \leqslant (q+1)^{n-\varrho}(q+2)^\varrho$, *where q and ϱ are defined by $m-1 = qn+\varrho$, $0 \leqslant \varrho < n$.*

In Nehari [6] one finds the last statement of 3), in [7] a result corresponding to the first statement of 3). A complete proof of our theorem, except for 6), has been given in [5], Chapt. 4, § 5. On the following pages the basic lines of the proof with some alterations are represented in detail. Other parts are omitted and the reader is referred to [5]. (Added in proof: see also [9].)

Proof. (a) The two properties of $\mathfrak{C}_m$ stated in 1) are immediate consequence of the same properties of $\mathfrak{P}$, $0 \in \mathfrak{C}_m$ follows from the facts that the constant 1 is in $\mathfrak{P}$ as well as $f(z) = 1 + \sum_{\mu=1}^{m} a_\mu z^\mu$ if the coefficients a_μ are restricted to a certain neighbourhood of 0.

(b) Any hyperplane in C^m (with the complex coordinates $t_1, \ldots, t_m$) may be represented as

$$H\colon \operatorname{Re}\sum_{\mu=1}^{m} \gamma_\mu t_\mu = c = cst \text{ with } \sum_{\mu=1}^{m} |\gamma_\mu|^2 = 1,\ c \geqslant 0,$$

and if $c > 0$, this representation is unique. c is the distance of H from 0, $\gamma := (\gamma_\mu)_{\mu=1}^m$ characterizes, for variable c, a set of parallel hyperplanes, covering a halfspace in C^m. For simplicity we suppose $\gamma_m \neq 0$; otherwise the results of our theorem hold with m_0 instead of m where m_0 is the largest number such that $\gamma_{m_0} \neq 0$.

For fixed γ there is exactly one $c =: c_\gamma$ such that $H =: H_\gamma$ is a supporting hyperplane for $\mathfrak{C}_m$, and it is easy to see that

$$c_\gamma = \max_{f \in \mathfrak{P}} \operatorname{Re}\sum_{\mu=1}^{m} \gamma_\mu a_\mu. \tag{4}$$

So to find the supporting hyperplane corresponding to γ, we have to solve the extremal problem (4), and we want to characterize the corresponding functions in $\mathfrak{P}$. Instead of (4) we may write

$$\operatorname{Re}\Big(c_\gamma - \sum_{\mu=1}^{m} \gamma_\mu a_\mu\Big) \begin{cases} \geqslant 0 & \text{for } f \in \mathfrak{P}, \\ = 0 & \text{for some } f \in \mathfrak{P}. \end{cases} \tag{5}$$

(c) Consider at first the subclass $\mathfrak{P}' \subset \mathfrak{P}$ with f holomorphic on $\bar{D}$. Then the expression under Re in (5), with any c instead of c_γ may be represented by a residue integral:

$$c - \sum_{\mu=1}^{m} \gamma_\mu a_\mu = \frac{1}{2\pi i}\int_{\partial D} f(z) R'(z)\,dz \tag{6}$$

where

$$R'(z) = c/z - \sum_{\mu=1}^{m} \gamma_\mu / z^{\mu+1} + S(z) \tag{7}$$

with S holomorphic on $\bar{D}$.

We can find the lower bound 0 for the real part of the right hand side of (6), using $\operatorname{Re} f(z) \geqslant 0$ and ∂D if for the differential $R'(z)\,dz$

$$i^{-1}R(z)\,dz \geqslant 0 \quad \text{on } \partial D \tag{8}$$

holds. To construct such a differential we introduce the following (multivalued) functions in D:

G: $\operatorname{Re} G = g$, where g is Green's function with singularity 0.

Q: $\operatorname{Re} Q = q$, where $q - \operatorname{Re}\sum_{\mu=1}^{m} \mu^{-1}\gamma_\mu z^{-\mu}$ is harmonic on $\bar{D}$, and $q(z) = 0$ for $z \in \partial D$. Existence and uniqueness of q are proved along the same pattern as for g.

H_ν: $\operatorname{Re} H_\nu = h_\nu$, h_ν the harmonic measure of K_ν, $\nu = 1, \ldots, n$.

Set

$$R := -cG + Q + \sum_{\nu=1}^{n-1} \beta_\nu H_\nu \tag{9}$$

with arbitrary constants $\beta_\nu \in \boldsymbol{R}$. Then

$$i^{-1}dR \in \boldsymbol{R} \quad \text{on } \partial D. \tag{10}$$

As $i^{-1}dG < 0$ on ∂D, we may choose c, for a fixed system $(\beta_\nu)_{\nu=1}^{n-1}$, such that (8) holds. As

$$c = \frac{1}{2\pi i}\int_{\partial D} dR(z)$$

we infer for this case: $c > 0$. From (6), (3), (1) (b) it follows:

$$\mathrm{Re}\Big(c - \sum_{\mu=1}^{m} \gamma_\mu a_\mu\Big) \geqslant 0.$$

This remains true for $c = c'_\gamma$ with

$$c'_\gamma := \inf c,$$

where inf refers to the set of c for which a differential dR with (8) exists; on account of $c > 0$, the inf exists. So we have for $f \in \mathfrak{P}'$ and, as $\mathfrak{P}'$ is dense in $\mathfrak{P}$, also for $f \in \mathfrak{P}$:

$$\text{(11)} \qquad \mathrm{Re} \sum_{\mu=1}^{m} \gamma_\mu a_\mu \leqslant c'_\gamma .$$

(d) If there is a function $f \in \mathfrak{P}$ such that equality in (11) holds, our extremal problem (5) with $c_\gamma = c'_\gamma$ has been solved. For the discussion of equality (11) we note first that our reasoning, starting with (6), holds also for functions $f \in \mathfrak{P}$ for which poles (necessarily of first order) on ∂D are admitted (regularity everywhere else on ∂D supposed) if each pole coincides with a zero of $R'(z)$. So we consider the class $\tilde{\mathfrak{P}}$, $\mathfrak{P}' \subset \tilde{\mathfrak{P}} \subset \mathfrak{P}$, consisting of functions holomorphic on $\bar{D}$, a finite number of poles (depending on the function) admitted on ∂D. We denote these poles for a particular function, by $\zeta_1, \ldots, \zeta_k$. By an easy application of the residue theorem we find, instead of (11):

$$\text{(12)} \qquad \mathrm{Re} \sum_{\mu=1}^{m} \gamma_\mu a_\mu \leqslant c'_\gamma - \tfrac{1}{2} \sum_{\varkappa=1}^{k} |b_\varkappa|\, |c_\varkappa|$$

where $b_\varkappa = R'(\zeta_\varkappa)$ and $c_\varkappa = \mathrm{res} f(\zeta_\varkappa)$. Denoting by $\tilde{\mathfrak{P}}_0$ the class of functions in $\tilde{\mathfrak{P}}$ with $\mathrm{Re} f(z) = 0$ for $z \in \partial D \setminus \{\zeta_\varkappa\}_{\varkappa=1}^{k}$ we see: If

(α) dR_0 is a differential (7) with $c = c'_\gamma$ and (8), and

(β) $P \in \tilde{\mathfrak{P}}_0$ is a function whose poles are zeros of dR_0,

then we have equality in (11). Vice versa, if a differential dR_0 exists, then (β) is necessary for equality in (11), if we consider only functions in $\tilde{\mathfrak{P}}$.

If $P \in \tilde{\mathfrak{P}}_0$, then $P(D)$ covers each point in the right halfplane the same number k of times if k is the number of poles on ∂D. So the general characterization of these extremal functions given in 3) is proved.

As each function $P \in \tilde{\mathfrak{P}}_0$ necessarily has at least one pole on each K_ν, $\nu = 1, \ldots, n$, consistency of (α) and (β) requires that dR_0 has at least one zero on each K_ν.

(e) The proof that dR_0 exists and, further, is unique, and that it fulfils the requirement just stated rests on the following

LEMMA. *For any real c, there exists exactly one differential (7) for which $i^{-1}dR \geqslant 0$ on K_ν, $\nu = 1, \ldots, n-1$ with equality in at least one point on each of these K_ν.*

Proof. Let $i^{-1}dR$ be any differential as defined by (7) with (10). Then, with $z = z(s)$, s arc length on ∂D:

$$i^{-1} R'(z)dz = i^{-1} \frac{\partial R}{\partial s} ds$$

and (see (9)):

$$\text{(13)} \qquad i^{-1} \frac{\partial R}{\partial s} = -cu(s) + v(s) + \sum_{\nu=1}^{n-1} \beta_\nu u_\nu(s)$$

with

$$u := -\frac{\partial g}{\partial n}, \quad v := \frac{\partial q}{\partial n}, \quad u_\nu := -\frac{\partial h_\nu}{\partial n},$$

where $\partial/\partial n$ means differentiation with respect to the interior normal. We consider the restriction of the function (13) on any of the boundary components K_μ, $\mu = 1, \ldots, n-1$, and we set, with $\beta = (\beta_1, \ldots, \beta_{n-1})$

$$\text{(14)} \qquad \omega_\mu(s; \beta) := -cu(s) + v(s) + \sum_{\nu=1}^{n-1} \beta_\nu u_\nu(s) \quad \text{for} \quad z(s) \in K_\mu .$$

We write also $u_\nu(z)$ with $z \in \partial D$ instead of $u_\nu(s)$, $\omega_\mu(z; \beta)$ (with $z \in K_\mu$) instead of $\omega_\mu(s; \beta)$. Further we put

$$\text{(15)} \qquad \tau_\mu(\beta) := \min_{z \in K_\mu} \omega_\mu(z; \beta), \quad \mu = 1, \ldots, n-1.$$

With this notation, the assertion to be proved is: There exists exactly one $\beta^{(0)} \in \boldsymbol{R}^{n-1}$ such that

$$\tau_\mu(\beta^{(0)}) = 0, \quad \mu = 1, \ldots, n-1.$$

With $\tau := (\tau_1, \ldots, \tau_{n-1})$ (15) defines a mapping

$$\tau = \Phi(\beta), \quad \Phi: \boldsymbol{R}^{n-1} \mapsto \boldsymbol{R}^{n-1}.$$

We claim that Φ is one-to-one. If this has been proved, there is, in particular, exactly one $\beta^{(0)}$ with $\Phi(\beta^{(0)}) = 0$. ■

First we show that Φ is locally one-to-one, i.e. that its Jacobian $\det J \neq 0$. Let $s = \psi_\mu(\beta)$ be the parameter of a point $z_\mu \in K_\mu$ where $\omega_\mu(s; \beta)$ adopts its minimum:

$$\omega_\mu(\psi_\mu(\beta); \beta) = \tau_\mu, \quad \mu = 1, \ldots, n-1.$$

Then the generic element of J is:

$$\frac{\partial \tau_\mu}{\partial \beta_\nu}(\beta) = \frac{\partial \omega_\mu}{\partial s}(\psi_\mu(\beta); \beta) \cdot \frac{\partial \psi_\mu}{\partial \beta_\nu}(\beta) + \frac{\partial \omega_\mu}{\partial \beta_\nu}(\psi_\mu(\beta); \beta).$$

Here, on account of the minimum property of τ_μ the first term on the right is 0, and so

$$\frac{\partial \tau_\mu}{\partial \beta_\nu}(\beta) = \frac{\partial \omega_\mu}{\partial \beta_\nu}(\psi_\mu(\beta); \beta) = u_\nu(\psi_\mu(\beta)) = u_\nu(z_\mu)$$

with some $z_\mu \in K_\mu$. But

$$u_\nu(z_\mu) = -\frac{\partial h_\nu}{\partial n}(z_\mu) < 0 \quad \text{for} \quad \nu \neq \mu$$

and

$$\sum_{\nu=1}^{n-1} u_\nu(z_\mu) = -u_n(z_\mu) > 0.$$

These properties imply, according to a theorem of Furthwängler, [4] (see also [5], p. 136): $\det J > 0$.

Further we show that Φ is surjective. Suppose the contrary and consider some $\tau^{(0)} \in \partial\Phi(\boldsymbol{R}^{n-1})$. On account of the property of Φ just proved $\tau^{(0)} \notin \Phi(\boldsymbol{R}^{n-1})$. So there exists a sequence $(\tau^i)_{i=1}^{\infty}$, $\tau^{(i)} \to \tau^{(0)}$ with $\tau^{(i)} = \Phi(\beta^{(i)})$ such that $(\beta^{(i)})_{i=1}^{\infty}$ is not bounded. But as the matrix $\big(u_\nu(z_\mu)\big)_{\mu,\nu=1}^{n-1}$ is not singular for any system $(z_\mu)_{\mu=1}^{n-1}$ we realize by (14) that

$$\omega := \big(\omega_1(z_1; \beta), \ldots, \omega_{n-1}(z_{n-1}; \beta)\big)$$

is, for any such system, unbounded for unbounded β, and so, also τ is not bounded. This contradiction completes the proof that Φ is surjective.

Further we show that Φ is injective. Assumption of the contrary means: there is an open arc C in $\boldsymbol{R}^{n-1}$, connecting two points, say $\beta^{(1)}$ and $\beta^{(2)}$, $\beta^{(1)} \neq \beta^{(2)}$, such that $\Phi(C)$ is closed. $\Phi(C)$ is homotopic to $\Phi(\beta^{(1)}) = \Phi(\beta^{(2)}) =: \tau^{(1)}$. A deformation of $\Phi(C)$ to $\tau^{(1)}$ may be carried through in small steps, each in a domain which is a one-to-one image under Φ of some domain in the space (β). Each step furnishes a curve $C^{(\varkappa)}$, $C^{(0)} = C$, $C^{(k)} = \tau^{(1)}$, if k is the number of steps, and each $C^{(\varkappa)}$ has a well-defined preimage. This contradicts $\beta^{(1)} \neq \beta^{(2)}$.

(f) Consider the differential dR_0 which we find according to our lemma on p. 82, if $c = c'_\gamma$. We claim:

$$dR_0 \geqslant 0 \quad \text{also for } z \in K_n$$

with equality in at least one point. For the proof suppose first the existence of a point $\zeta_n \in K_n$ such that $dR_0(\zeta_n) < 0$, and choose one zero ζ_ν of dR_0 on K_ν for $\nu = 1, \ldots, n-1$. Then there exists a function $P \in \tilde{\mathfrak{P}}_0$ exactly with the poles ζ_ν, $\nu = 1, \ldots, n$ (see e.g. [5], p. 133), and analogously to (12) we find:

$$\operatorname{Re}\sum_{\mu=1}^{m} \gamma_\mu a_\mu = c'_\gamma + |b_1||c_1| > c'_\gamma$$

contradicting (11).

Suppose on the other hand $dR_0(z) > 0$ all over K_n. Then, if in (13) β_ν, $\nu = 1, \ldots, n-1$, is replaced by $\beta_\nu + \delta$, $\delta > 0$, we add a term $\delta \sum_{\nu=1}^{n-1} u_\nu(s) = -\delta u_n(s)$, which is < 0 on K_n, but > 0 on K_ν, $\nu = 1, \ldots, n-1$. If δ is small enough, the resulting differential is still > 0 on K_n, and it is also > 0 on each K_ν, $\nu = 1, \ldots, n-1$,

and that means that it is possible to lessen the coefficient $c = c'_\gamma$ of u in (13), keeping $dR_0 > 0$ all over ∂D, and this contradicts the definition of c'_γ.

So we have proved:

Lemma. *Among the differentials $i^{-1}dR$, defined by (9) with (8), there is exactly one, $i^{-1}dR_0$ with minimal $c = c'_\gamma$, and dR_0 has at least one zero on each K_ν, $\nu = 1, \ldots, n$.*

(g) In (d) we have found a characterization of the extremal functions for (4) as far as they are contained in $\tilde{\mathfrak{P}}$. There remains the problem whether there are other extremal functions in $\mathfrak{P}$ (for which our reasoning based on a boundary integral cannot be applied).

The proof that there are no such functions rests on an approximation lemma, stating that $\tilde{\mathfrak{P}}_0$ is dense in $\mathfrak{P}$. For the details see [5], p. 165–168.

(h) We further have to exploit the characterization of the extremal functions given in (d). Let j denote the number of different zeros of dR_0 on ∂D. By (f) we know: $j \geqslant n$. An extremal function P has at most j poles, and so $P(D)$ covers the right halfplane at most j times. We set $j = n+r$. So, the zeros of dR_0 supposed to be known, we know the positions of the possible poles of P and $\operatorname{Re}P(z) = 0$ on ∂D, the poles excepted. The argument of the residue in a pole is fixed by the latter requirement (it is the argument of the interior normal on ∂D in the pole), but the modulus is free. So we have j nonnegative constants at our disposal, but there are n side conditions, $n-1$ for the singlevaluedness of the resulting function, one for the normalization at 0. So the manifold of these functions is of dimension $j-n = r$. For the details of this proof, as well as for the proofs of 4) and 5) the reader is referred to [5].

(i) To prove the statement in 3): $r \leqslant m-1$ note that the zeros of dR_0 on ∂D are of even order, so at least of order 2, whereas each pole of an extremal function P is of order 1. So, if $P^{(1)}$ and $P^{(2)}$ are two extremal functions belonging to different points of Π^r, the integrand in

$$(16) \qquad I := \frac{1}{2\pi i}\int_{\partial D} P^{(1)}(z)P^{(2)}(z)dR_0(z)$$

is holomorphic on ∂D, and as $P^{(1)}$ and $P^{(2)}$ are imaginary (except the poles), $i^{-1}dR_0$ is real on ∂D, we find that I is real. To evaluate (16) by the residue theorem we write:

$$P^{(\iota)}(z) = \sum_{\mu=0}^{\infty} a_\mu^{(\iota)} z^\mu, \quad \iota = 1, 2; \quad a_0^{(\iota)} = 1,$$

$$R'_0(z) = -\sum_{\mu=0}^{m} \frac{\gamma_\mu}{z^{\mu+1}} + S(z), \quad \gamma_0 = -c_\gamma \quad \text{(cf. (7))}.$$

So we find:

(17) $$\operatorname{Im}\sum_{\mu+\nu=0}^{m}\gamma_{\mu+\nu}a_\mu^{(1)}a_\nu^{(2)} = 0.$$

We introduce the matrix:

(18) $$\Gamma = \begin{bmatrix} 0 & \gamma_1 & \dots & \gamma_{m-1} & \gamma_m \\ \gamma_1 & \gamma_2 & \dots & \gamma_m & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_m & 0 & \dots & 0 & 0 \end{bmatrix}$$

with $\det\Gamma = \pm\gamma_m^{m+1} \neq 0$. With the notation

$$*a^{(\iota)} := (a_0^{(\iota)}, a_1^{(\iota)}, \dots, a_m^{(\iota)}) = (1, a^{(\iota)})$$

(17) may be written as:

(19) $$\operatorname{Im}{*a^{(1)}}\Gamma{*a^{(2)\mathrm{T}}} = 0.$$

Let $a^{(1)}$ vary through $r+1$ independent points of $\Pi^{(r)}$, $a^{(0)}$, $a^{(1)}, \dots, a^{(r)}$, and set $*a^{(\varrho)} = (1, a^{(\varrho)})$, $\varrho = 0, \dots, r$. We denote the matrix of type $(r+1, m+1)$ with the lines $*a^{(\varrho)}$, $\varrho = 0, \dots, r$ by A. Instead of the right hand factor $*a^{(2)}$ in (19) we write

(20) $$\tau = (t_0, t_1, \dots, t_m) = (t_0, t)$$

and we set

$$A\Gamma =: B.$$

Then we have for $t \in \Pi^{(r)}$ the equations

(21) $$\operatorname{Im}B\tau^{\mathrm{T}} = 0,$$

(22) $$t_0 = 1.$$

Set $B = B'+iB''$, $\tau = \tau'+i\tau''$, B', B'', τ', τ'' real, then (21) is

(23) $$B'\tau''^{\mathrm{T}}+B''\tau'^{\mathrm{T}} = 0.$$

Introducing the $(r+1, 2m+2)$-matrix and $(2m+2)$-vector resp.:

$$\tilde{B} := (B', B''), \quad \tilde{\tau} := (\tau'', \tau')$$

we may write (23) as

(24) $$\tilde{B}\tilde{\tau}^{\mathrm{T}} = 0.$$

The real rank of A is $r+1$, and the same is true for B and $\tilde{B}$. So the dimension of the set of solutions of (24) is $2(m+1)-(r+1) = 2m-r+1$. Linear independence of vectors $\tilde{\tau}$ and of the corresponding vectors τ with respect to the field of reals are equivalent, and so the dimension of the set of solutions of (21) is $l = 2m-r+1$. (22), i.e. $t_0'' = 0$, $t_0' = 1$ $(t = t'+it'',\ t_0 = t_0'+it_0'')$ reduces the dimension by 2. Consider first the equation $t_0'' = 0$. It must be proved that it is independent from the equations (23). Write these in the form

$$B'(t_0'', t'')^{\mathrm{T}}+B''(t_0', t')^{\mathrm{T}} = 0$$

and $t_0'' = 0$ as

$$e(t_0'', t'')^{\mathrm{T}}+n(t_0', t')^{\mathrm{T}} = 0$$

with $e = (1, 0, \dots, 0)$, $n = (0, \dots, 0)$. The contrary of our assertion would mean that the vector (e, n) is linearly dependent on the lines of (B', B''). But the first column of B' is

$$\operatorname{Re}(a_1^{(\varrho)}\gamma_1+\dots+a_m^{(\varrho)}\gamma_m) = c_\gamma, \quad \varrho = 0, \dots, r$$

and the last column is $\operatorname{Re}(a_0^{(\varrho)}\gamma_m) = \operatorname{Re}\gamma_m$, and the last column of B'' is $\operatorname{Im}\gamma_m$ in each line. These facts, together with $\gamma_m \neq 0$ preclude the above assumption.

That also $t_0' = 1$ means a reduction of dimension by 1 is trivial. So $l-2 = 2m-r-1$ is the dimension of the linear manifold represented by the vectors t satisfying (21), (22) with (20). It contains $\Pi^{(r)}$ and therefore: $r \leqslant 2m-r-1$, i.e. $r \leqslant m-1$. ∎

(k) For the proof of 6) note that a vertex e of $\Pi^{(r)}$ corresponds to a function with just one pole on each K_ν, $\nu = 1, \dots, n$. Each pole must be a zero of dR_0 on ∂D; the system of these zeros is fixed and their number is $j = n+r$; suppose j_ν of them are on K_ν, $\nu = 1, \dots, n$. Then there are $\prod_{\nu=1}^{n} j_\nu$ different extremal functions with just one pole on each K_ν, and this is the number of vertices of $\Pi^{(r)}$. So we have the combinatorial problem: to partition a number $j \geqslant n$ into n terms: $j = \sum_{\nu=1}^{n} j_\nu$, $j_\nu \geqslant 1$ for $\nu = 1, \dots, n$, such that $K = \prod_{\nu=1}^{n} j_\nu$ is minimal or maximal respectively. We claim: the minimum is attained if the partition is as unbalanced as possible, i.e. for a partition $\mathscr{P}_1$ with $j_\iota = j-n+1$ for one index ι, $j_\nu = 1$ for $\nu \neq \iota$. The maximum is attained if the partition is as well balanced as possible, i.e. if, with $j = qn+\varrho$, $q \in N$, $0 \leqslant \varrho < n$, $j_\iota = q$ for $n-\varrho$ indices, $j_\iota = q+1$ for ϱ indices ι. We denote such a partition by $\mathscr{P}_2$. To prove the first statement consider a partition not a $\mathscr{P}_1$ and assume (without restriction) that j_1 is its maximal term:

$$j = j_1+\nu j_2+\dots, \quad 2 \leqslant j_2 \leqslant j_1;$$

replace it by

$$j = (j_1+1)+(\nu-1)j_2+(j_2-1)+\dots$$

Then K becomes

$$K' = \frac{(j_1+1)(j_2-1)}{j_1 j_2}K = \frac{j_1 j_2-(j_1-j_2)-1}{j_1 j_2}K < K.$$

So, if the partition is not $\mathscr{P}_1$, it is possible to lessen K. To prove the second statement, consider a partition not a $\mathscr{P}_2$ and assume (without restriction) that j_1 is its smallest term and that $j_2 \geqslant j_1+2$:

$$j = j_1+j_2+\dots$$

If we replace it by

$$j = (j_1+1)+(j_2-1)+\dots$$

K becomes

$$K' = \frac{(j_1+1)(j_2-1)}{j_1 j_2} K = \frac{j_1 j_2 + (j_2 - j_1) - 1}{j_1 j_2} K > K.$$

This proves our statement.

In our case we have (see 5) and 3)):

$$m + \frac{n-1}{2} \leqslant j = n+r \leqslant m+n-1$$

and so we find for the number e of vertices of $\Pi^{(r)}$ the estimates in 6).

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CLASSICAL EXTREMAL PROBLEMS FOR UNIVALENT FUNCTIONS

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Denote by S the class of functions

$$(S) \qquad f(z) = c_0 + c_1 z + c_2 z^2 + \ldots,$$

regular and univalent in the unit disc D: $|z| < 1$.

Let $L(z_1, z_2)$ be the curve $z = z(s)$, $0 \leqslant s \leqslant \bar{s}$, $z_1 = z(0)$, $z_2 = z(\bar{s})$, $|z_1| < |z_2|$, for which $z'(s)$ and $r'(s) = |z(s)|'$ exist and are continuous except for a finite number of values of s. The parameter s denotes the length of the arc.

By $\mathscr{L}(z_1, z_2, f)$ denote the image of $L(z_1, z_2)$ by means of $f(z) \in S$. $\bar{L}(z_1, z_2)$ and $\bar{\mathscr{L}}(z_1, z_2, f)$ denote the lengths of $L(z_1, z_2)$ and $\mathscr{L}(z_1, z_2, f)$, respectively.

THEOREM I. *If $f(z) \in S$ and $|z_1| < |z_2| < 1$, then*

$$(1) \qquad \frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leqslant \frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2},$$

where the upper estimate holds true if $r'(s) \geqslant 0$.

For $|z| \leqslant r < 1$, one obtains

THEOREM I*. *If $f(z) \in S$ and $|z_1| < |z_2| \leqslant r < 1$, then*

$$(1^*) \qquad \frac{1-r}{(1+r)^3} \leqslant \frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1+r}{(1-r)^3},$$

where the upper estimate holds true if $r'(s) \geqslant 0$.

As a corollary we get:

THEOREM $\bar{\text{I}}$. *If $f(z) \in S$ and $|z_1| < |z_2| \leqslant r < 1$, then*

$$(2) \qquad \frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \leqslant \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2},$$

where the left inequality holds if the segment joining the points $f(z_1)$ and $f(z_2)$ lies entirely in the image $f(D)$ of the unit disc by means of $f(z)$, while the right inequality holds if, on the segment joining z_1 with z_2, $|z|$ only increases or only decreases.