ON THE MAXIMUM PRINCIPLE
FOR THE QUOTIENT OF NORMS OF MAPPINGS

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Introduction

In [2] the maximum principle for the quotient of bicylinder norms of mappings having common zeros has been proved. This result, obtained in $C^n$, can relatively easily be generalized to $C^n$ for polycylinder norms.

The object of the present paper is to prove an analogous general theorem for a wider class of norms, not necessarily polycylinder ones. Norms in the numerator and in the denominator may differ from each other. This is an essential generalization of the result from [2], requiring the application of non-trivial facts from the theory of plurisubharmonic functions.

In consequence, there have been obtained results of stability type for mappings having the same zeros, and a lemma of Schwarz type, being in some cases a strengthening of a result of J. Siciak [5].

1. Notation

In this paper $C, C^n$ will denote, respectively, the field of complex numbers and the $n$-dimensional complex space. For $z = (z_1, \ldots, z_n) \in C^n$ we set

$$|z| = \max |z_i|, \quad ||z|| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$ 

Besides, for $w \in C^{n+1}$, we introduce the notation $w' = (w_1, \ldots, w_n)$ and $w = (w', w_{n+1}).$ Other notations will be taken analogously as in [2].

2. Fundamental notions

Let $\Omega$ be a bounded domain in $C^n$. In the further part of the paper we shall assume that

(a) $H = (h_1, \ldots, h_n)$ is a mapping holomorphic on $\Omega$ and continuous on $\partial \Omega$;

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this mapping has no zeros on $\partial G$, whereas in $\Omega$ it has isolated zeros in the sense of [2] (i.e., for any $z^0$ such that $H(z^0) = 0$ there exists a positive integer $\nu$ such that all $h_j (j = 1, \ldots, n)$ have a zero of order $\nu$ at $z^0$ and homogeneous parts of degree $\nu$ of these functions vanish simultaneously at $z^0$ only; $\nu$ is called the order of $z^0$).

(b) $F = (f_1, \ldots, f_m)$ is a mapping holomorphic on $\Omega$ and continuous on $\overline{\Omega}$. If $z^0$ is an isolated zero of order $\nu$ of $H$, then each $f_j (j = 1, \ldots, m)$ has a zero of order no less than $\nu$ at $z^0$.

c (c) $L$ is a real non-negative continuous and absolutely homogeneous function defined on $C^n$, i.e.,

$$L(\lambda u) = |\lambda| L(u), \quad u \in C^n, \quad \lambda \in C,$$

and it is plurisubharmonic as well.

d (d) $M$ is a real non-negative continuous and absolutely homogeneous function defined on $C^n$, satisfying the condition

$$M(z) > 0 \quad \text{for} \quad z \neq 0 \in C^n.$$

Under the above assumptions the function $\Phi$ is defined by the formula

$$\Phi(z) = \lim_{t \to 1} L(F(z^t)) / M(H(z^t)) \quad \text{for} \quad z \in \overline{\Omega}.$$

3. Auxiliary results

Let $S = \{z \in \overline{\Omega}; h_1(z) = \cdots = h_n(z) = 0\}$. It is easily seen that the set $S$ is finite and that

$$\Phi(z) = L(F(z)) / M(H(z)) \quad \text{for} \quad z \in \overline{\Omega} \setminus S.$$ 

Let $z^0 \in S'$ be an isolated zero of order $\nu$. Then $h_j$ expands in a series of homogeneous polynomials

$$h_j(z) = \sum_{k=0}^\infty Q_{j,k}(z-z^0)$$

in some neighborhood of $z^0$ and the system $Q_{j,k}(z) = 0, j = 1, \ldots, n$, has the trivial solution only. Let us introduce the notation $Q = (Q_1, \ldots, Q_n).$ In view of the above and condition (2), we have $M \cdot Q(z) \neq 0$ for $z \neq 0$. According to (b), for every $j, f_j$ expands in a series of homogeneous polynomials of the form

$$f_j(z) = \sum_{k=0}^\infty P_{j,k}(z-z^0), \quad j = 1, \ldots, m.$$

Let $F' = (P_1, \ldots, P_m)$.

**PROPERTY 1.** If $z^0 \in S'$ is the above-mentioned zero, then there exists a point $z^*$ such that $|z^*| = 1$ and

$$\Phi(z^*) = L(F(z^*)) / M(Q(z^*)).$$

**PROPERTY 2.** The function $\Phi$ is bounded.

**PROPERTY 3.** If $z^0 \in S'$, then there exist a positive integer $p$, a holomorphic curve $\varphi$, and a continuous subharmonic function $\psi$, such that

$$\Phi(z^0 + P\varphi(t)) = \psi(t), \quad |\varphi(t)| \neq 0 \quad \text{for} \quad t < 0.$$

Indeed, analogously as in [2] we prove for $n > 2$ that, if $z^0 \in S'$ and $a = Q(z^0)$, where $|z^0| = 1$, then there exist a positive integer $p$ and a holomorphic curve $\varphi(t)$, $|t| < 0$, such that $\varphi(0) = z^0$, and

$$H(z^0 + P\varphi(t)) = a t^p, \quad |\varphi(t)| \neq 0 \quad \text{for} \quad |t| < 0.$$ 

For $|t| < 0$. We also show easily (cf. [2]) that, if $z^0$ is a zero of order $\nu$ of the function $f_j$, then $f_j(z^0 + w, w') = w_j, f_j(w)$, where $f_j$ is a holomorphic function, while $|w, w'|$ — sufficiently small. Let $F' = (f_1, \ldots, f_m)$. We define the function $\psi$ by the formula

$$\psi(t) = (1 / M(Q(z^*))) L(F'(\varphi(t))), \quad |t| < 0.$$

4. The maximum principle

In this section we shall prove

**THEOREM 1.** The function $\Phi$ satisfies in $\Omega$ the maximum principle in the sense that

$$\sup_{z \in \Omega} \Phi(z) = \sup_{z \in \Omega} \Phi(z).$$
Proof. Without loss of generality we may assume that the function \( \Phi \) attains its upper bound \( \hat{M} \) at a point \( \hat{z} \in \Omega \), because otherwise there is nothing to prove.

Consider the first case when \( \hat{z} \notin S' \). Then, there exists a point \( \hat{z} \notin \Omega \setminus S' \) such that \( \Phi(\hat{z}) = \hat{M} \). Indeed, by \( (b) \) for \( x' = \hat{z} \) and the assumption made above, the function \( \phi \) attains its maximum for \( t = 0 \). Consequently, in virtue of the maximum principle for subharmonic functions, it is constant in the neighbourhood \( |t| < \varepsilon \), i.e., \( \Phi(t) = \hat{M} \) for \( |t| < \varepsilon \). It is sufficient to put \( \hat{z} = \varepsilon + t \hat{e} \) for some \( t \neq 0 \).

In view of the above, we may assume that the function \( \Phi \) attains its maximum at a point \( \hat{z} \notin S' \). Let \( H(\hat{z}) = (b_1, \ldots, b_n) = b \). Without loss of generality we may assume that \( b_1 \neq 0 \). Consider in \( \Omega \) the analytic set \( S = \{ z \in \Omega : h_2(z) = (b_2/b_1)h_1(z), \ldots, h_n(z) = (b_n/b_1)h_1(z) \} \). Let \( S' \) be an irreducible component of the set \( S \) in \( \Omega \), containing the point \( \hat{z} \), and \( S_1 = S \setminus S' \). This set is closed and connected in \( \Omega \setminus S' \) since \( S' \) is finite. Consider on this set the function \( g(z) = (b_1/M(b_1))L_1(1/h_1(z)\Phi(z)) \). Note that on \( S_1 \), the functions \( g \) and \( \Phi \) coincide. Consequently, the function \( g \) attains at \( \hat{z} \) its upper bound. On the other hand, on this set the mapping \( (1/h_1)^p \) is holomorphic (\( h_1(z) \neq 0 \) for \( z \in S_1 \)), that is, \( g \), as a superposition of a plurisubharmonic function with a holomorphic mapping, is a plurisubharmonic function on \( S_1 \), i.e., for every \( z \in S_1 \) \( g \) is a plurisubharmonic function in some neighbourhood (in \( C^2 \)) of \( \hat{z} \). Then, by the maximum principle for plurisubharmonic functions on analytic sets (see [3], p. 272), \( g(z) = \hat{M} \) on \( S_1 \) locally in some neighbourhood of \( \hat{z} \). Hence, on account of the connectivity of \( S_1 \) and the continuity of \( g \), it follows that \( g(z) = M \) for \( z \in S_1 \). Since the set \( S_1 \) is dense in \( S_1 \), therefore, according to the definition of \( \Phi \) as an upper limit, we also have \( \Phi(z) = M \) for \( z \in S_1 \). Hence, furthermore, in virtue of the Remmert–Stein theorem (see [4], p. 81), we have that \( \Phi(z) = \hat{M} \) arbitrarily close to \( \partial \Omega \), and so, in view of the continuity of \( \Phi \) in the neighbourhood of \( \partial \Omega \), it attains its upper bound on the boundary of \( \Omega \). This completes the proof.

5. Stability of mappings and Schwarz’s lemma

Let us first give a simple corollary from Theorem 1.

Corollary 1. If \( H, F, L, M \) satisfy the assumptions of Section 2, and

\[
L(F(z)) \leq AM(H(z))
\]

for \( z \in \partial \Omega \), then this inequality remains true for \( z \in \partial D \).

Let us further assume that \( m = n \) and that \( F \) and \( H \) satisfy conditions (a) and (b) simultaneously. Let \( S' = \{ z \in \Omega : f_1(z) = 0, \ldots, f_m(z) = 0 \} \).

Under the assumptions made at present, directly from Corollary 1 we obtain

Theorem 2. If \( S' = S' \), \( F \) and \( H \) have isolated zeroes of the same order, and \( L(F(z)) = M(H(z)) \) for \( z \in \partial \Omega \), then this equality remains true for \( z \in \partial D \).

Corollary 2. If \( S' \) is not empty, \( L = M = |z| \), and \( |F(z)| = |H(z)| \) for \( z \in \partial \Omega \), then

\[
F = a \cdot H,
\]

where \( a \) is an automorphism of the unit polycylinder, such that \( a(0) = 0, 0 \in C^n \).

The proof runs analogously as in [2].

Corollary 3. If the mappings \( F \) and \( H \) possess at least one single zero, \( L = M = |z| \), and \( |F(z)| = |H(z)| \) for \( z \in \partial \Omega \), then

\[
F = a \cdot H,
\]

where \( a \) is a linear unitary transformation.

Indeed, let \( z_0 \) be the above-mentioned single zero, and \( r > 0 \) a number so small that in the ball \( |w| < r \) there should exist mappings \( F^{-1}, H^{-1} \) inverse to \( F \) and \( H \). Then, by Theorem 2, \( |w| = |F \circ H^{-1}(w)| \) for \( |w| < r \), besides, \( F \circ H^{-1} \) is a biholomorphic mapping satisfying the condition \( F \circ H^{-1}(0) = F(z_0') = 0 \). Consequently, \( F \circ H^{-1} \) is a linear unitary transformation of \( C^n \) onto \( C^n \) (see [6], p. 547). Denote it by \( a \). Hence, \( F(z) = a \cdot H(z) \) in some neighbourhood of the point \( z_0' \), and thus in the entire domain \( \Omega \).

(e) Assume that \( L \) and \( M \) satisfy conditions (a) and (b) simultaneously, and \( M \) is a plurisubharmonic function. Let \( H \) satisfy condition (a), and that \( M \) and \( H(z) = 1 \) for \( z \in \partial \Omega \). Let \( F \) be a holomorphic mapping in \( \Omega \) such that the coordinates of this mapping have at points of the set \( S' \) zeros of at least the same order as \( H \).

Theorem 3. Under assumption (c), there exists a positive constant \( A \) such that \( L(F(z)) \leq A \) for \( z \in \partial \Omega \), then

\[
L(F(z)) \leq AM(H(z)) \quad \text{for} \quad z \in \partial \Omega.
\]

Proof. Note that \( M \) is a plurisubharmonic function in \( \partial \Omega \). Consequently, we have \( M(H(z)) \leq 1 \) for \( z \in \partial \Omega \). Take any point \( z_0' \in \partial \Omega \). If \( M \circ H(z_0') = 1 \), inequality \( (9) \) is obvious for \( z = z_0' \). So, assume that \( M \circ H(z_0') < 1 \), where \( 0 < e^s < 1 \). Take any number \( s \) satisfying the inequality \( 0 < s < e^s \), and an open set \( \{ z \in \Omega : M \circ H(z) < 1 - e \} \). Let \( D' \) be the component of this set, containing \( z_0' \). Then the function \( \Phi(z) \) defined by formula \( (7) \) satisfies in \( D' \) the maximum principle, i.e., \( \Phi(z) \leq A(1 - e^{-s}) \) for \( z \in D' \). Hence \( L(F(z)) \leq A(1 - e^{-s})M \circ H(z) \). Passing with \( s \to 0 \), we get \( L(F(z)) \leq AM(H(z)) \). For \( z \in \partial \Omega \), we obtain inequality \( (9) \) directly. This concludes the proof of the theorem.

(f) Assume that \( M \) satisfies the assumptions of the generalized Schwarz’s lemma. Let us define the set \( \Omega_M = \{ z \in C^n : M(z) < 1 \} \). This set, as it can easily be verified, is a bounded domain. Let \( M \) satisfy assumption (a).

Under the above assumptions there takes place

Corollary 4. If \( F \) is a holomorphic mapping in \( \Omega_M \), \( F(0) = 0 \), and there exists a constant \( A \) such that \( L(F(z)) \leq A \) for \( z \in \partial \Omega \), then

\[
L(F(z)) \leq AM(z) \quad \text{for} \quad z \in \Omega.
\]
Indeed, put in Theorem 3 $H(z) = (z_1, \ldots, z_n)$. Then, by the continuity of $M$, we have $M \ast H(z) = 1$ for $z \in \partial \Omega_M$. From this and Theorem 3 we obtain (10).

This result is some analogue to the lemma of [5] for $\nu = 1$.

Assume that $L$ satisfies the same assumptions as $M$ in (f). Then $\Omega_L = \{z \in \mathbb{C}^n : L(z) < 1\}$ is a bounded domain. Let $m = n$.

Directly from Corollary 4 we obtain

**Corollary 5** (cf. [5]). If $F$ is a biholomorphic mapping of $\Omega_M$ onto $\Omega_L$, and $F(0) = 0$, then

$$L \ast F(z) = M(z) \quad \text{for} \quad z \in \Omega_L.$$

6. Concluding remarks

The results obtained in the preceding section are connected with those of J. Siciak. In some cases Theorem 3 is a generalization of the lemma of [5]. For example, when the functions $T$ and $S$ of [5] satisfy, respectively, such conditions as $L$ and $M$ do, Theorem 3 is stronger than Theorem 3 of [5]. Namely, $\Omega$ may be any domain, not necessarily circular, the function $M \ast H$ may possess more than one zero, in contradistinction to the function $S$ of [5].

**References**


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