COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

ON THE MAXIMUM PRINCIPLE FOR THE QUOTIENT OF NORMS OF MAPPINGS

JACEK CHĄDZYŃSKI

Institute of Mathematics, University of Łódź
Banacha 22, PL-90-238 Łódź, Poland

Introduction

In [2] the maximum principle for the quotient of bicylinder norms of mappings having common zeros has been proved. This result, obtained in $\boldsymbol{C}^2$, can relatively easily be generalized to $\boldsymbol{C}^n$ for polycylinder norms.

The object of the present paper is to prove an analogous general theorem for a wider class of norms, not necessarily polycylinder ones. Norms in the numerator and in the denominator may differ from each other. This is an essential generalization of the result from [2], requiring the application of non-trivial facts from the theory of plurisubharmonic functions.

In consequence, there have been obtained results of stability type for mappings having the same zeros, and a lemma of Schwarz type, being in some cases a strengthening of a result of J. Siciak [5].

1. Notation

In this paper $\boldsymbol{C}$, $\boldsymbol{C}^n$ will denote, respectively, the field of complex numbers and the n-dimensional complex space. For $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$ we set

$$|z| = \max|z_j|, \quad ||z|| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}.$$

Besides, for $w \in \boldsymbol{C}^{n+1}$, we introduce the notation $w' = (w_1, \ldots, w_n)$ and $w = (w', w_{n+1})$. Other notations will be taken analogously as in [2].

2. Fundamental notions

Let Ω be a bounded domain in $\boldsymbol{C}^n$. In the further part of the paper we shall assume that

(a) $H = (h_1, \ldots, h_n)$ is a mapping holomorphic on Ω and continuous on $\overline{\Omega}$;

this mapping has no zeros on $\partial\Omega$, whereas in Ω it has isolated zeros in the sense of [2] (i.e., for any z^0 such that $H(z^0) = 0$ there exists a positive integer ν such that all h_j $(j = 1, \ldots, n)$ have a zero of order ν at z^0 and homogeneous parts of degree ν of these functions vanish simultaneously at z^0 only; ν is called the order of z^0).

(b) $F = (f_1, \ldots, f_m)$ is a mapping holomorphic on Ω and continuous on $\overline{\Omega}$. If z^0 is an isolated zero of order ν of H, then each f_j $(j = 1, \ldots, m)$ has a zero of order no less than ν at z^0.

(c) L is a real non-negative continuous and absolutely homogeneous function defined on C^m, i.e.,

(1) $$L(\lambda u) = |\lambda| L(u), \quad u \in C^m, \quad \lambda \in C,$$

and it is plurisubharmonic as well.

(d) M is a real non-negative continuous and absolutely homogeneous function defined on C^n, satisfying the condition

(2) $$M(z) > 0 \quad \text{for} \quad z \neq 0 \in C^n.$$

Under the above assumptions the function Φ is defined by the formula

(3) $$\Phi(z) = \overline{\lim_{\substack{\zeta \to z \\ \zeta \in \Omega}}} L(F(\zeta))/M(H(\zeta)) \quad \text{for} \quad z \in \overline{\Omega}.$$

3. Auxiliary results

Let $S' = \{z \in \overline{\Omega}\colon h_1(z) = \ldots = h_n(z) = 0\}$.

It is easily seen that the set S' is finite and that

(4) $$\Phi(z) = L(F(z))/M(H(z)) \quad \text{for} \quad z \in \overline{\Omega} \setminus S'.$$

Let $z^0 \in S'$ be an isolated zero of order ν. Then h_j expands in a series of homogeneous polynomials

(5) $$h_j(z) = \sum_{l=\nu}^{\infty} Q_{jl}(z - z^0)$$

in some neighbourhood of z^0 and the system $Q_{j\nu}(\xi) = 0$, $j = 1, \ldots, n$, has the trivial solution only. Let us introduce the notation $Q^\nu = (Q_{1\nu}, \ldots, Q_{n\nu})$. In view of the above and condition (2), we have $M \circ Q^\nu(\xi) \neq 0$ for $\xi \neq 0$. According to (b), for every j, f_j expands in a series of homogeneous polynomials of the form

(6) $$f_j(z) = \sum_{l=\nu}^{\infty} P_{jl}(z - z^0), \quad j = 1, \ldots, m.$$

Let $P^\nu = (P_{1\nu}, \ldots, P_{m\nu})$.

PROPERTY 1. *If $z^0 \in S'$ is the above-mentioned zero, then there exists a point ξ^* such that $|\xi^*| = 1$ and*

(7) $$\Phi(z^0) = L(P^\nu(\xi^*))/M(Q^\nu(\xi^*)).$$

Indeed, take any sequence of points $z^k \to z^0$ and put $|z^k - z^0| = \varrho_k$ and $z^k - z^0 = \varrho_k \xi^k$. It can be seen that $|\xi^k| = 1$. At the cost of choosing a subsequence we may assume that $\xi^k \to \hat{\xi}$. Of course, $|\hat{\xi}| = 1$. After easy calculations, by the absolute homogeneity of L, M, and (5), (6), we get

$$\lim_{k\to\infty} L(F(z^k))/M(H(z^k)) = L(P^\nu(\hat{\xi}))/M(Q^\nu(\hat{\xi})).$$

On account of the arbitrariness of the sequence (z^k), we obtain

$$\Phi(z^0) \leqslant \sup_{|\xi|=1} L(P^\nu(\xi))/M(Q^\nu(\xi)).$$

Let ξ^* be a point at which $L \circ P^\nu/M \circ Q^\nu$ attains its upper bound on the boundary of the polycylinder $|\xi| = 1$. Hence

$$\Phi(z^0) \leqslant L(P^\nu(\xi^*))/M(Q^\nu(\xi^*)).$$

For the sequence of points $z^{*k} = (1/k)\xi^*$, we have

$$\lim_{k\to\infty} \Phi(z^{*k}) = L(P^\nu(\xi^*))/M(Q^\nu(\xi^*)) \leqslant \Phi(z^0).$$

Hence we get (7).

Directly from (4), (7) and the definition of Φ we obtain

PROPERTY 2. *The function Φ is bounded.*

PROPERTY 3. *If $z^0 \in S'$, then there exist a positive integer p, a holomorphic curve φ, and a continuous subharmonic function ψ, such that*

(8) $$\Phi(z^0 + t^p\varphi(t)) = \psi(t), \quad |\varphi(t)| \neq 0$$

for $|t| < \tau$.

Indeed, analogously as in [2] we prove for $n > 2$ that, if $z^0 \in S'$ and $a = Q^\nu(\xi^*)$, where $|\xi^*| = 1$, then there exist a positive integer p and a holomorphic curve $\varphi(t)$, $|t| < \tau$, such that $\varphi(0) = \xi^*$, and

$$H(z^0 + t^p\varphi(t)) \equiv at^{p\nu}, \quad |\varphi(t)| \neq 0$$

for $|t| < \tau$. We also show easily (cf. [2]) that, if z^0 is a zero of order ν of the function f_j, then $f_j(z^0 + w_{n+1}w') = w_{n+1}^\nu f_j^*(w)$, where f_j^* is a holomorphic function, while $|w_{n+1}w'|$ — sufficiently small. Let $F^* = (f_1^*, \ldots, f_m^*)$. We define the function ψ by the formula

$$\psi(t) = \big(1/M(Q^\nu(\xi^*))\big) L\big(F^*(\varphi(t), t^p)\big) \quad \text{for} \quad |t| < \tau.$$

4. The maximum principle

In this section we shall prove

THEOREM 1. *The function Φ satisfies in Ω the maximum principle in the sense that*

$$\sup_{z\in\overline{\Omega}} \Phi(z) = \sup_{z\in\partial\Omega} \Phi(z).$$

Proof. Without loss of generality we may assume that the function Φ attains its upper bound $\tilde{M}$ at a point $\tilde{z} \in \Omega$, because otherwise there is nothing to prove.

Consider the first case when $\tilde{z} \in S'$. Then, there exists a point $\check{z} \in \Omega \setminus S'$ such that $\Phi(\check{z}) = \tilde{M}$. Indeed, by (8) for $z^0 = \check{z}$ and the assumption made above, the function ψ attains its maximum for $t = 0$. Consequently, in virtue of the maximum principle for subharmonic functions, it is constant in the neighbourhood $|t| < \tau$, i.e., $\psi(t) = \tilde{M}$ for $|t| < \tau$. It is sufficient to put $\check{z} = \tilde{z} + t^p\varphi(t)$ for some $t \neq 0$.

In view of the above, we may assume that the function Φ attains its maximum at a point $\check{z} \notin S'$. Let $H(\check{z}) = (b_1, \ldots, b_n) = b$. Without loss of generality we may assume that $b_1 \neq 0$. Consider in Ω the analytic set $S = \{z \in \Omega\colon h_2(z) = (b_2/b_1)h_1(z), \ldots, h_n(z) = (b_n/b_1)h_1(z)\}$. Let S_1 be an irreducible component of the set S in Ω, containing the point $\check{z}$, and $\tilde{S}_1 = S_1 \setminus S'$. This set is closed and connected in $\Omega \setminus S'$ since S' is finite. Consider on this set the function $g(z) = (|b_1|/M(b))L\big((1/h_1(z))F(z)\big)$. Note that on $\tilde{S}_1$ the functions g and Φ coincide. Consequently, the function g attains at $\check{z}$ its upper bound. On the other hand, on this set the mapping $(1/h_1)F$ is holomorphic ($h_1(z) \neq 0$ for $z \in \tilde{S}_1$), that is, g, as a superposition of a plurisubharmonic function with a holomorphic mapping, is a plurisubharmonic function on $\tilde{S}_1$, i.e., for every $z \in \tilde{S}_1$, g is a plurisubharmonic function in some neighbourhood (in C^n) of this point. Then, by the maximum principle for plurisubharmonic functions on analytic sets (see [3], p. 272), $g(z) = \tilde{M}$ on $\tilde{S}_1$ locally in some neighbourhood of $\check{z}$. Hence, on account of the connectivity of $\tilde{S}_1$ and the continuity of g, it follows that $g(z) = \tilde{M}$ for $z \in \tilde{S}_1$. Since the set $\tilde{S}_1$ is dense in S_1, therefore, according to the definition of Φ as an upper limit, we also have $\Phi(z) = \tilde{M}$ for $z \in S_1$. Hence, furthermore, in virtue of the Remmert–Stein theorem (see [4], p. 81), we have that $\Phi(z) = \tilde{M}$ arbitrarily close to $\partial\Omega$, and so, in view of the continuity of Φ in the neighbourhood of $\partial\Omega$, it attains its upper bound on the boundary of Ω. This completes the proof.

5. Stability of mappings and Schwarz's lemma

Let us first give a simple corollary from Theorem 1.

COROLLARY 1. *If H, F, L, M satisfy the assumptions of Section 2, and*

$$L(F(z)) \leqslant AM(H(z))$$

for $z \in \partial\Omega$, then this inequality remains true for $z \in \overline{\Omega}$.

Let us further assume that $m = n$ and that F and H satisfy conditions (a) and (b) simultaneously. Let $S'' = \{z \in \Omega\colon f_1(z) = 0, \ldots, f_n(z) = 0\}$.

Under the assumptions made at present, directly from Corollary 1 we obtain

THEOREM 2. *If $S' = S''$, F and H have isolated zeros of the same order, and $L(F(z)) = M(H(z))$ for $z \in \partial\Omega$, then this equality remains true for $z \in \overline{\Omega}$.*

COROLLARY 2. *If S' is not empty, $L = M = |\cdot|$, and $|F(z)| = |H(z)|$ for $z \in \partial\Omega$, then*

$$F = a \circ H,$$

where a is an automorphism of the unit polycylinder, such that $a(0) = 0$, $0 \in C^n$.

The proof runs analogously as in [2].

COROLLARY 3. *If the mappings F and H possess at least one single zero, $L = M = \|\cdot\|$, and $\|F(z)\| = \|H(z)\|$ for $z \in \partial\Omega$, then*

$$F = a \circ H,$$

where a is a linear unitary transformation.

Indeed, let z^0 be the above-mentioned single zero, and $r > 0$ a number so small that in the ball $\|w\| < r$ there should exist mappings F^{-1}, H^{-1} inverse to F and H. Then, by Theorem 2, $\|w\| = \|F \circ H^{-1}(w)\|$ for $\|w\| \leqslant r$, besides, $F \circ H^{-1}$ is a biholomorphic mapping satisfying the condition $F \circ H^{-1}(0) = F(z^0) = 0$. Consequently, $F \circ H^{-1}$ is a linear unitary transformation of C^n onto C^n (see [6], p. 547). Denote it by a. Hence, $F(z) = a \circ H(z)$ in some neighbourhood of the point z^0, and thus in the entire domain Ω.

(e) Assume that L and M satisfy conditions (c) and (d) simultaneously, and M is a plurisubharmonic function. Let H satisfy condition (a), and $M \circ H(z) = 1$ for $z \in \partial\Omega$. Let F be a holomorphic mapping in Ω such that the coordinates of this mapping have at points of the set S' zeros of at least the same order as H.

THEOREM 3 (Generalized Schwarz's lemma). *If L, M, F, H satisfy assumption* (e), *and there exists a positive constant A such that $L \circ F(z) \leqslant A$ for $z \in \Omega$, then*

$$L \circ F(z) \leqslant AM \circ H(z) \quad \text{for} \quad z \in \Omega. \tag{9}$$

Proof. Note that $M \circ H$ is a plurisubharmonic function in Ω. Consequently, we have $M \circ H(z) \leqslant 1$ for $z \in \Omega$. Take any point $z^0 \in \Omega$. If $M \circ H(z^0) = 1$, inequality (9) is obvious for $z = z^0$. So, assume that $M \circ H(z^0) = 1 - \varepsilon^*$, where $0 < \varepsilon^* < 1$. Take any number ε satisfying the inequality $0 < \varepsilon < \varepsilon^*$, and an open set $\{z \in \Omega\colon M \circ H(z) < 1 - \varepsilon\}$. Let Ω_ε^0 be the component of this set, containing z^0. Then the function Φ defined by formula (3) satisfies in $\overline{\Omega}_\varepsilon^0$ the maximum principle, i.e., $\Phi(z) \leqslant A/(1-\varepsilon)$ for $z \in \Omega_\varepsilon^0$. Hence $L \circ F(z^0) \leqslant (A/(1-\varepsilon))M \circ H(z^0)$. Passing with ε to zero, we get $L \circ F(z^0) \leqslant AM \circ H(z^0)$. For $z \in S'$, we obtain inequality (9) directly. This concludes the proof of the theorem.

(f) Assume that M satisfies the assumptions of the generalized Schwarz's lemma. Let us define the set $\Omega_M = \{z \in C^n\colon M(z) < 1\}$. This set, as it can easily be verified, is a bounded domain. Let L satisfy assumptions (d).

Under the above assumptions there takes place

COROLLARY 4. *If F is a holomorphic mapping in Ω_M, $F(0) = 0$, and there exists a constant A such that $L \circ F(z) \leqslant A$ for $z \in \Omega$, then*

$$L \circ F(z) \leqslant AM(z) \quad \text{for} \quad z \in \Omega. \tag{10}$$

Indeed, put in Theorem 3 $H(z) = (z_1, \ldots, z_n)$. Then, by the continuity of M, we have $M \circ H(z) = 1$ for $z \in \partial\Omega_M$. From this and Theorem 3 we obtain (10).

This result is some analogue to the lemma of [5] for $\nu = 1$.

Assume that L satisfies the same assumptions as M in (f). Then $\Omega_L = \{z \in \boldsymbol{C}^m: L(z) < 1\}$ is a bounded domain. Let $m = n$.

Directly from Corollary 4 we obtain

COROLLARY 5 (cf. [5]). *If F is a biholomorphic mapping of Ω_M onto Ω_L, and $F(0) = 0$, then*

$$L \circ F(z) = M(z) \quad \text{for} \quad z \in \Omega_L.$$

6. Concluding remarks

The results obtained in the preceding section are connected with those of J. Siciak. In some cases Theorem 3 is a generalization of the lemma of [5]. For example, when the functions T and S of [5] satisfy, respectively, such conditions as L and M do, Theorem 3 is stronger than the lemma of [5]. Namely, Ω may be any domain, not necessarily circular, the function $M \circ H$ may possess more than one zero, in contradistinction to the function S^ν of [5].

References

[1] J. Chądzyński, *Extremum principle for the quotient of plurisubharmonic functions (I)*, Bull. Soc. Sci. Lettres Łódź 27, 5 (1977), 13 pp.

[2] —, *Some generalization of the maximum principle for the quotient of plurisubharmonic functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 26 (1978), 695–699.

[3] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Englewood Cliffs 1965.

[4] M. Hervé, *Several complex variables. Local theory*, Oxford 1963.

[5] J. Siciak, *A generalization of Schwarz's lemma and of Hadamard's three circles theorem*, Colloq. Math. 21 (1964), 203–207.

[6] Б. В. Шабат, *Введение в комплексный анализ*, Москва 1969.

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979

COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

НЕКОТОРЫЕ ИНТЕГРАЛЬНЫЕ ФОРМУЛЫ В МНОГОМЕРНОМ КОМПЛЕКСНОМ АНАЛИЗЕ И ИХ ПРИЛОЖЕНИЯ

Ш. А. ДАУТОВ

Институт Физики Сибирского Отделения АН СССР
Академгородок, SU-660036 Красноярск 36, СССР

В этой статье приводится обзор некоторых работ красноярских математиков по теории функций многих комплексных переменных. Здесь рассматриваются 1) приложение формального решения уравнения $\bar\partial u = f$, пригодного для правых частей f, имеющих рост конечного порядка около границы области, к описанию множества нулей функций из класса Неванлинны–Джрбашяна; 2) различные приложения многомерных аналогов логарифмического вычета; 3) утверждение о том, что только голоморфные функции представимы интегралом Мартинелли–Бохнера.

Отметим, что в статье не приводятся все ссылки на предшествующие и близкие работы, а также полные доказательства. Всё это можно найти в оригинальных работах, а также в книгах [3] и [5].

1. Нули голоморфных функций конечного порядка

Здесь будут изложены результаты статьи [12]. Пусть D — ограниченная область в $\boldsymbol{C}^n$ с дважды гладкой границей, т.е. $D = \{z \in \boldsymbol{C}^n: \varrho(z) < 0\}$, где ϱ — функция класса C^2 в $\boldsymbol{C}^n$, причём $\operatorname{grad}\varrho|_{\partial D} \neq 0$. Через $N_\alpha(D)$ $(\alpha > 0)$ обозначим класс голоморфных в области D функций, для которых

$$(1.1) \qquad \int_D |\varrho(z)|^{\alpha-1} \ln^+|F(z)|\, d\sigma_{2n} < \infty.$$

Здесь $d\sigma_k$ — элемент k-мерного объёма Лебега.

В классической работе М. М. Джрбашяна [13], [14] приведен следующий результат: если D — единичный круг на комплексной плоскости, то при любом $\alpha > 0$ дискретная последовательность $\{a_j\}$ точек из D является множеством нулей функции из $N_\alpha(D)$ тогда и только тогда, когда

$$\sum_j (1-|a_j|)^{\alpha+1} < \infty.$$