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LINDELÖF-TYPE THEOREMS FOR QUASICONFORMAL AND QUASIREGULAR MAPPINGS

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1. Introduction

The starting point of this survey is the following classical theorem of Lindelöf ([13], p. 69): *If f: $B^2 \to \boldsymbol{R}^2$ is a conformal mapping or a bounded analytic function of the unit disc B^2 having a limit α through a curve terminating at a given boundary point $b \in \partial B^2$, then f has an angular limit α at b.*

Quasiregular mappings in $\boldsymbol{R}^n$, $n \geqslant 2$, have many properties similar to those of analytic functions in $\boldsymbol{R}^2$, and therefore one of the main themes in the study of these mappings has been to examine which theorems concerning analytic functions have their counterparts for quasiregular mappings. For the general theory of quasiregular mappings the reader is referred to [7]–[9], and [29], and to the expository articles [15], [19]. Here we shall present some recent results proved in [23], [24], [25], and [26], which are counterparts of Lindelöf's theorem. More precisely, all the results in the sequel are related to the following problem. We remark that a mapping is said to be quasiconformal if it is a quasiregular homeomorphism.

PROBLEM. Let f: $B^n \to \boldsymbol{R}^n$ be a quasiconformal or quasiregular mapping of the unit ball B^n, let $b \in \partial B^n$, $E \subset B^n$, and $b \in \overline{E}$. Suppose that the limit $\lim_{x \to b,\, x \in E} f(x) = \alpha$ exists. Find conditions on E and f which ensure that α is the angular limit of f at b.

Conditions on the size of the set E will be given by means of the so-called capacity densities, defined in terms of moduli of curve families, which we shall discuss at first. The study of the problem above is divided into three cases according as the mapping f is (a) quasiconformal, (b) locally K-quasiconformal, or (c) bounded and quasiregular. The main tool in the study of cases (a) and (b) is the modulus of a curve family, whereas for the case (c) we need a recent two-constants theorem of Rickman ([16], 4.22), based on some results of Maz'ja [11].

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The classes of quasiconformal and quasiregular mappings of B^n contain as proper subclasses, when the dimension $n = 2$, the classes of conformal mappings and analytic functions of B^2, respectively (cf. [15], [19]). It seems that some of the results, in particular those in (a) (cf. Theorem 3.1), are new even in the special case $n = 2$. A unified discussion of the topic of this survey is contained in [33].

2. Preliminary results

In this section we shall introduce some notation and terminology and discuss some properties of the modulus of a curve family, on which the results in Sections 3 and 4 are based.

2.1. For $x \in \mathbf{R}^n$, $n \geqslant 2$, we write $x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i$, where $e_1, \ldots, e_n$ is the standard orthonormal basis in $\mathbf{R}^n$. For $r > 0$ let $B^n(x, r) = \{y \in \mathbf{R}^n : |y-x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$. If $r > s > 0$, then we write $R(x, r, s) = B^n(x, r) \setminus \overline{B}^n(x, s)$ and $R(r, s) = R(0, r, s)$.

2.2. Quasiregular mappings. Let $f\colon B^n \to \mathbf{R}^n$ be a continuous mapping in the Sobolev space $W^1_{n,\mathrm{loc}}(B^n)$. Then f is said to be *K-quasiregular* (K-qr) if there exists a constant $K \in [1, \infty)$ such that the following two inequalities hold a.e. in B^n:

$$\max_{|h|=1} |f'(x)h|^n \leqslant K J_f(x),$$

$$J_f(x) \leqslant K \min_{|h|=1} |f'(x)h|^n.$$

Here J_f is the Jacobian determinant of f and $f'(x)\colon \mathbf{R}^n \to \mathbf{R}^n$ is the linear mapping $f'(x)e_i = \partial f(x)/\partial x_i$, $i = 1, \ldots, n$. Note that both f' and J_f exist a.e. in B^n, because $f \in W^1_{n,\mathrm{loc}}(B^n)$. The mapping f is said to be *(locally) K-quasiconformal* (K-qc) if it is a K-qr (local) homeomorphism. The mapping f is *quasiregular* (*quasiconformal*) if it is K-qr (K-qc) for some $K \in [1, \infty)$. By virtue of important results of Rešetnjak, a non-constant qr mapping is discrete, open, and sense-preserving. For terminology and references to Rešetnjak's work the reader is referred to [7], [29].

2.3. *Remark.* It follows that a mapping of B^2 is 1-qr if and only if it is analytic (cf. [15], [19]). Since, in addition, many properties of analytic functions have their analogues for qr mappings (cf. [15], [19]), one can regard the theory of qr mappings of B^n as an n-dimensional counterpart of the theory of analytic functions of B^2.

2.4. Modulus of a curve family. Let Γ be a family of curves in $\mathbf{R}^n$. Denote by m the n-dimensional Lebesgue measure in $\mathbf{R}^n$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\varrho} \int_{\mathbf{R}^n} \varrho^n dm,$$

where ϱ runs through all non-negative Borel functions $\varrho\colon \mathbf{R}^n \to \mathbf{R}^1 \cup \{\infty\}$ with $\int_\gamma \varrho\, ds \geqslant 1$ for all locally rectifiable $\gamma \in \Gamma$.

2.5. LEMMA. *The modulus is an outer measure in the space of all curve families in $\mathbf{R}^n$.*

The basic properties of the modulus, like Lemma 2.5, can be found in Väisälä's book ([18], Ch. 1). If E, F, and G are subsets of $\mathbf{R}^n$, we write $\Delta(E, F; G) = \{\gamma\colon [0, c) \to G\colon \gamma \text{ continuous}, \gamma(0) \in E, \gamma(t) \to F \text{ as } t \to c\}$. The exact value of the modulus can be calculated only for very few curve families. Therefore various estimates are of importance. In the following examples we give some estimates, which will be used in the sequel.

2.6. EXAMPLES. (1) Let $b > a > 0$. If $E \subset \overline{B}^n(a)$ and $F \subset \mathbf{R}^n \setminus B^n(b)$, then

$$M\big(\Delta(E, F; \mathbf{R}^n)\big) \leqslant \omega_{n-1}\left(\log \frac{b}{a}\right)^{1-n},$$

and here ω_{n-1} is the $(n-1)$-dimensional measure of S^{n-1} (cf. [18], 7.5).

(2) Suppose that $b > a > 0$ and $E, F \subset \mathbf{R}^n$ with $S^{n-1}(r) \cap E \neq \emptyset \neq S^{n-1}(r) \cap F$ for $a < r < b$. By [18], 10.9, there exists a positive constant c_n depending only on n such that $M\big(\Delta(E, F; R(b, a))\big) \geqslant c_n \log(b/a)$.

(3) The modulus has the following *symmetry property* (cf. [23], 4.3): If $E, F \subset B^n$, then $M\big(\Delta(E, F; B^n)\big) \geqslant M\big(\Delta(E, F; \mathbf{R}^n)\big)/2$.

In addition to the lower bounds for the modulus given above, we shall apply the following lemma, called the *comparison principle* for the modulus (cf. Näkki [12], 3.3). The lemma is a modification of a result of Martio, Rickman, and Väisälä ([8], 3.11).

2.7. LEMMA. *Let $b > a > 0$, let F_1, F_2, and F_3 be three sets in $\mathbf{R}^n$ with $F_1, F_2 \subset \overline{B}^n(a)$, $F_3 \subset \mathbf{R}^n \setminus B^n(b)$, and let $\Gamma_{ij} = \Delta(F_i, F_j; \mathbf{R}^n)$. Then the following estimate holds*:

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log(b/a)\}.$$

Here c_n is the constant in 2.6 (2).

2.8. CAPACITY DENSITIES. For $E \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, and $r > 0$ we write

$$M(E, r, x) = M\big(\Delta(S^{n-1}(x, 2r), \overline{B}^n(x, r) \cap E; \mathbf{R}^n)\big).$$

The *lower* and *upper capacity densities* of E at x are defined by

$$\underline{\operatorname{cap\,dens}}(E, x) = \liminf_{r \to 0} M(E, r, x),$$

$$\overline{\operatorname{cap\,dens}}(E, x) = \limsup_{r \to 0} M(E, r, x).$$

2.9. *Remark.* From a result of Ziemer [27] it follows that for compact E one can define $M(E, r, x)$ by (cf. [10])

$$M(E, r, x) = \inf_u \int_{\mathbf{R}^n} |\nabla u|^n dm,$$

where u runs through all $C^\infty(R^n)$ functions with $\operatorname{spt} u \subset B^n(x, 2r)$ and $u(x) \geqslant 1$ for $x \in \bar{B}^n(x, r) \cap E$.

The modulus inequalities in the following lemma will be the main tool in Sections 3 and 4. For the inequalities in (1), (2), and (3) the reader is referred to [18], [14], and [7], respectively.

2.10. LEMMA. *Let f: $B^n \to R^n$ be a continuous mapping, let Γ be a curve family in B^n, and let $f\Gamma = \{f \circ \gamma \colon \gamma \in \Gamma\}$.*

(1) *If f is K-qc, then $M(\Gamma)/K \leqslant M(f\Gamma) \leqslant KM(\Gamma)$.*

(2) *If f is K-qr, then $M(f\Gamma) \leqslant KM(\Gamma)$.*

(3) *If f is K-qr and the elements of Γ are in a Borel set A in B^n and $N(y, f, A) = \operatorname{card} f^{-1}(y) \cap A$, $N(f, A) = \sup\{N(y, f, A) \colon y \in R^n\}$, and if $N(f, A) < \infty$, then*

$$M(\Gamma) \leqslant KN(f, A)M(f\Gamma).$$

The *cluster set* of a mapping f: $B^n \to R^n$ at $b \in \partial B^n$ is $C(f, b) = \bigcap_U \overline{f(B^n \cap U)}$, where U runs through all neighborhoods of b. It is clear that $C(f, b)$ is a non-empty compact connected set. $C(f, b)$ consists of one point if and only if f has a limit at b.

2.11. LEMMA. *Let f: $B^n \to R^n$ be qr, let (b_k) be a sequence in B^n with $b_k \to b \in \partial B^n$ and $f(b_k) \to 0$, and let $C(f, b) \subset \partial fB^n$. Fix $t \in (0, 1)$ and write $E = \bigcup B^n\big(b_k, t(1-|b_k|)\big)$. Then $f(x) \to 0$ when x approaches b through the set E. If there exists a number $s \in (0, 1)$ such that $1-|b_k| > s|b_k - b|$ for all $k = 1, 2, \ldots$, then $\operatorname{cap} \underline{\operatorname{dens}}\,(E, b) \geqslant D(n, s, t)$, and here $D(n, s, t)$ is a positive constant depending only on n, s, and t.*

Proof. Suppose that there exists a sequence (a_k) in E with $a_k \to b$, $f(a_k) \to \beta \neq 0$. After relabeling, if necessary, we may assume that $a_k \in B^n\big(b_k, t(1-|b_k|)\big) = B_k$ for each k. Write $\Gamma_k' = \Delta(fB_k, \partial fB^n; fB^n)$. Let Γ_k be the family of the maximal liftings of the paths in Γ_k' starting at B_k (for terminology, cf. [9], 3.11, 3.12). By 2.6 (1) and Lemma 2.10 (2) we get

$$M(\Gamma_k') \leqslant M(f\Gamma_k) \leqslant KM(\Gamma_k) \leqslant K\omega_{n-1}\left(\log\frac{1}{t}\right)^{1-n}.$$

Since $C(f, b) \subset \partial fB^n$ it follows from 2.6 (2) that $M(\Gamma_k') \to \infty$, which contradicts the inequality above. For the proof of the second part of the lemma we assume $1-|b_k| > s|b - b_k|$, $k = 1, 2, \ldots$ Then $F = \bigcup B^n\big(b_k, st(|b - b_k|)\big) \subset E$ and employing the notation in 2.8 we obtain by Lemma 2.5 and 2.6 (2) $M(E, |b_k - b|, b) \geqslant M(F, |b_k - b|, b) \geqslant c_n \log(1 + st) > 0$.

2.12. *Remarks.* The first part of Lemma 2.11 is based on some ideas of Bagemihl and Seidel [1]. It is easily seen that the condition $C(f, b) \subset \partial fB^n$ cannot be dropped (cf. [1]; [23], 6.4). If the condition $C(f, b) \subset \partial fB^n$ is not assumed, then one can prove a weaker result of this kind (cf. [1], [25], [33]).

3. Quasiconformal mappings

A mapping f: $B^n \to R^n$ is said to have an *angular limit* α at $b \in \partial B^n$ if for each $s \in (0, 1)$ the following holds: If (b_k) is a sequence in B^n with $b_k \to b$ and $1-|b_k| > s|b_k - b|$ for all k, then $f(b_k) \to \alpha$.

3.1. THEOREM. *Let f: $B^n \to R^n$ be qc, let $b \in \partial B^n$, and for $\varepsilon > 0$ let $\delta_\varepsilon = \operatorname{cap} \underline{\operatorname{dens}}\,\big(f^{-1}B^n(\varepsilon), b\big)$. If $\limsup_{\varepsilon \to 0} \delta_\varepsilon (\log(1/\varepsilon))^{n-1} = \infty$, then f has an angular limit 0 at b.*

Proof. Suppose that this is not the case. Then there exist a number $s \in (0, 1)$ and a sequence (b_k) in B^n with $b_k \to b$, $1-|b_k| > s|b_k - b|$, $k = 1, 2, \ldots$, such that $f(b_k) \to \beta \neq 0$. Since f is injective it is obvious that $C(f, b) \subset C(f, \partial B^n) = \partial fB^n$. By Lemma 2.11 there is an integer k_0 such that if $F = \bigcup_{k \geqslant k_0} B^n\big(b_k, \tfrac{1}{2}(1-|b_k|)\big)$, then $fF \subset R^n \setminus B^n(|\beta|/2)$ and $M(F, |b_k - b|, b) \geqslant D(n, s, \tfrac{1}{2})/2 = D > 0$ for all $k \geqslant k_0$. Write $\Gamma_\varepsilon = \Delta(E_\varepsilon, F; B^n)$. For each $\varepsilon > 0$ choose $k_\varepsilon \geqslant k_0$ with $M(E_\varepsilon, \varrho_\varepsilon, b) \geqslant \delta_\varepsilon/2$, where $\varrho_\varepsilon = |b_{k_\varepsilon} - b|$. Apply 2.6 (2), (3) and Lemma 2.7 with $F_1 = E_\varepsilon \cap \bar{B}^n(b, \varrho_\varepsilon)$, $F_2 = F \cap \bar{B}^n(b, \varrho_\varepsilon)$, and $F_3 = S^{n-1}(b, 2\varrho_\varepsilon)$ to obtain

$$M(\Gamma_\varepsilon) \geqslant 2^{-1}3^{-n}\min\{\delta_\varepsilon/2, D, c_n \log 2\}. \tag{3.2}$$

By 2.6 (1) we get for $\varepsilon \in (0, |\beta|/2)$

$$M(f\Gamma_\varepsilon) \leqslant \omega_{n-1}\left(\log\frac{|\beta|}{2\varepsilon}\right)^{1-n},$$

which by (3.2) and Lemma 2.10 (1) yields a contradiction unless $\delta_\varepsilon \to 0$ when $\varepsilon \to 0$. Thus there is, in view of (3.2), a number $r_1 \in (0, |\beta|/2)$ with $M(\Gamma_\varepsilon) \geqslant 3^{-n-2}\delta_\varepsilon$ for $\varepsilon \in (0, r_1)$, and we get by Lemma 2.10 (1)

$$\delta_\varepsilon\left(\log\frac{|\beta|}{2\varepsilon}\right)^{n-1} \leqslant 3^{n+2}K\omega_{n-1}$$

for $\varepsilon \in (0, r_1)$. Letting $\varepsilon \to 0$ yields a contradiction.

3.3. COROLLARY. *Let f: $B^n \to R^n$ be qc, let $b \in \partial B^n$, let $E \subset B^n$ with $b \in \bar{E}$, and let $\lim_{x \to b,\, x \in E} f(x) = 0$. If $\operatorname{cap} \underline{\operatorname{dens}}(E, b) = \delta > 0$, then f has angular limit 0 at b.*

Proof. The proof follows from Theorem 3.1, since here $\delta_\varepsilon \geqslant \delta$ for all $\varepsilon > 0$.

3.4. COROLLARY. *Let f: $B^n \to R^n$ be qc, let $b \in \partial B^n$, let $E \subset B^n$ be a connected set with $b \in \bar{E}$, and let $f(x) \to 0$ as $x \to b$ and $x \in E$. Then f has angular limit 0 at b.*

Proof. From 2.6 (2) it follows that the condition in 3.3 is satisfied with $\delta = c_n \log 2$.

3.5. *Remarks.* (1) Corollary 3.4 follows also from a result of F. W. Gehring ([3], p. 21).

(2) There are very thin sets E satisfying $\operatorname{cap} \underline{\operatorname{dens}}(E, b) > 0$. By a result of Wallin (cf. [23], 2.5 (3)) it is possible that even the Hausdorff dimension of E is zero.

(3) It has been shown in [23], Section 5, that Theorem 3.1 is a qc counterpart of a theorem of J. L. Doob ([2], Thm 4) concerning bounded analytic functions (cf. also T. Hall [5], Thm II).

(4) One can show that in the assumptions of Corollary 3.3, the condition $\operatorname{cap}\underline{\operatorname{dens}}\,(E, b) > 0$ cannot be replaced by $\operatorname{cap}\overline{\operatorname{dens}}\,(E, b) > 0$ (cf. [23], 6.5).

3.6. *Remark.* With minor modifications of the proof, one can extend Theorem 3.1 to cover the case of *boundary-preserving* qr mappings as well (cf. [20], Section 4; [22]). A qr mapping $f\colon B^n \to R^n$ is said to be boundary-preserving if $C(f, \partial B^n) = \partial f B^n$. It is clear that qc mappings, i.e. injective qr mappings, are boundary-preserving.

4. Locally K-quasiconformal mappings in space

Throughout the entire section we assume that $f\colon B^n \to R^n$ is locally K-qc, $n \geqslant 3$, $b \in \partial B^n$, and $\varphi \in (0, \pi/2)$. We shall present here a counterpart of Theorem 3.1 for these mappings. The assumption $n \geqslant 3$ is made because the following lemma of Martio, Rickman, and Väisälä ([9], 2.3), essential for the sequel, holds only for dimensions $n \geqslant 3$ (cf. [9], 2.11).

4.1. Lemma. *There is a constant $\psi(n, K) \in (0, 1)$ depending only on n and K such that $f|B^n(\psi(n, K))$ is injective.*

4.2. Corollary. *For every $r \in (0, 1)$ there is a constant $c(n, K, r)$ depending only on n, K, and r such that $N(f, \bar{B}^n(r)) \leqslant c(n, K, r)$.*

We denote by $K(b, \varphi)$ the cone $\{z \in R^n\colon (b|b-z) > |b-z|\cos\varphi\}$. Here $(u|v)$ is the inner product $\sum_{i=1}^{n} u_i v_i$ of vectors $u, v \in R^n$.

4.3. Theorem. *Suppose that $E \subset K(b, \varphi) \cap B^n$, $b \in \bar{E}$, and that $\lim_{x\to b,\, x\in E} f(x) = 0$. If $\operatorname{cap}\underline{\operatorname{dens}}(E, b) = \delta > 0$, then f has an angular limit 0 at b.*

Idea of proof. Choose $r_0 \in (0, \cos\varphi)$ such that $M(E, s, b) \geqslant 2\delta/3$ for all $s \in (0, r_0]$. Fix $\lambda > 1$ with $\log\lambda = (\delta/3\omega_{n-1})^{1/(1-n)}$. For $s \in (0, r_0]$ let $A_\lambda^\varphi(s) = K(b, \varphi) \cap R(b, s, s/\lambda)$. Then it follows from Example 2.6 (1) that $M(E \cap A_\lambda^\varphi(s), s, b) \geqslant \delta/3$. The proof now follows from a long calculation, where we use Lemma 2.7, and from the fact that there is a constant $C > 0$ depending only on n, K, φ, and δ such that $N(f, \bar{A}_\lambda^\varphi(s)) \leqslant C$ for all $s \in (0, r_0]$ (cf. Corollary 4.2).

4.4. Corollary. *Suppose that $\lim_{t\to 1,\, t\in(0,1)} f(tb) = 0$. Then f has an angular limit 0 at b.*

4.5. *Remarks.* (1) Theorem 4.3 and Corollary 4.4 fail to hold for $n = 2$. A counterexample is provided by the function $g(z) = \exp(-(1-z)^{-4})$, $z \in B^2$, which is a qr local homeomorphism having a radial limit 0 at $z = 1$. However, the function g does not have an angular limit at $z = 1$, since $\lim_{z\to 1,\, z\in L} g(z) = \infty$ when $L = \{x + i(1-x)\colon 0 < x < 1\}$.

(2) It seems to be an open problem whether the assumption $E \subset K(b, \varphi) \cap B^n$ in Theorem 4.3 is necessary (cf. 5.2).

We shall now show that Theorem 4.3 can be improved if the condition $C(f, b) \subset \partial f B^n$ holds. For this purpose we shall need the following estimate of the maximal multiplicity, which is a consequence of Corollary 4.2. This estimate is sharp when φ is fixed (cf. [25], 2.15).

4.6. Lemma. *For $\lambda \geqslant 2$ and $s \in (0, \cos\varphi)$ let $A_\lambda^\varphi(s) = K(b, \varphi) \cap R(b, s, s/\lambda)$. There exists a constant $d(n, K, \varphi) > 0$ depending only on n, K, and φ such that $N(f, A_\lambda^\varphi(s)) \leqslant d(n, K, \varphi)\log\lambda$.*

The next theorem is a counterpart of Theorem 3.1. Note that here δ_ε has a meaning slightly different from the one in 3.1.

4.7. Theorem. *Let $E_\varepsilon = K(b, \varphi) \cap f^{-1}B^n(\varepsilon)$ and $\delta_\varepsilon = \operatorname{cap}\underline{\operatorname{dens}}\,(E_\varepsilon, b)$ for $\varepsilon > 0$. If $C(f, b) \subset \partial f B^n$ and $\limsup_{\varepsilon\to 0} \delta_\varepsilon^{n/(n-1)}\left(\log\frac{1}{\varepsilon}\right)^{n-1} = \infty$, then f has an angular limit 0 at b.*

Idea of proof. The proof makes use of ideas in the proofs of Lemma 2.11, Theorem 3.1, and Theorem 4.3. More specifically, one chooses for each $\varepsilon > 0$ a number $\lambda_\varepsilon \geqslant 2$ such that $\log\lambda_\varepsilon = (\delta_\varepsilon/3\omega_{n-1})^{1/(1-n)}$ and applies the method in 3.1 to the domains $D_\varepsilon = A_{\lambda_\varepsilon}^\varphi(s_\varepsilon)$ when $s_\varepsilon \in (0, \cos\varphi)$ is small. While doing so one applies Lemma 2.10 (3) and the upper bound $N(f, D_\varepsilon) \leqslant \text{const}\cdot\log\lambda_\varepsilon = \text{const}\cdot\delta_\varepsilon^{1/(1-n)}$ (cf. Lemma 4.6). Using some ideas of Gehring and Väisälä ([4], Lemma 8.2) one can show that there exists a number $K \geqslant 1$ and for each $\varepsilon > 0$ a K-qc mapping $h_\varepsilon\colon R^n \to R^n$ with $h_\varepsilon^1 D_\varepsilon = B^n$. Now one can use 2.10 (1), 2.6 (3) and proceed as in the proof of Theorem 3.1.

4.8. *Remark.* The results in this section can be proved, with minor changes, for such qr mappings $f\colon B^n \to R^n$, $n \geqslant 2$, that have the following property: There are numbers $p \in [1, \infty)$ and $t \in (0, 1)$ such that $N(f, B^n(x, t(1-|x|))) \leqslant p$ for all $x \in B^n$ (cf. [25]). Locally K-qc mappings of B^n, $n \geqslant 3$, have this property by Lemma 4.1. Note that the function g in 4.5 (1) fails to have this property.

5. Bounded quasiregular mappings

In this final section we shall present some counterparts of the results in Sections 3 and 4 for the case of bounded qr mappings of B^n when the dimension $n \geqslant 2$. No proofs will be given here, for details the reader is referred to [26]. The methods in [26] are different from those in Sections 3 and 4 and are based on a local version of a two-constants theorem for qr mappings, which Rickman ([16], 4.22) recently derived from some estimates of solutions of elliptic differential equations due to Maz'ja [11].

Throughout this section we shall assume that $f\colon B^n \to B^n$, $n \geqslant 2$, is a qr mapping and $b \in \partial B^n$. The next result follows from a theorem of Martio and Rickman [6].

5.1. LEMMA. *If the radial limit* $\lim_{t\to 1,\, t\in(0,1)} f(tb) = \alpha$ *exists, then f has an angular limit α at b.*

5.2. *Remark.* It has been an open question whether, in the assumptions of Lemma 5.1, the limit along the radius terminating at b can be replaced by a limit through a curve in B^n terminating at b. This question was recently answered in the negative by Rickman [17], who constructed, for a given $n \geqslant 3$, a qr mapping $g\colon B^n \to B^n$ such that there exist infinitely many curves $\gamma_1, \gamma_2, \ldots$ in B^n terminating at $e_1 \in \partial B^n$ and distinct points $\alpha_1, \alpha_2, \ldots$ in R^n such that $g(x) \to \alpha_j$ as $x \to e_1$ and $x \in \gamma_j$ but such that g does not have an angular limit at e_1. Note that g is not a local homeomorphism in Rickman's construction (cf. Remark 4.5 (2)).

The following generalization of Lemma 5.1 was proved in [26].

5.3. THEOREM. *Let* $\varphi \in (0, \pi/2)$, $E \subset K(b, \varphi) \cap B^n$, $b \in \overline{E}$, *and suppose that* $\lim_{x\to b,\, x\in E} f(x) = \alpha$. *If* $\operatorname{cap}\underline{\operatorname{dens}}(E, b) > 0$, *then f has an angular limit α at b.*

Note that Theorem 5.3 resembles Theorem 4.3, but the assumptions on the mapping are different in these results.

Rickman's example, where none of the curves γ_j is contained in a cone $K(e_1, \varphi)$, shows that the non-tangentiality condition $E \subset K(b, \varphi)$ in Theorem 5.3 cannot be dropped when the dimension $n \geqslant 3$. On the other hand, it is possible to drop the mentioned condition for all dimensions $n \geqslant 2$ if the set E contains a piece of an $(n-1)$-dimensional surface which has certain regularity properties at b (cf. Rickman [17]). Finally, we mention the following weaker result, which can be proved by means of Poleckii's inequality, Lemma 2.10 (2) (cf. [21]).

5.4. THEOREM. *Let G be a domain in B^n with* $b \in \overline{G} \subset B^n \cup \{b\}$. *If the limit* $\lim_{x\to b,\, x\in\partial G\setminus\{b\}} f(x) = \alpha$ *exists, then also* $\lim_{x\to b,\, x\in\overline{G}\setminus\{b\}} f(x) = \alpha$.

5.5. *Remark.* The boundedness condition in 5.1, 5.3, and 5.4 can be weakened (cf. [6], [26], [21], [33]).

5.6. Addendum. Since this paper was submitted for publication, several results connected with this topic have been proved (cf. [31], [32], [30]). In [30] and [32] results applicable to the coordinate functions f_j of a qc or qr mapping $f = (f_1, \ldots, f_n)$ were proved. A unified treatise of these questions is given in [33]. Granlund, Lindqvist, and Martio have proved some interesting results in [28].

5.7. THEOREM ([31]). *For $n \geqslant 2$ and $K \geqslant 1$ there exists a positive number $\mu = \mu(n, K)$ with the following property. Let $f\colon B^n \to B^n$ be a bounded K-qr mapping, $b \in \partial B^n$, let $\beta\colon (0, 1) \to (0, 1)$ be a continuous increasing function with $\beta(t) \to 0$ as $t \to 0$, let $\gamma\colon (0, 1] \to B^n$ be a curve with $\gamma(t) \to b$ as $t \to 0$ and $1 - |\gamma(t)| = \beta(t)|\gamma(t) - b|$ for all $t \in (0, 1]$. Denote $M(t) = |f(\gamma(t)) - \alpha|$, $t \in (0, 1)$. If $M(t)$ decreases towards 0 as $t \to 0$, and, in addition,*

$$\lim_{t\to 0} \beta(t)^\mu \log\frac{1}{M(t)} = \infty, \tag{5.8}$$

then f has an angular limit α at b.

This result implies, in particular, that the function g in Rickman's example (cf. 5.2) cannot tend to α_j along γ_j at the rate implied by (5.8). If (5.8) is replaced by a somewhat stronger assumption, then, as shown in [31], f will be identically equal to α.

References

[1] F. Bagemihl and W. Seidel, *Sequential and continuous limits of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. A I 280 (1960), 1–17.

[2] J. L. Doob, *The boundary values of analytic functions*, Trans. Amer. Math. Soc. 34 (1932), 153–170.

[3] F. W. Gehring, *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I 336/11 (1963), 1–21.

[4] F. W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. 114 (1965), 1–70.

[5] T. Hall, *Sur la mesure harmonique de certains ensembles*, Ark. Mat. Astr. Fys. 25A, 28 (1937), 1–8.

[6] O. Martio and S. Rickman, *Boundary behavior of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 507 (1972), 1–17.

[7] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, ibid. 448 (1969), 1–40.

[8] —, —, —, *Distortion and singularities of quasiregular mappings*, ibid. 465 (1970), 1–13.

[9] —, —, —, *Topological and metric properties of quasiregular mappings*, ibid. 488 (1971), 1–31.

[10] O. Martio and J. Sarvas, *Density conditions in the n-capacity*, Indiana Univ. Math. J. 26 (1977), 761–776.

[11] V. G. Maz'ja, *On the continuity at a boundary point of the solution of quasilinear elliptic equations* (Russian), Vestnik Leningrad. Univ. 25 (13) (1970), 42–55.

[12] R. Näkki, *Extension of Loewner's capacity theorem*, Trans. Amer. Math. Soc. 180 (1973), 229–236.

[13] R. Nevanlinna, *Analytic Functions*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 162, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[14] E. A. Poleckii, *The modulus method for non-homeomorphic quasiconformal mappings* (Russian), Mat. Sb. 83 (1970), 261–272.

[15] S. Rickman, *On the theory of quasiregular mappings*, in; *The Proceedings of the Rolf Nevanlinna Symposium on Complex Analysis, Silivri 1976*, pp. 79–87, Baski Atölyesi, Istanbul 1978.

[16] —, *On the number of omitted values of entire quasiregular mappings*, J. Analyse Math. 37 (1980), 100–117.

[17] —, *Asymptotic values and angular limits of quasiregular mappings of a ball*, Ann. Acad. Sci. Fenn. Ser. A I, 5 (1980), 185–196.

[18] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[19] J. Väisälä, *A survey of quasiregular maps in* R^n, The Proceedings of the International Congress of Mathematicians, Helsinki, Finland, August 15–23, 1978, Vol. II, pp. 685–691.
[20] M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 11 (1976), 1–44.
[21] —, *On the Iversen-Tsuji theorem for quasiregular mappings*, Math. Scand. 41 (1977), 90–98.
[22] —, *On angular limits of closed quasiregular mappings*, The Proceedings of the first Finnish–Polish Summer School in Complex Analysis, Podlesice, Poland, June 16–22, 1977, Part II, Ed. by J. Ławrynowicz and O. Lehto, pp. 69–74, University of Łódź, Łódź 1978.
[23] —, *On the existence of angular limits of n-dimensional quasiconformal mappings*, Ark. Mat. 18 (1980), 157–180.
[24] —, *Lower bounds for the moduli of path families with applications to non-tangential limits of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 4 (1979), 279–291.
[25] —, *On the boundary behavior of locally K-quasiconformal mappings in space*, ibid. 5 (1980), 79–95.
[26] —, *Capacity densities and angular limits of quasiregular mappings*, Trans. Amer. Math. Soc. 263 (1981), 343–354.
[27] W. P. Ziemer, *Extremal length and p-capacity*, Michigan Math. J. 16 (1969), 43–51.
[28] S. Granlund, P. Lindqvist, and O. Martio, *Phragmén–Lindelöf's and Lindelöf's theorems*, in preparation.
[29] Ju. G. Rešetnjak, *Spatial mappings of bounded distortion* (Russian), Izdat. Nauka, Novosibirsk, 1982.
[30] M. Vuorinen, *On the Harnack constant and the boundary behavior of Harnack functions*, Ann. Acad. Sci. Fenn. Ser. A I 7 (1982), 259–277.
[31] —, *Some uniqueness theorems for quasiregular mappings*, in preparation.
[32] —, *On functions with a finite or locally finite Dirichlet integral*, in preparation.
[33] —, *The boundary behavior of quasiregular mappings*, in preparation.

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