The Schwarz inequality then yields

$$d^2(u_1-u_0)^2 \leqslant \int_{u_0}^{u_1}\int_{\beta'_u} dv\,du \cdot \int_{u_0}^{u_1}\int_{\beta'_u} \left|\frac{dz}{dw}\right|^2 dv\,du \leqslant |S'|_E \cdot |S|_E = |S| \cdot |S|_E \leqslant |D| \cdot \pi.$$

Letting $u_1 \to u_\infty$, we get

$$d^2(u_\infty-u_0)^2 \leqslant \pi \cdot |D|,$$

hence $u_\infty < \infty$: The trajectory γ necessarily has finite length in the metric $|f(z)|^{\frac{1}{2}}|dz|$.

In the general case, if γ is an arbitrary geodesic ray, we subdivide it into its straight open arcs γ_i, bounded by z_0 and the zeroes of f on γ. The orthogonal strips S_i of the arcs γ_i do not overlap, and we can use the above argument for every S_i. We get, by taking the square root,

$$d(u_i-u_{i-1}) \leqslant \sqrt{|S_i|} \cdot \sqrt{|S_i|_E},$$

hence

$$d(u_n-u_0) = d\sum_{i=1}^{n}(u_i-u_{i-1}) \leqslant \sum_{i=1}^{n}\sqrt{|S_i|} \cdot \sqrt{|S_i|_E}.$$

Applying again the Schwarz inequality yields

$$d^2(u_n-u_0)^2 \leqslant \sum_{i=1}^{n}|S_i| \cdot \sum_{i=1}^{n}|S_i|_E \leqslant |D| \cdot \pi.$$

The bound is independent of n, therefore

$$d^2(u_\infty-u_0)^2 \leqslant \pi \cdot |D|,$$

as before.

To finish the proof we remark that there exists a sequence of disjoint subarcs γ_n of γ with endpoints $z_n^{(1)} \to \zeta_1$, $z_n^{(2)} \to \zeta_2$, as both ζ_1 and ζ_2 belong to the cluster set of γ. The lengths of these arcs satisfy $\sum |\gamma_n| \leqslant |\gamma| < \infty$. We conclude $d(\zeta_1, \zeta_2) \leqslant \lim_{n\to\infty} |\gamma_n| = 0$, contradicting $d(\zeta_1, \zeta_2) > 0$.

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RELATIONS BETWEEN THE BOUNDARY VALUES AND PERIODS FOR GENERALIZED ANALYTIC FUNCTIONS IN R^n

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The basic idea of the concept of generalized analytic functions is the following: the class of holomorphic functions is to be replaced by a wider class in such a way that fundamental properties of holomorphic functions should retain their validity. It is possible to define generalized analytic functions of that kind either in C, C^n or in R^n.

In this paper the following approach to the concept of generalized analytic functions is applied: Instead of the real part u and the imaginary part v of a holomorphic function we regard a pair (u, ω_u), where u is a real-valued function and ω_u is a closed differential form depending on u and its first order derivatives. The aim of the paper is to discuss the interrelation between the periods of ω_u and the boundary values of u. This theory generalizes the well-known facts about the dependence of periods of a holomorphic function on the boundary values of its real part. We remark that the associated partial differential equations of second order are not necessarily self-adjoint.

1. A concept of generalized analytic functions in R^n

In order to define holomorphic functions or their generalizations there are three ways of approach. Firstly, one can define holomorphic and generalized analytic functions as vectors $(u_1, \ldots, u_m)$ with components u_i fulfilling a given first order partial differential system of elliptic type (this type of definition includes e.g. the notion of pseudo-analytic functions in L. Bers' sense). The second way is to define harmonic differential forms. This idea consists in replacing the real and imaginary parts of a holomorphic function by a differential form and its dual. The basic idea of the third generalization is to replace the imaginary part only by a differential form. The precise definition is the following:

Let ω_u be a differential form of degree $n-1$ depending on a real-valued function u and its first order derivatives. It will be assumed that the second order equation

$$d\omega_u = 0$$

is a partial differential equation of elliptic type. Then the pair (u, ω_u) is called a *generalized analytic function*, if ω_u is closed.

EXAMPLE. If ω_u is linear with respect to u, then it can be written in the form

$$(1)\qquad \omega_u = \sum_{i,j} (-1)^{i(n-i)} \left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u + c_i\right) dx_{i+1} \wedge \ldots \wedge dx_{i-1},$$

where a_{ij}, b_i, c_i are functions depending on $x_1, \ldots, x_n$ (the alternating product $dx_{i+1} \wedge \ldots \wedge dx_{i-1}$ stands for $dx_{i+1} \wedge \ldots \wedge dx_n \wedge dx_1 \wedge \ldots \wedge dx_{i-1}$). The differential form ω_u is *closed* iff u is a solution of the second order equation

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u + c_i\right) = 0.$$

In the case $n = 2$ from the equation $d\omega_u = 0$ follows the existence of a function v such that $\omega_u = dv$. If one regards multiply connected domains, then v possesses additive periods on the boundary components, in general. The function $f = u + iv$ is a solution of the first order system

$$\frac{\partial v}{\partial x_1} = a_{21}\frac{\partial u}{\partial x_1} + a_{22}\frac{\partial u}{\partial x_2} + b_2 u + c_2,$$

$$\frac{\partial v}{\partial x_2} = -a_{11}\frac{\partial u}{\partial x_1} - a_{12}\frac{\partial u}{\partial x_2} - b_1 u - c_1.$$

For $a_{ij} = \delta_{ij}$, $b_i = 0$, $c_i = 0$ the function f is holomorphic.

2. Canonical representation of generalized analytic functions in the sense above

A generalized analytic function (u, ω_u) possesses the so-called *canonical representation* if the coefficients b_i, c_i of the differential form ω_u defined by (1) fulfil certain additional conditions. Firstly, it may be assumed without restriction of generality that each c_i vanishes identically (see the concluding Remark 5). Consequently, it is sufficient to regard differential forms ω_u of type

$$\omega_u = \sum_{i,j} (-1)^{i(n-i)} \left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u\right) dx_{i+1} \wedge \ldots \wedge dx_{i-1}$$

in a given bounded domain in R^n. The following conditions are imposed on the domain G and the coefficients of ω_u (λ is a fixed real number, $0 < \lambda < 1$):

(a) The domain G belongs to A^1_λ; this means that the boundary ∂G possesses representations belonging to $\mathscr{C}^1_\alpha$.

(b) The functions a_{ij} belong to $\mathscr{C}^2_\alpha(\bar{G})$, the b_i belong to $\mathscr{C}^1_\alpha(\bar{G})$.

(c) The differential equation $d\omega_u = 0$ is uniformly elliptic. Thus there exist constants k_1, k_2 (with $0 < k_1 < k_2$) such that for arbitrary λ_i the condition

$$k_1 \sum_i \lambda_i^2 \leqslant \sum_{i,j} a_{ij}\lambda_i\lambda_j \leqslant k_2 \sum_i \lambda_i^2$$

is fulfilled in the closure $\bar{G}$ of the domain G.

In order to get the canonical form of ω_u we write u as the product of two real-valued functions ϱ and μ, $u = \varrho \cdot \mu$. Denote $a_{ij}\varrho$ by A_{ij} and $b_i\varrho + a_{ij}\dfrac{\partial \varrho}{\partial x_j}$ by B_i. Fixing the function ϱ we can regard the differential form $\omega_{\varrho\mu}$ as a new differential form Ω_μ depending on the function μ, namely

$$\Omega_\mu = \sum_{i,j} (-1)^{i(n-i)} \left(A_{ij}\frac{\partial \mu}{\partial x_j} + B_i \mu\right) dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

The new differential form Ω_μ can be regarded as a new representation of the given differential form ω_u. It is called *canonical* if the coefficients B_i fulfil the condition

$$(2)\qquad \sum_i \frac{\partial B_i}{\partial x_i} = 0.$$

The meaning of this condition is the following. The differential equation $d\Omega_\mu = 0$ does not contain the function μ itself. Each solution μ of $d\Omega_\mu = 0$ satisfies, consequently, the maximum-minimum-principle.

From the definition of the coefficients B_i it follows immediately that condition (2) is fulfilled iff the auxiliary function ϱ is a solution of the differential equation

$$(3)\qquad \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \varrho}{\partial x_j} + b_i\varrho\right) = 0.$$

Consider the differential equation adjoint to (3):

$$(4)\qquad \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial \sigma}{\partial x_j}\right) - \sum_i b_i \frac{\partial \sigma}{\partial x_i} = 0.$$

Since the coefficient at σ vanishes identically, it follows that the maximum-minimum-principle holds for the adjoint differential equation. The Dirichlet boundary value problem with identically vanishing boundary values possesses, consequently, only the trivial solution. Using Fredholm's theorem for second order equations of elliptic type we see that the Dirichlet boundary value problem with arbitrarily prescribed boundary values possesses a uniquely determined solution for each of the two equations (3) and (4) (see, for instance, C. Miranda [5]). In particular, there exists a uniquely determined solution ϱ of equation (3) with the boundary value 1 on the whole boundary of G. This solution ϱ is different from zero everywhere in the closure $\bar{G}$ of G. In the case of $n = 2$ the proof of this assertion was given in

[7], [9]; the proof is based on the fact that the zeros of $u^2+\operatorname{grad}^2 u$ are isolated (see P. Hartman and A. Wintner [2]). Since the zeros of $u^2+\operatorname{grad}^2 u$ are isolated also in the case $n \geqslant 3$, the proof of the inequality $\varrho \neq 0$ given in [7], [10] remains valid in the case $n \geqslant 3$, as well.

From the inequality $\varrho \neq 0$ it follows that the differential equation $d\Omega_\mu = 0$ is also uniformly elliptic.

3. The matrix of periods of generalized analytic functions in R^n

We assume that the boundary of G consists of a finite number of components S_j, $j = 1, \ldots, q$. For any closed differential form Ω of degree $n-1$ the integral

$$\int_{S_j} \Omega = \pi_j[\Omega]$$

is called the *period* of Ω on the boundary component S_j. By virtue of Stokes' theorem we get immediately

$$\sum_j \pi_j[\Omega] = 0. \tag{5}$$

Let μ_k be the uniquely determined solution of the boundary value problem

$$d\Omega_{\mu_k} = 0, \quad \mu_k = \delta_{kj} \text{ on } S_j.$$

The functions μ_k are generalized harmonic measures of the boundary components S_k. The matrix Π consisting of the periods of the generalized harmonic measures μ_k is called the *matrix of periods* of the generalized analytic function (μ, Ω_μ):

$$\Pi = (\pi_j[\Omega_{\mu_k}]).$$

Together with the differential form Ω_μ we inspect the differential form

$$\Omega_\mu^{\text{red}} = \sum_{i,j} (-1)^{i(n-i)} A_{ij} \frac{\partial \mu}{\partial x_i} dx_{i+1} \wedge \ldots \wedge dx_{i-1},$$

which is called the *reduced differential form*. Moreover, we define the so-called *adjoint differential form*

$$\Omega_\nu^* = \sum_{i,j} (-1)^{i(n-i)} \left(A_{ji} \frac{\partial \nu}{\partial x_j} - B_i \nu \right) dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

The reason for this notation is the fact that the differential equations

$$d\Omega_\mu = 0 \quad \text{and} \quad d\Omega_\nu^* = 0$$

are adjoint to each other. Denote by Π^* the matrix of periods of Ω_ν^*. Finally, write

$$\int_{S_j} \Omega_{\mu_k}^{\text{red}} = C_{jk}.$$

The matrix

$$\Pi_{\text{red}} = (C_{jk})$$

is called the *reduced matrix of periods* of Ω_μ (we remark that, in general, Ω_μ^{red} is not closed). Then the following theorem holds:

THEOREM 1. *The matrix of periods of the adjoint differential form Ω_ν^* is equal to the transposed matrix of periods of the differential form Ω_μ:*

$$\Pi^* = \Pi_{\text{red}}^{\text{trans}}.$$

For the proof we introduce the so-called *mixed differential form*

$$\tilde{\Omega}_{\mu,\nu} = \nu \Omega_\mu^{\text{red}} - \mu \Omega_\nu^*.$$

It is easy to check that $\tilde{\Omega}_{\mu,\nu}$ is closed if Ω_μ and Ω_ν^* are closed. From the definition of $\tilde{\Omega}_{\mu,\nu}$ it follows that

$$\pi_j[\tilde{\Omega}_{\mu_k,\nu_l}] = \delta_{lj} C_{jk} - \delta_{kj} \pi_j[\Omega_{\nu_l}^*].$$

Applying (5) to $\tilde{\Omega}_{\mu_k,\nu_l}$ we obtain

$$0 = C_{lk} - \pi_k[\Omega_{\nu_l}^*],$$

and the theorem is proved.

Since the solutions μ_k fulfil the maximum-minimum-principle, the signs of the C_{jk} are given by

$$\operatorname{sign} C_{jk} = (-1)^{1+\delta_{jk}} \tag{6}$$

(see [7]). The same relation is valid for the reduced matrix of periods in the case of the adjoint differential form. On the other hand, it follows from the theorem just proved that

$$\Pi = (\Pi_{\text{red}}^*)^{\text{trans}}.$$

Consequently, we have

COROLLARY 1.

$$\operatorname{sign} \pi_j[\Omega_{\mu_k}] = (-1)^{1+\delta_{jk}}.$$

For the specific case of $\mu \equiv 1$, the differential form Ω_μ is equal to

$$\Omega_1 = \sum_{i,j} (-1)^{i(n-i)} B_i dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

Therefore in the general case the differential form Ω_μ can be written as

$$\Omega_\mu = \Omega_\mu^{\text{red}} + \mu \Omega_1.$$

Setting $\mu = \mu_k$ we get

$$\pi_j[\Omega_{\mu_k}] = C_{jk} + \delta_{jk} \pi_j[\Omega_1].$$

Using Corollary 1 and taking $j = k$ we hence conclude that

$$C_{kk} + \pi_k[\Omega_1] > 0.$$

Now we look for a solution μ of $d\Omega_\mu = 0$ with constant values on each boundary component, assuming that Ω_μ possesses arbitrarily prescribed periods d_j (by virtue of (5) we assume $\sum d_j = 0$). Such a solution μ can be represented in the form

$$\mu = \sum_k c_k \mu_k.$$

In order to determine the unknown real coefficients c_j it is enough to solve the following linear algebraic system:

$$\sum_k c_k \pi_j[\Omega_{\mu_k}] = d_j.$$

Using Corollary 1 we arrive, by means of purely algebraic considerations, at the following results (see [6], [7], [9]):

COROLLARY 2. *For given d_j with $\sum d_j = 0$ there exist solutions μ of $d\Omega_\mu = 0$ with constant boundary values on each boundary component S_j, such that*

$$\pi_j[\Omega_\mu] = d_j.$$

The solution μ is uniquely determined if we prescribe arbitrarily one of the constant boundary values of μ.

COROLLARY 3. *If all d_j are equal to zero, then the corresponding c_k have the same signs.*

Applying the maximum-minimum-principle, we obtain as a consequence of Corollary 3 the following

COROLLARY 4. *Let μ be a solution of the partial differential equation $d\Omega_\mu = 0$ with constant boundary values on each boundary component S_j of G. If all periods of Ω_μ vanish, then we have either $\mu > 0$ or $\mu \equiv 0$ or $\mu < 0$ in $\bar{G}$.*

4. Applications

If ϱ is a solution of the differential equation (3) and μ is a solution of $d\Omega_\mu = 0$, then $u = \varrho\mu$ is a solution of $d\omega_u = 0$. Since $\varrho > 0$ in $\bar{G}$ and $\varrho = 1$ on ∂G, the Corollaries 2, 3 and 4 to Theorem 1 are valid for the differential form ω_u itself. On the other hand, the maximum-minimum-principle holds for $d\Omega_\mu = 0$. This means that each boundary value problem $d\Omega_{\tilde{\mu}} = 0$ in G, $\tilde{\mu} = g$ on ∂G, is uniquely solvable. To the given numbers d_j with $\sum_j d_j = 0$ there correspond constants c_j such that

$$d\Omega_{\hat{\mu}} = 0,$$
$$\hat{\mu} = c_j \quad \text{on} \quad S_j,$$
$$\pi_j[\Omega_{\hat{\mu}}] = d_j - \pi_j[\Omega_{\tilde{\mu}}].$$

Consequently, the function $\mu = \tilde{\mu} + \hat{\mu}$ fulfils the conditions

$$d\Omega_\mu = d\Omega_{\tilde{\mu}+\hat{\mu}} = d\Omega_{\tilde{\mu}} + d\Omega_{\hat{\mu}} = 0,$$
$$\mu = g + c_j \quad \text{on } S_j, \quad \pi_j[\Omega_\mu] = d_j.$$

Hence the following theorem holds:

THEOREM 2. *Let g be given boundary data and d_j given real numbers fulfilling the condition $\sum_j d_j = 0$. Then there exist constants c_j and a function u such that the following conditions are fulfilled*

$$d\omega_u = 0 \quad \text{in} \quad G, \quad u = g + c_j \quad \text{on} \quad S_j, \quad \pi_j[\omega_u] = d_j.$$

The function u is uniquely determined if we prescribe one of the constants c_j.

The meaning of Theorem 2 is the following: if we admit the boundary values $g + c_j$ instead of g, then there exist solutions u of the partial differential equation $d\omega_u = 0$ fulfilling the side-conditions $\pi_j[\omega_u] = d_j$.

Using the notation of generalized analytic function in $\boldsymbol{R}^n$ we reformulate Theorem 2 in the following way:

Let g be a given function on the boundary ∂G of domain G possessing q boundary components S_j, $j = 1, \ldots, q$. Let d_j be given numbers fulfilling the condition $\sum_j d_j = 0$. For given coefficients a_{ij}, b_i there exists a generalized analytic function (u, ω_u) such that

(a) *the "real part" u assumes the value $g + c_j$ on the boundary component S_j, $j = 1, \ldots, q$;*

(b) *the "imaginary part" ω_u possesses the period d_j on the boundary component S_j, $j = 1, \ldots, q$.*

The constants are subject to choice. They are uniquely determined if we prescribe one of them.

Now let

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_i b_i \frac{\partial}{\partial x_i}$$

be a given linear differential operator of second order. Using the coefficients b_i of L we define the following differential form (of degree $n-1$):

$$\omega_L = \sum_i (-1)^{i(n-i)} b_i dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

Then for the differential operator αL the corresponding differential form $\omega_{\alpha L}$ is given by

$$\omega_{\alpha L} = \sum_{i,j} (-1)^{i(n-i)} \left(\alpha b_i - a_{ji}\frac{\partial \alpha}{\partial x_j}\right) dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

We say that L possesses *globally Bianchi's canonical form*, if L can be written as

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial}{\partial x_j}\right) \tag{7}$$

in the whole closure $\bar{G}$. On the other hand, it was proved in [10] that L possesses globally Bianchi's canonical form whenever the corresponding differential form ω_L is exact. Consequently, the product αL possesses Bianchi's *canonical* form, if $\omega_{\alpha L}$ is exact. By virtue of the second de Rham's theorem a differential form is *exact*, if its periods are equal to zero. This means that αL possesses Bianchi's canonical form, if $(\alpha, \omega_{\alpha L})$ is a generalized analytic function the periods of its imaginary part are equal to zero. Using the theorems proved above we obtain the following

THEOREM 3. *Let G be a given bounded domain in R^n belonging to A^1_α. Let, moreover, L be a given linear, uniformly elliptic differential operator of second order*

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_i b_i \frac{\partial}{\partial x_i},$$

where $a_{ij} \in \mathscr{C}^2_\alpha(\bar{G})$, $b_i \in \mathscr{C}^1_\alpha(\bar{G})$. Denote by S_k, $k = 1, \ldots, q$, the boundary components of the given domain G. Then there exists a function α with constant values each boundary component S_k, such that αL can be written globally as Bianchi's canonical form.

5. Concluding remarks

Remark 1. If the differential equation contains the function u itself, then it is not always possible to write the differential equation in Bianchi's canonical form (7).

To see this, consider the differential equation

$$\Delta u + \gamma \cdot u = 0. \tag{8}$$

If it were possible to write this equation in the form (7), the adjoint equation $L^*[v] = 0$ would be a differential equation without v. For $L^*[v] = 0$ the maximum-minimum-principle ought to hold. Hence it would follow that the Dirichlet's boundary value problem for $L^*[v] = 0$ and, consequently, also for the differential equation (8) itself, is uniquely solvable. This contradicts the well-known properties of the differential equation (8).

Remark 2. For the construction of the canonical representation of generalized analytic function in R^n we need an auxiliary function ϱ fulfilling the partial differential equation (3). In order to prove the existence of a canonical representation we must know the following property of the differential equation (3): if a solution ϱ has positive boundary values, then it is different from zero everywhere in $\bar{G}$. The proof of this property is trivial if we know that (3) possesses any positive solution. Indeed: let u_1 be a given positive solution of (3). Then $u = u_1 u_2$ is a solution of (3) iff u_2 is a solution of a differential equation not containing u_2 itself. Therefore the maximum-minimum-principle holds for u_2 and every Dirichlet's boundary value problem is uniquely solvable. In particular, there exists a solution u_2 such that $u = u_1 u_2$ has boundary values 1. Since $u_2 \neq 0$ it is $u \neq 0$ everywhere in $\bar{G}$, too.

Remark 3. In the theory of dual differential forms (theory of Hodge, see [4]) the associated partial differential equations are self-adjoint. But the concept of generalized analytic functions defined above includes the case of non-self-adjoint partial differential equations too.

Remark 4. The proof of Theorem 1 given above is a modification and simplification of the proof given in [9]. The special case of two real variables ($n = 2$) has been first considered in [7]. A detailed proof for this case is given also in the book [11].

Remark 5. Let ω_u be the inhomogeneous differential form (1). The corresponding homogeneous differential form will be denoted by $\tilde{\omega}_u$,

$$\tilde{\omega}_u = \sum_{i,j} (-1)^{i(n-i)}\left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u\right) dx_{i+1} \wedge \ldots \wedge dx_{i-1}.$$

Then

$$\omega_{u+u_0} = \tilde{\omega}_u + \omega_{u_0}.$$

If u_0 is a specific solution of the inhomogeneous partial differential equation

$$d\omega_{u_0} = 0$$

and if u is a solution of the homogeneous equation

$$d\tilde{\omega}_u = 0,$$

then $u + u_0$ is a solution of the differential equation

$$d\omega_u = 0.$$

Using, for instance, fundamental solutions one can construct a specific solution of the inhomogeneous differential equation. The solution u of the homogeneous equation can be chosen so as to have

$$u + u_0 = g + c_i \quad \text{on} \quad S_j,$$

$$\pi_j[\omega_{u+u_0}] = d_j,$$

where g and d_j are prescribed, $\sum_j d_j = 0$. Therefore Theorem 2 is also valid for inhomogeneous differential forms.

Remark 6. Instead of domains in R^n we may consider domains contained in a manifold of dimension n.

Remark 7. The problem regarded in Theorem 2 (Dirichlet's boundary value problem with side-conditions) is a so-called free Dirichlet's boundary value problem (see, for instance, the paper [6] by A. D. Myshkis; non-self-adjoint differential equations in R^2 are dealt with in [8]).

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SOLUTIONS WITH PRESCRIBED PERIODS ON THE BOUNDARY COMPONENTS OF NON-LINEAR ELLIPTIC SYSTEMS OF FIRST ORDER IN MULTIPLY CONNECTED DOMAINS IN THE PLANE

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Let G be a domain in the plane. Dirichlet's boundary value problem for holomorphic functions is the following: to determine a holomorphic function in G for which the real part assumes given boundary values g on the boundary ∂G. The solution is uniquely determined, if one prescribes the value of the imaginary part at one point z_0. Only in the case of simply-connected domains the solution is necessarily single-valued. But in the case of multiply connected domains the solution of Dirichlet's boundary value problem possesses (purely imaginary) periods on the boundary components γ_j, in general. However a single-valued solution to Dirichlet's boundary value problem exists, if we replace the given boundary data g on γ_j by $g+c_j$, where the c_j are suitably chosen constants.

An analogous assertion holds in the case of linear elliptic systems (see [2]). The aim of the present paper is to prove that an analogous theorem is valid also for general non-linear elliptic systems on the plane.

In the second part of the paper we consider systems permitting solutions with additive periods. For such systems we prove the existence of a solution possessing arbitrarily prescribed periods on the boundary components.

We will be concerned with the differential equation

$$(*)\qquad \frac{\partial w}{\partial z^*} = F\left(z, w, \frac{\partial w}{\partial z}\right),$$

where the right-hand side $F(z, w, h)$ fulfils the following conditions (λ is a fixed real number, $0 < \lambda < 1$, $w = (w_1, \ldots, w_m)$, $h = (h_1, \ldots, h_m)$, $F = (F_1, \ldots, F_m)$):

$$|F_j(z_2, w, h) - F_j(z_1, \tilde{w}, \tilde{h})| \leqslant l\Big(|z_2 - z_1|^\lambda + \sum_j |w_j - \tilde{w}_j| + \sum_j |h_j - \tilde{h}_j|\Big),$$

$$\|F(\cdot, w, h) - F(\cdot, \tilde{w}, \tilde{h})\|_\lambda \leqslant L_1 \|w - \tilde{w}\|_\lambda + L_2 \|h - \tilde{h}\|_\lambda.$$