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A REMARK ON HARTOGS' DOUBLE SERIES

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In a yet unpublished manuscript *On Möbius' matrices*, in connection with his investigations concerning the critical exponent, V. Pták posed the following question:

Suppose we are given a sequence f_n of functions holomorphic in the disc $D_1 = \{z \in C;\ |z| < 1\}$. Suppose further, that for each $z \in D_1$ and each $w \in D_2 = \{w \in C;\ |w| < 1\}$ the series $f(z, w) = \sum_{n=0}^{\infty} f_n(z) w^n$ is convergent. Does it follow that $f(z, w)$ is holomorphic in $D_1 \times D_2$ as a function of two complex variables?

In this note we give a negative answer to this question by refining in a natural way the construction given in the example on the pages 11–12, footnote, of the classical work of F. Hartogs [1].

1. First we construct a sequence $\{f_n(z)\}$ of polynomials with the following properties:

(i) $f_n(z) \to 0$ for every $z \in C$,

(ii) $\left|f_m\left(\frac{1}{2^n}\right)\right| < \frac{1}{2^m}$ for $m \neq n;\quad m, n = 1, 2, \ldots,$

(iii) $\left|f_n\left(\frac{1}{2^n}\right)\right| > 2^n$ for $n = 1, 2, \ldots$

Define for $n = 1, 2, \ldots$

$$A_n = \left\{z \in C;\, |z| \leqslant n,\ \operatorname{Re} z \leqslant \frac{1}{2^n} - \frac{1}{2^{n+2}} \text{ or } \operatorname{Re} z \geqslant \frac{1}{2^n} + \frac{1}{2^{n+2}}\right\},$$

$$B_n = \left\{z \in C;\, |z| \leqslant n,\ \frac{1}{2^n} - \frac{1}{2^{n+3}} \leqslant \operatorname{Re} z \leqslant \frac{1}{2^n} + \frac{1}{2^{n+3}}\right\}.$$

A_n, B_n are disjoint compact sets. Let $\tilde{A}_n$, $\tilde{B}_n$ be disjoint open neighborhoods of A_n, B_n, respectively. The function $\varphi_n(z) = 0$ on $\tilde{A}_n$, $\varphi_n(z) = 2^{n+1}$ on $\tilde{B}_n$ is holomorphic in $\tilde{A}_n \cup \tilde{B}_n$. Since the complement of $A_n \cup B_n$ is connected, by Runge's theorem there exists a polynomial $f_n(z)$ such that

$$|f_n(z)| < \frac{1}{2^n} \quad \text{on} \quad A_n \tag{1}$$

and

(2) $$|f_n(z)| > 2^n \quad \text{on} \quad B_n.$$

We assert that the sequence $\{f_n(z)\}$, $n = 1, 2, \ldots$, fulfills all the conditions (i), (ii), (iii).

(i) It follows from the construction of A_n that, for every $z \in C$, there exists a n_z such that $z \in A_n$ for $n \geqslant n_z$ and, consequently, $|f_n(z)| < 1/2^n$ from (1) for $n \geqslant n_z$.

(ii) For $m < n$ clearly $\frac{1}{2^n} < \frac{1}{2^m} - \frac{1}{2^{m+2}}$, for $m > n$ clearly $\frac{1}{2^n} > \frac{1}{2^m} + \frac{1}{2^{m+2}}$, hence $\frac{1}{2^n} \in A_m$ for $m \neq n$ and (ii) follows from (1).

(iii) $1/2^n \in B_n$ for $n = 1, 2, \ldots$, hence (iii) follows from (2).

2. Define

$$f(z, w) = \sum_{n=1}^{\infty} f_n(z) w^n.$$

From (i) it follows that the sequence $\{f_n(z_0)\}$, $n = 1, 2, \ldots$, is bounded for every $z_0 \in C$. Hence the series $f(z, w)$ converges for every $(z, w) \in C \times D_2$ and the function $f(z_0, w)$ is holomorphic in D_2 for every $z_0 \in C$. Therefore all conditions required by Pták are fulfilled for $f(z, w)$.

3. Now we show that for every fixed $w_0 \in D_2$, $w_0 \neq 0$, the function $f(z, w_0)$ is not bounded and so not holomorphic in any neighborhood of $z = 0$. Thus take such a w_0 and denote $|w_0| = r_0$, $0 < r_0 < 1$. Choose n_0 so that $1/2^{n_0} < r_0$ and estimate $|f(1/2^{nn_0}, w_0)|$ for $n = 1, 2, \ldots$ From (ii), (iii) it follows

$$\left|f\left(\frac{1}{2^{nn_0}}, w_0\right)\right| \geqslant \left|f_{nn_0}\left(\frac{1}{2^{nn_0}}\right) w_0^n\right| - \sum_{m \neq nn_0}^{\infty} \left|f_m\left(\frac{1}{2^{nn_0}}\right) w_0^m\right|$$

$$> 2^{nn_0} r_0^n - \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} > (2^{n_0} r_0)^n - 1 \to \infty \quad \text{for} \quad n \to \infty.$$

4. From the assertion in 3 it follows that $f(z, w)$ is not holomorphic in any neighborhood of the point $(0, 0) \in D_1 \times D_2$.

Reference

[1] F. Hartogs, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen*, Math. Ann. 62 (1906), 1–88.

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THE COEFFICIENT PROBLEM FOR FUNCTIONS WITH POSITIVE REAL PART IN A FINITELY CONNECTED DOMAIN*

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We consider the following problem: Let D denote a domain of finite order n of connectivity; set

$$\partial D = \sum_{\nu=1}^{n} K_\nu$$

where the components K_ν are supposed to be proper continua. Without restriction of generality we suppose that $0 \in D$, $\infty \notin \bar{D}$ (closure of D), and that each K_ν is an analytic curve. Let $\mathfrak{P}$ denote the following family of functions:

(1) $f \in \mathfrak{P}$ if and only if (a) f is holomorphic in D; (b) $\operatorname{Re} f(z) > 0$ for $z \in D$; (c) $f(0) = 1$.

If

(2) $$f(z) = 1 + \sum_{\mu=1}^{\infty} a_\mu z^\mu$$

is the power series development of $f \in \mathfrak{P}$ near 0, the problem is to characterize the set

(3) $$\mathfrak{C}_m = \{a_1, \ldots, a_m\}_{f \in \mathfrak{P}} \subset C^m$$

for any m and, in particular, the functions $P \in \mathfrak{P}$ for which

$$a := (a_1, \ldots, a_m) \in \partial\mathfrak{C}_m$$

(extremal functions).

We call $\mathfrak{C}_m$ the mth Carathéodory-body of $\mathfrak{P}$, for it was Carathéodory who, for the special case $D = U$, the unit disc, solved the problem in 1907, [1]. The solution was carried on to a very elegant algebraic characterization of $\partial\mathfrak{C}_m$ by Toeplitz, Carathéodory and E. Fischer in 1911, see [8], [2], [3]. We present here a sol-

* A two hours lecture with this title was given at the Banach Center by the author on April 28, 1979. This article gives a modified (§§ (e), (f), (i)) and extended (§ (k)) version.