(see, for instance, the paper [6] by A. D. Myshkis; non-self-adjoint differential equations in R^2 are dealt with in [8]).

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Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979

COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

SOLUTIONS WITH PRESCRIBED PERIODS ON THE BOUNDARY COMPONENTS OF NON-LINEAR ELLIPTIC SYSTEMS OF FIRST ORDER IN MULTIPLY CONNECTED DOMAINS IN THE PLANE

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Let G be a domain in the plane. Dirichlet's boundary value problem for holomorphic functions is the following: to determine a holomorphic function in G for which the real part assumes given boundary values g on the boundary ∂G. The solution is uniquely determined, if one prescribes the value of the imaginary part at one point z_0. Only in the case of simply-connected domains the solution is necessarily single-valued. But in the case of multiply connected domains the solution of Dirichlet's boundary value problem possesses (purely imaginary) periods on the boundary components γ_j, in general. However a single-valued solution to Dirichlet's boundary value problem exists, if we replace the given boundary data g on γ_j by $g+c_j$, where the c_j are suitably chosen constants.

An analogous assertion holds in the case of linear elliptic systems (see [2]). The aim of the present paper is to prove that an analogous theorem is valid also for general non-linear elliptic systems on the plane.

In the second part of the paper we consider systems permitting solutions with additive periods. For such systems we prove the existence of a solution possessing arbitrarily prescribed periods on the boundary components.

We will be concerned with the differential equation

$$(*)\qquad \frac{\partial w}{\partial z^*} = F\left(z, w, \frac{\partial w}{\partial z}\right),$$

where the right-hand side $F(z, w, h)$ fulfils the following conditions (λ is a fixed real number, $0 < \lambda < 1$, $w = (w_1, \ldots, w_m)$, $h = (h_1, \ldots, h_m)$, $F = (F_1, \ldots, F_m)$):

$$|F_j(z_2, w, h) - F_j(z_1, \tilde{w}, \tilde{h})| \leqslant l\Big(|z_2 - z_1|^\lambda + \sum_j |w_j - \tilde{w}_j| + \sum_j |h_j - \tilde{h}_j|\Big),$$

$$\|F(\cdot, w, h) - F(\cdot, \tilde{w}, \tilde{h})\|_\lambda \leqslant L_1\|w - \tilde{w}\|_\lambda + L_2\|h - \tilde{h}\|_\lambda.$$

Here $||\cdot||_\lambda$ denotes the norm in the space $\mathscr{C}_\lambda(\bar{G})$ (see [3]). Analogously, by $||\cdot||_{1,\lambda}$ we denote the norm of functions possessing Hölder-continuous first order derivatives in $\bar{G}$. We will need the usual operators T_G and Π_G (see I.N. Vekua [1]).

Denote (see [3]) by $\Phi_{(w,h)}$ the holomorphic solution of the boundary value problem

$$\operatorname{Re}\Phi_{(w,h)} = -\operatorname{Re} T_G F(\cdot, w, h) \quad \text{on } \partial G,$$
$$\operatorname{Im}\Phi_{(w,h)}[z_0] = -\operatorname{Im} T_G F(\cdot, w, h)[z_0].$$

The function $\Phi_{(w,h)}$ depending on the pair (w,h) possesses purely imaginary additive periods d_j on the boundary components. As regards the dependence of the periods d_j on the pair (w,h) the following auxiliary theorem holds:

AUXILIARY THEOREM. *Let d_j and $\tilde{d}_j$ be the periods corresponding to the pairs (w,h) and $(\tilde{w},\tilde{h})$, respectively. Then there exists a constant such that*

$$|d_j-\tilde{d}_j| \leqslant \text{const}\cdot||(w,h)-(\tilde{w},\tilde{h})||_\lambda.$$

Proof. Since

$$||T_G F(\cdot, w, h) - T_G F(\cdot, \tilde{w}, \tilde{h})||_{1,\lambda} \leqslant ||T_G||\cdot||(w,h)-(\tilde{w},\tilde{h})||_\lambda,$$

one can estimate the boundary values of

$$T_G F(\cdot, w, h) - T_G F(\cdot, \tilde{w}, \tilde{h})$$

and of the first order derivatives of the last expression by $||(w,h)-(\tilde{w},\tilde{h})||_\lambda$. By virtue of Schauder's theorem the norm $||\operatorname{Re}\Phi_{(w,h)} - \operatorname{Re}\Phi_{(\tilde{w},\tilde{h})}||_{1,\lambda}$ can be also estimated by $||(w,h)-(\tilde{w},\tilde{h})||_\lambda$.

On the one hand, the periods of $\operatorname{Im}\Phi_{(w,h)}$ can be represented by the integral

$$d_j = \int_{\gamma_j} \frac{\partial \operatorname{Re}\Phi_{(w,h)}}{\partial n}\, ds.$$

An analogous representation holds for $\tilde{d}_j$ and, consequently, for $d_j-\tilde{d}_j$. In view of the estimate for $||\operatorname{Re}\Phi_{(w,h)} - \operatorname{Re}\Phi_{(\tilde{w},\tilde{h})}||_{1,\lambda}$ proved above the assertion follows immediately from the last representation.

On the other hand, for given d_j there exists a holomorphic function whose real part has constant value on every boundary component γ_j. This function is uniquely determined, if it is required that its imaginary part should vanish at a chosen point z_0 and, moreover, if one of the constant boundary values of the real part is zero. Moreover, the function in question is a linear combination of a finite number of holomorphic functions (which are connected with the harmonic measures of the boundary components), and depends on the periods d_j continuously. Subtracting this linear combination from $\Phi_{(w,h)}$ one gets a function $\hat{\Phi}_{(w,h)}$ possessing the following properties:

(a) $\operatorname{Re}\hat{\Phi}_{(w,h)} = g + \text{const}$ on every γ_j,

(b) $\operatorname{Im}\hat{\Phi}_{(w,h)}[z_0] = c$,

(c) The periods of $\operatorname{Im}\hat{\Phi}_{(w,h)}$ on γ_j are 0, and thus $\hat{\Phi}_{(w,h)}$ is single-valued.

It is easy to prove that the following estimate holds:

$$||\hat{\Phi}_{(w,h)} - \hat{\Phi}_{(\tilde{w},\tilde{h})}||_{1,\lambda} \leqslant \text{const}\cdot||(w,h)-(\tilde{w},\tilde{h})||_\lambda.$$

Analogously we proceed with the holomorphic function Ψ (see [3]), such that $\operatorname{Re}\Psi$ assumes the prescribed boundary values g on ∂G. The function $\hat{\Psi}$ possesses purely imaginary periods on the boundary components. Repeating the construction (given above) of a holomorphic function with prescribed purely imaginary periods one gets, finally, a holomorphic function $\hat{\Psi}$ fulfilling the following conditions:

(a) $\operatorname{Re}\hat{\Psi} = g + \text{const}$ on every γ_j,

(b) $\operatorname{Im}\hat{\Psi}[z_0] = 0$,

(c) $\hat{\Psi}$ is single-valued.

The function $\hat{\Psi}$ is uniquely determined, if one of the constants is prescribed. Replacing Ψ and $\Phi_{(w,h)}$ in the definition of the operator $\tilde{T}$ given in [3] by $\hat{\Psi}$ and $\hat{\Phi}_{(w,h)}$, respectively, we define an operator $\hat{T}$ having the following properties:

If (w,h) is a fixed element of T, then w is a solution of the differential equation (*) and satisfies the conditions:

(a) $\operatorname{Re} w = g + c_j$ on γ_j,

(b) $\operatorname{Im} w[z_0] = c$,

(c) w is single-valued.

The proof of this assertion is analogous to the considerations in [3], Chapter 11.3. In this way we get the following

THEOREM 1. *Let the constants L_1, L_2 be small enough. Then there exists a single-valued solution w of the equation (*), which is a solution to the following boundary value problem*:

(a) $\operatorname{Re} w = g + c_j$ *on* γ_j,

(b) $\operatorname{Im} w[z_0] = c$,

where g and c are given data. The solution is uniquely determined, if one of the constants c_j is prescribed.

Remark. The limitation of L_1 can be replaced by a limitation of the diameter $\operatorname{diam}(G)$ of the domain considered (see [3], p. 232).

Now we assume, additionally that the right-hand side $F(z,w,h)$ of the equation (*) fulfils the condition

$$(**) \qquad F(z, w+i\alpha, h) = F(z, w, h)$$

for every real α. This means that $w+i\alpha$ is a solution whenever w is a solution. Supposing (**), one can construct solutions of (*) possessing arbitrarily prescribed purely imaginary additive periods on the boundary components γ_j. In order to construct such a solution, the function $\hat{\Psi}$ constructed above is to be replaced by a holomorphic multiply-valued function with prescribed purely imaginary additive periods on the boundary components.

The space in which we look for a solution of (*) is the space of all functions $w = \hat{\Psi} + w_0$, where w_0 is a single-valued function belonging to $\mathscr{C}^1_\lambda(\bar{G})$. Instead of the operator T of [3] we use the operator constructed above. This operator maps the space under consideration into itself. Hence follows

THEOREM 2. *Assume that the right-hand side of the partial differential equation* (*) *fulfils condition* (**). *Then for given data* g, c *and* d_j (*where* $\sum_j d_j = 0$) *there exists a solution* w *satisfying the following conditions*:

(a) $\operatorname{Re} w = g + c_j$ *on* γ_j,

(b) $\operatorname{Im} w[z_0] = c$,

(c) *the period of* $\operatorname{Im} w$ *on* γ_j *is equal to* d_j.

The solution is uniquely determined, if one of the constants c_j *is prescribed.*

It is clear that we need a limitation of L_1 and L_2 as an assumption for the validity of Theorem 2. Again one can replace limitation of L_1 by limitation of $\operatorname{diam}(G)$.

Denote $\|F(\cdot, 0, 0)\|_\lambda$ by M. Using the triangle inequality, we infer from the second assumption about the right-hand side $F(z, w, h)$ that

$$\|F(\cdot, w, h)\|_\lambda \leqslant M + L_1 \|w\|_\lambda + L_2 \|h\|_\lambda .$$

This means that the norm of $F(\cdot, w, h)$ is bounded in a certain polycylinder

$$\mathscr{D} = \{(w, h)\colon \|w\|_\lambda \leqslant R_1, \|h\|_\lambda \leqslant R_2\}.$$

Therefore the norms $\|\hat{\Phi}_{(w,h)}\|_{1,\lambda}$, and consequently the functions $\hat{\Phi}_{(w,h)}$, are bounded. On the other hand, to the functions $\hat{\Psi}$ defined above one can add an arbitrary constant. Choosing this constant sufficiently large we make the constructed solution w possess a positive real part. This proves ([1]) the following

THEOREM 3. *There exists a solution* w *of the problem defined in Theorem* 2 *such that* $\operatorname{Re} w$ *is positive in* $\overline{G}$.

Remark. Since w is multiply-valued, the imaginary part is unbounded, in general.

As a special case of Theorem 3 we get

COROLLARY. *There exist single-valued solutions* w (*i.e. with all periods* d_j *vanishing*) *with a positive real part and constant boundary values on every boundary component.*

For the linear system

$$\frac{\partial v}{\partial y} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} + b_1 u,$$
$$-\frac{\partial v}{\partial x} = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} + b_2 u$$

the following stronger result was proved in [2], p. 188:

Let $w = u + iv$ be a solution such that u has constant boundary values on every boundary component γ_j. Then u is either positive or negative or vanishes identically.

([1]) Here we omit the discussion of the dependence of L_1, L_2 and M on R_1 and R_2.

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Presented to the Semester
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