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ON THE METRIC $|f(z)|^{\lambda}|dz|$ WITH HOLOMORPHIC f

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Introduction

Let f be a holomorphic function in a plane domain D, and let λ be a positive real number. We consider the metric induced by the line element $|dw| = |f(z)|^{\lambda}|dz|$. For the special value $\lambda = 1/2$ it is the metric $|f(z)|^{1/2}|dz|$ which is associated with the quadratic differential f (f transforms such that $f(z)dz^2$ is invariant). This metric, first introduced and investigated by O. Teichmüller on compact Riemann surfaces, plays an important role for extremal problems in connection with conformal and quasiconformal mappings. However, most of the basic properties of the metric and its geodesies as well as the proofs carry over to the general case $\lambda > 0$, except that one can, of course, no longer speak of horizontal and vertical trajectories: There are no distinquished line elements for arbitrary λ such as $dw^2 = f(z)dz^2 > 0$ (horizontal) and $dw^2 = f(z)dz^2 < 0$ (vertical) in the case $\lambda = 1/2$.

The area element which corresponds to the line element $|f(z)|^{\lambda}|dz|$ is $|f(z)|^{2\lambda}dxdy$, and it is of special interest to consider the metrics with finite total area $\iint_D |f(z)|^{2\lambda}dxdy$. These are the metrics associated with the space $A_{2\lambda}$, the analytic functions with finite $L_{2\lambda}$-norm, $0 < \lambda < \infty$.

In the sequel, the considerations will mostly be restricted to simply connected domains and holomorphic f, not necessarily of finite $L_{2\lambda}$-norm.

1. Local properties of the metric

1.1. For the investigation of the metric and, of course, in order to define it on a Riemann surface, the line element must be invariant under a conformal or even just an analytic substitution. Let f be holomorphic in the domain D of the z-plane and let $z = h(\tilde{z})$ be an analytic mapping of a domain $\tilde{D}$ into D. We want to define $\tilde{f}$ in $\tilde{D}$ such that, for $z = h(\tilde{z})$,

$$(1) \qquad |\tilde{f}(\tilde{z})|^{\lambda}|d\tilde{z}| = |f(z)|^{\lambda}|dz|,$$

or, equivalently

$$|\tilde{f}(\tilde{z})| = |f(z)|\left|\frac{dz}{d\tilde{z}}\right|^{1/\lambda}.$$

Thus, necessarily

(2) $$\tilde{f}(\tilde{z}) = f(h(\tilde{z}))h'(\tilde{z})^{1/\lambda}.$$

If h is conformal ($h'(\tilde{z}) \neq 0$ in $\tilde{D}$) and $\tilde{D}$ is simply connected, one can fix the argument of h' and thus choose a single valued branch of $h'^{1/\lambda}$. Therefore $\tilde{f}$ is uniquely determined in $\tilde{D}$, up to a factor of modulus one. Conversely, the function (2) evidently satisfies the invariance relation (1).

In particular, if $\lambda = 1/2$, $\tilde{f}(\tilde{z}) = f(h(\tilde{z}))h'^2(\tilde{z})$, which is uniquely determined for any analytic mapping h.

1.2. Just as in the case of a quadratic differential it is possible to introduce, locally, a natural or distinguished parameter, which allows a particularly simple form of the weight function f. Let us first consider a regular point z_0 of the metric, i.e. a point such that $f(z_0) \neq 0$. We choose a single valued branch of f^λ in a neighborhood of z_0. The function

$$w = F(z) = \int_{z_0}^{z} f(t)^\lambda dt$$

has derivative $F'(z_0) = f(z_0)^\lambda \neq 0$ and hence maps some properly chosen neighborhood U of z_0 conformally onto a disk $|w| < r$. The line element of the metric becomes $|dw| = |f(z)|^\lambda |dz|$, i.e. it is the Euclidean line element in the F-plane. Therefore, any two points w_1 and w_2 in $|w| < r/2$ can be joined by a unique shortest arc, namely the straight line segment connecting the two points. Its F^{-1}-image is the unique shortest arc connecting $z_1 = F^{-1}(w_1)$ with $z_2 = F^{-1}(w_2)$ compared with all connecting arcs within the entire domain D.

1.3. Let now z_0 be a critical point of the metric, i.e. a zero of f. Let $z_0 = 0$,

$$f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots, \quad a_n \neq 0, n \geqslant 1$$

in a neighborhood of z_0. Assume now that ζ, $\zeta = 0 \leftrightarrow z = 0$, is a new parameter, such that

$$f(z)^\lambda dz = g(\zeta)^\lambda d\zeta = A\zeta^{n\lambda} d\zeta,$$

where A is a properly chosen constant. By integration we get

$$F(z) = \int_0^z f(t)^\lambda dt = A\int_0^\zeta t^{n\lambda} dt = \frac{A}{n\lambda+1}\zeta^{n\lambda+1}.$$

Choosing $A = n\lambda+1$ we get the particularly simple representation $F(z) = \zeta^{n\lambda+1}$, $z \leftrightarrow \zeta$.

On the other hand,

$$f(z)^\lambda = z^{n\lambda}(a_n + a_{n+1}z + \ldots)_\lambda = z^{n\lambda}(b_0 + b_1 z + \ldots),$$

where the bracket is a single valued branch of $(a_n + a_{n+1}z + \ldots)^\lambda$. Therefore

$$F(z) = \frac{b_0}{n\lambda+1}\cdot z^{n\lambda+1} + \frac{b_1}{n\lambda+2}\cdot z^{n\lambda+2} + \ldots = z^{n\lambda+1}(c_0 + c_1 z + \ldots),$$

$$c_k = \frac{b_k}{n\lambda+k+1}, \quad k = 0, 1, 2, \ldots$$

We have, necessarily,

$$\zeta^{n\lambda+1} = z^{n\lambda+1}(c_0 + c_1 z + \ldots)$$

and hence

$$\zeta = z(c_0 + c_1 z + \ldots)^{1/(n\lambda+1)} = d_1 z + d_2 z^2 + \ldots, \quad d_1 = c_0^{1/(n\lambda+1)}.$$

The last series clearly converges and provides the conformal transformation we are looking for. It is therefore possible to introduce a distinguished parameter ζ in the neighborhood of a critical point of the metric such that $w = F(\zeta) = \zeta^{n\lambda+1}$, $dw = f(z)^\lambda dz = (n\lambda+1)\zeta^{n\lambda} d\zeta$. It is uniquely determined up to a factor of norm one.

1.4. Using the distinguished parameter we can now find the shortest lines in the neighborhood of a critical point z_0. Let γ be an arc joining ζ_1 and ζ_2. If $|\zeta_1|$ and $|\zeta_2|$ are sufficiently small, there is an arc joining ζ_1 and ζ_2 within $|\zeta| < \varrho$ which is shorter than any arc leaving the neighborhood. We therefore only have to consider arcs γ in $|\zeta| < \varrho$, and without restricting the generality we can assume ζ_1 and $w_1 = F(\zeta_1)$ to be on the positive real axis. The sectors $0 \leqslant \arg\zeta \leqslant \pi/(n\lambda+1)$, $0 \geqslant \arg\zeta \geqslant -\pi/(n\lambda+1)$ are mapped onto the upper and lower half circles respectively. It is now easy to realize that in the w-plane (where the length element is the Euclidean element $|dw|$) the shortest line is the straight line segment between w_1 and w_2, if ζ_2 is in one of the closed adjacent sectors, and it consists of the two radii with endpoints w_1 and w_2 in all the other cases. The second statement follows, because any curve γ joining ζ_1 with ζ_2 has to cut one of the radii $\arg\zeta = \pm\pi/(n\lambda+1)$ and its w-image therefore cuts the negative real axis before it proceeds to w_2. We conclude that there always exists a unique shortest connection between ζ_1 and ζ_2. If this connection consists of two radii, their angle at zero must be $\geqslant \pi/(n\lambda+1)$ (angle condition; it is equal to $2\pi/(n+2)$ in the case $\lambda = 1/2$).

2. Trajectories and geodesics

2.1. A curve γ: $z = \gamma(t)$ in D, defined and continuously differentiable in some open interval of the real t-axis and with non vanishing derivative, is called a *straight arc* with respect to the metric $|f(z)|^\lambda |dz|$, if $f(\gamma(t)) \neq 0$ for all t (i.e. γ does not pass through a critical point of the metric) and $\arg f(\gamma(t))^\lambda \gamma'(t) = \text{const}$ for some (and hence every) continuous branch of $f(\gamma(t))^\lambda$; equivalently, in the integrated form, $w = F(\gamma(t))$ is a straight open line segment in the w-plane. A straight arc will always be assumed to be parametrized by the arc length with respect to the metric $|f(z)|^{\lambda}|dz|$, which is the Euclidean length on the corresponding rectilinear segment in the w-plane. (We call it the natural or distinguished parameter.) If this interval lies on the real axis, the natural parameter is $\pm u+\text{const}$, $u = \operatorname{Re} w$.

A maximal straight arc, i.e. which is not properly contained in another straight arc, is called a *trajectory*. Through a regular point z_0 of the metric there exists, in every direction, a uniquely determined trajectory. We get it (in the natural parametrization) by reversing a local function element F near z_0 and continuing $z = F^{-1}(w)$ along the straight line of the given direction in the w-plane. Since $dz/dw = 1/f(z)^{\lambda} \neq 0$, F^{-1} is locally a conformal mapping. Again assuming, for notational convenience, that the straight line in the w-plane is horizontal and that the original function element F satisfies $F(z_0) = 0$, the trajectory is the curve $F^{-1}(u)$, defined in some (maximal) open interval $-\infty \leqslant u_{-\infty} < u < u_{\infty} \leqslant \infty$. The uniqueness of the trajectory follows from the local uniqueness, which is guaranteed by the local homeomorphisms F.

A *trajectory ray* is the restriction of F^{-1} onto a half line, e.g. $0 \leqslant u < u_{\infty}$.

A trajectory ray can tend to a zero of f, in a well determined direction (as is seen using the distinguished parameter in the neighborhood of the zero). It is then called a critical, otherwise a regular, trajectory ray. If a trajectory contains at least one critical ray, it is called critical itself.

A *geodesic* γ: $z = \gamma(t)$ is a maximal locally shortest line. The latter means that the image of the interval $[t_1, t_2]$ is shortest, if t_1 and t_2 are sufficiently close to each other. A geodesic can again be parametrized by the arc length in the metric $|f(z)|^{\lambda}|dz|$.

A trajectory is an open subarc of a geodesic. If one of its rays tends to a zero z_0, the geodesic continuation at this point is to a certain extent arbitrary; it only has to satisfy the angle condition, which means that both angles between the two arcs have to be $\geqslant \pi/(n\lambda+1)$. The extreme continuations to the left and to the right are uniquely determined.

A geodesic ray which does not tend to the boundary of D must have infinite length. For, let it be defined in $0 \leqslant u < u_{\infty}$, u being the natural parameter, and let $u_n \to u_{\infty}$, $\gamma(u_n) = z_n \to z \in D$. If z is a regular point, it has a neighborhood U (a disk in the metric $|f(z)|^{\lambda}|dz|$) such that every geodesic arc which has a point in common with U can be prolonged to ∂U. Therefore there is a positive number d such that every subarc of a geodesic which passes sufficiently near z contains an interval of length at least d, and hence u_n is contained in a horizontal interval I_n of the same Euclidean length. Without restricting the generality we can assume that the intervals I_n are disjoint, hence $u_{\infty} = \infty$. But the same argument works for a critical point z, as is seen using the distinguished parameter near z.

2.2. THEOREM (Uniqueness of geodesic connections). *Let f be holomorphic in a simply connected domain D. Then, any two points z_1 and z_2 of D can be joined by at most one geodesic arc with respect to the metric $|f(z)|^{\lambda}|dz|$.*

Proof. The proof rests on the argument principle and is in fact the same as Teichmüller's original proof for quadratic differentials.

Let γ be a simple geodesic polygon, i.e. a Jordan curve with finitely many distinguished points (vertices), the sides of which are geodesic arcs (the analogue of a Euclidean polygon). But it is convenient to count also the zeroes of f on γ as vertices. Therefore the sides of the polygon are in fact straight arcs: $\arg dw = \arg f(z)^{\lambda}dz = \lambda \arg f(z)+\arg dz = \text{const}$ along any side.

The argument principle applied to f and the interior D_0 of γ (with possible zeroes of f on the boundary) yields

$$\frac{1}{2\pi}\int_{\gamma} d\arg f(z) = \sum n_i+\sum n_j\frac{\theta_j}{2\pi}, \tag{3}$$

where the n_i are the orders of the zeroes of f in D_0, while the $n_j \geqslant 0$ are the orders of the zeroes of f at the vertices z_j of γ, with θ_j the interior angles between the adjacent sides. The integral $\int_{\gamma}$ is to be understood as the sum of the integrals along the sides γ_j of γ. This integral can now be computed, using $d\arg f(z)^{\lambda}dz = \lambda d(\arg f(z))+ +d(\arg dz) = 0$ along each side of γ. Moreover,

$$\int_{\gamma} d(\arg dz) \equiv \sum_j \int_{\gamma_j} d(\arg dz) = 2\pi-\sum_j(\pi-\theta_j). \tag{4}$$

Multiplying (3) by $2\pi\lambda$ and adding it to (4) gives

$$\begin{aligned} 0 &= 2\pi\lambda\sum_i n_i+\lambda\sum_j n_j\theta_j+2\pi+\sum_j(\theta_j-\pi) \\ &= 2\pi\lambda\sum_i n_i+2\pi+\sum_j\{(\lambda n_j+1)\theta_j-\pi\}. \end{aligned}$$

We conclude that $2\pi+\sum_j\{(\lambda n_j+1)\theta_j-\pi\} \leqslant 0$, hence

$$\sum_j\{\pi-(\lambda n_j+1)\theta_j\} \geqslant 2\pi. \tag{5}$$

Let now γ_1 and $\gamma_2 \neq \gamma_1$ be two geodesic arcs connecting z_1 and z_2. By passing to subarcs, if necessary, we can assume that the two arcs together form a simple closed geodesic polygon. The two interior angles θ_1 and θ_2 at z_1 and z_2 respectively are positive, as straight arcs with a common endpoint and zero angle coincide. At all the other vertices the angle condition for geodesics holds, i.e. $(\lambda n_j+1)\theta_j \geqslant \pi$. Therefore, on the left hand side of inequality (5), all the terms are smaller or equal to zero, except for two which are strictly less than π, so the sum is smaller than 2π, a contradiction.

2.3. There are several

COROLLARIES. (a) *For a holomorphic f in a simply connected domain there are no closed geodesics*: The angle condition would hold everywhere or, another argument, any two points on it would be connected by two different geodesic arcs.

(b) *There are no geodesic loops.* This means that no geodesic can intersect itself (the angle condition would hold everywhere except for one point).

(c) *Two different geodesic rays starting at the same point evidently cannot meet again.* A similar result is the following: Let β be a closed straight arc with (regular) endpoints z_1 and z_2, and let γ_1 and γ_2 be trajectory rays with initial points z_1 and z_2 respectively and forming a right angle with β. Then γ_1 and γ_2 cannot meet. Otherwise there would be a first intersection z_0 and the respective subarcs would form a simple closed polygon with inner angles $\pi/2$ or $3\pi/2$ at z_1 and z_2 and $\theta_0 > 0$ at z_0. Here, $n_j = 0$ at every vertex, and thus inequality (5) would be vexed. But the argument works just as well for geodesic arcs if the inner angles at z_1 and z_2 satisfy $\theta_j(\lambda n_j+1) \geqslant \pi/2$, $j = 1, 2$.

(d) *Any two points z_1 and z_2 in D can be joined by at most one shortest line* (because it is necessarily a geodesic).

The following is less immediate.

(e) THEOREM. *Let f be holomorphic in a simply connected domain. Then, any geodesic ray γ with respect to the metric $|f(z)|^{\lambda}|dz|$, $\lambda > 0$, tends to the boundary.*

Proof. Let γ be a geodesic ray which does not tend to the boundary of D. Let it be parametrized by the natural parameter t, $0 \leqslant t < \infty$ (i.e. the arc length with respect to the metric $|f(z)|^{\lambda}|dz|$), $t = 0$ corresponding to its initial point z_0. Then there is a sequence of values $t_n \uparrow \infty$ such that $z_n = \gamma(t_n)$ tends to some point $z \in D$. Let U be a closed disk around z (with respect to the natural parameter). As γ is maximal, we can assume (by passing to a subsequence, if necessary) that the subarc between any two consecutive points z_n, z_{n+1} leaves U. On the other hand, if z_n and z_{n+1} are sufficiently close to z, they can be joined by a shortest, hence a geodesic arc α which stays in U. This contradicts the uniqueness theorem.

2.4. It is of course not true that two arbitrary points z_1 and z_2 in D can be joined by a shortest arc, as $f \equiv 1$ in a non convex domain shows. But if one only considers connecting arcs within a fixed geodesic polygon, then there always exists a unique (relatively) shortest connection. The same is true (for an absolutely shortest connection) if the boundary of D is further away from one of the two points than the other point.

THEOREM (Existence of shortest connection). *Let z_1 and z_2 be two arbitrary points in D, and let γ_0 be a simple closed geodesic polygon (the sides are straight arcs) containing z_1 and z_2 in its interior D_0, with $\overline{D}_0 \subset D$. Then there is a unique shortest connection γ between z_1 and z_2 in $\overline{D}_0$.*

Proof. The proof proceeds as in [2] for the metric associated with a quadratic differential. It is easy to see, by local considerations (using the distinguished parameter), that the theorem holds for any two points in $\overline{D}_0$ which are sufficiently close to each other. We now start with a minimal sequence (γ_n) of connecting arcs. Let $a_n = \int_{\gamma_n} |f(z)|^{\lambda}|dz|$, $a_n \to a_0 = \inf_{\{\gamma\}} \int_{\gamma} |f(z)|^{\lambda}|dz|$, where the infimum is taken over all rectifiable arcs connecting z_1 and z_2 within $\overline{D}_0$. Let N be a fixed number such that any two points in $\overline{D}_0$ of distance a/N, where $a \geqslant a_n$ for all n, can be joined by a unique relatively shortest arc in $\overline{D}_0$. Subdividing the parameter interval $0 \leqslant t \leqslant a_n$ for γ_n into N subintervals of equal length, we can choose a subsequence of (γ_n) such that the images of the subdividing points converge. Connecting their limits by relatively shortest arcs in $\overline{D}_0$ clearly provides a shortest connection γ of z_1 and z_2 in $\overline{D}_0$. If γ' is another shortest connection, we can proceed as before to show that $\gamma = \gamma'$. Of course, the angle condition can now be vexed at a vertex of γ_0. But the change of the angle by the obstacle γ_0 is such that the interior angle of a closed polygon is magnified, which even strengthens the inequality and hence the contradiction.

In this theorem, D does not necessarily have to be simply connected, as D_0 is. But in the addendum in the introduction of this section D must be simply connected. The curves γ_n of the minimal sequence then stay in a compact subset, which assures the convergence.

2.5. COROLLARY. *Let f be holomorphic in a simply connected domain D. Then for the metric $|f(z)|^{\lambda}|dz|$, every subarc of a geodesic is the unique shortest connection of its two endpoints.*

Proof. Let γ be a geodesic arc connecting z_1 and z_2, and let γ_0 be a simple closed geodesic polygon the interior D_0 of which contains γ (such a polygon can always easily be constructed). There is a relatively shortest connection γ' of z_1 and z_2 in $\overline{D}_0$. As γ and γ' are both relative geodesics in $\overline{D}_0$ (γ is an absolute geodesic, hence a fortiori a relative geodesic), they must coincide, as was shown in the preceding section.

2.6. A consequence of this result is that, under certain conditions, there is no geodesic loop starting from a boundary point z_0 and returning to it. Let D be simply connected, and let γ be such a loop. It necessarily is a simple closed curve. Let z_1 and z_2 be points on γ tending to z_0 and such that the sequence of subintervals $\gamma_0 = [z_1, z_2]$ of γ is increasing and exhausts γ. Then

$$\varliminf_{z_1, z_2 \to z_0} d(z_1, z_2) \geqslant |\gamma| = \int_{\gamma} |f(z)|^{\lambda}|dz|,$$

with $d(z_1, z_2) = \inf_{\{\beta\}} \int_{\beta} |f(z)|^{\lambda}|dz|$, $\{\beta\}$ ranging over all curves connecting z_1 and z_2 in D.

If $\varliminf_{z_1, z_2 \to z_0} d(z_1, z_2) = 0$, i.e. if there are points z_1, z_2 arbitrary close to z_0 which can be joined by arbitrary short arcs β, we get a contradiction. For this again it is sufficient that z_0 has a neighborhood U of finite area $|U| = \iint_U |f(z)|^{2\lambda} dx dy$, or even that $|D_0| = \iint_{D_0} |f(z)|^{2\lambda} dx dy < \infty$. This follows from an application of the Schwarz inequality to the arcs β_r between the two ends of γ on the circles $|z - z_0| = r$.

The length of such an arc is

$$l(r) = \int_{\beta_r} |f(z_0+re^{i\theta})|^{\lambda} r\,d\theta .$$

Therefore

$$l^2(r) \leqslant r \cdot 2\pi \int_{\beta_r} |f(z_0+re^{i\theta})|^{2\lambda} r\,d\theta ,$$

and the left hand side in the inequality

$$\int_0^r \frac{l^2(r)}{r}\,dr \leqslant 2\pi \iint |f(z_0+re^{i\theta})|^{2\lambda} r\,dr\,d\theta \leqslant 2\pi |D_0|$$

must converge, which shows that $l(r_n) \to 0$ for a properly chosen sequence $r_n \to 0$.

3. Strips of parallel trajectories

3.1. For many purposes it is necessary to consider not only an individual trajectory, but the set of all parallel trajectories through an interval, i.e. a strip of parallel trajectories. Let f be holomorphic in a simply connected domain D and let β be a straight open arc. It is the image of a rectilinear open interval β' in the w-plane by a branch of F^{-1} defined in an open neighborhood of β'. As D is assumed to be simply connected, β is a Jordan arc. We now continue $z = F^{-1}(w)$ analytically along the straight lines α' orthogonal to β'. The set S' of points which are reached by this continuation process is open. For, if $w_1 \in \alpha'$ belongs to it and w_0 is the common point of α' and β', then the function element F^{-1} which is defined in a neighborhood of w_0 can be continued to w_1 along a finite chain of disks, and the same chain also serves for the continuation along the neighboring lines. Therefore a neighborhood of w_1 is also contained in S'. Evidently, S' is connected. But it is also simply connected, as any closed curve in S' can be contracted along the lines α' to the interval β' and then along β' to a point. As it is defined, $z = F^{-1}(w)$ is single valued on S'. But it is also injective. For, let $F^{-1}(w_1) = F^{-1}(w_2) = z_0$ for $w_1 \neq w_2$. If the two points w_1 and w_2 are on the same line α', its subinterval $[w_1, w_2]$ is mapped by F^{-1} onto a geodesic loop, which is impossible. If they are on two different lines α'_1 and α'_2 respectively, the intersections w'_1 and w'_2 of these lines with β' are different and so are their images $z_i = F^{-1}(w'_i)$, $i = 1, 2$, on β. The subinterval $[z_1, z_2]$ of β together with the subintervals $[z_0, z_1]$ and $[z_0, z_2]$ of the trajectories α_1 and α_2 respectively form a geodesic triangle. The interior angles on the basis $[z_1, z_2]$ are $\pi/2$ or $3\pi/2$, while the angle on the top vertex is positive. This is impossible as was shown in Section 2.3 (c). We conclude that F^{-1} is a conformal mapping of S' onto a simply connected domain S swept out by the trajectories which cut β orthogonally. S is called the strip of trajectories orthogonal to β or just the orthogonal strip of β. As it is simply connected and does not contain zeroes of f, the integral $F(z) = \int f(z)^{\lambda} dz$ has single valued branches in S.

Any two branches can only differ by a factor $e^{k\lambda 2\pi i}$, k an integer. Therefore any branch of F maps S conformally onto a Euclidean strip which is congruent to S' and has Euclidean area

$$\iint_S |F'(z)|^2 dx\,dy = \iint_S |f(z)|^{2\lambda} dx\,dy = \iint_{S'} du\,dv ,$$

which is the area of S with respect to the metric $|f(z)|^{\lambda}|dz|$. Let us denote Euclidean length and area with two bars with a subscript E, whereas length and area with respect to the metric $|f(z)|^{\lambda}|dz|$ are denoted without subscript. Then the above equation says

$$|S| = |S'|_E .$$

3.2. Let us now assume that

$$|D| = \iint_D |f(z)|^{2\lambda} dx\,dy \quad \text{and} \quad |D|_E = \iint_D dx\,dy$$

are finite. Let β be a straight interval, and let S be the orthogonal strip of β. For convenience of notation we assume that the F-image β' of β is vertical and hence S' is a horizontal strip in the $w = u+iv$-plane. Let $l(v) = |\alpha_v| = \int_{\alpha_v} |f(z)|^{\lambda}|dz|$ be the length of the trajectory α_v corresponding to the subinterval α'_v of the horizontal $\operatorname{Im} w = v$ in S'. $l(v)$ is equal to the Euclidean length of this interval, in our notation $l(v) = |\alpha'_v|_E$. As

$$\int l(v)\,dv = \iint_{S'} du\,dv = \iint_S |f(z)|^{2\lambda} dx\,dy \leqslant |D|$$

is finite, a.a. trajectories orthogonal to β have finite length (in the metric $|f(z)|^{\lambda}|dz|$). The same statement can be made about their Euclidean length:

$$|\alpha_v|_E \equiv l_E(v) = \int_{\alpha_v} |dz| = \int_{\alpha'_v} \left|\frac{dz}{dw}\right| du, \quad z = F^{-1}(w).$$

Therefore

$$\int l_E(v)\,dv = \iint_{S'} \left|\frac{dz}{dw}\right| du\,dv,$$

and an application of the Schwarz inequality yields

$$\left(\int l_E(v)\,dv\right)^2 \leqslant \iint_{S'} du\,dv \cdot \iint_{S'} \left|\frac{dz}{dw}\right|^2 du\,dv \leqslant |D| \cdot \iint_S dx\,dy \leqslant |D| \cdot |D|_E ,$$

which is finite. Thus $|\alpha_v|_E$ must be finite for a.a. values of v. We conclude that a.e. trajectory orthogonal to any fixed straight interval β has finite Euclidean length. There are at most denumerably many critical trajectories orthogonal to β, as no two rays can meet at the same zero and f has at most denumerably many zeroes. Therefore, in particular, a.e. trajectory which cuts β orthogonally is a cross cut both ends of which converge to well defined boundary points. If D is the disk, we will show that every geodesic ray converges to a boundary point. The same is then true, by conformal mapping, for any Jordan domain.

3.3. In this section we generalize the length inequality for geodesic arcs in the following way: We consider a closed straight arc β and orthogonal trajectories α_1 and α_2 through its endpoints. Then the length of any curve γ joining α_1 with α_2 is at least equal to $|\beta|$. The same is more generally true for geodesic arcs instead of just straight arcs.

THEOREM (Divergence principle). *Let β be a closed geodesic arc with endpoints z_1 and z_2, and let α_1 and α_2 be geodesic rays with initial points z_1 and z_2 respectively. Let the angles between the rays α_i and β be at least $\pi/2(\lambda n_i+1)$, where $n_i \geqslant 0$ is the order of the zero of f at z_i, $i = 1, 2$. Then the length of any curve γ joining a point $\zeta_1 \in \alpha_1$ to a point $\zeta_2 \in \alpha_2$ is $|\gamma| \geqslant |\beta|$.*

Proof. The proof is the same as in [3] for quadratic differentials. We first choose a simple geodesic polygon γ_0 the interior D_0 of which contains β and γ as well as the two subintervals $[z_1, \zeta_1]$ and $[z_2, \zeta_2]$ of α_1 and α_2 respectively. $\overline{D}_0$ contains at most finitely many zeroes of f. We now mark the two endpoints of β and the possible zeroes of f on β. Through every other point z of β there is a well defined trajectory which cuts β orthogonally. Each of its rays either tends to a zero of f in $\overline{D}_0$ (without leaving $\overline{D}_0$) or else has a first intersection with γ_0. We now also mark the initial points of these finitely many (relatively) critical rays on β. By the markings, β is subdivided into finitely many straight intervals β_i. Every orthogonal trajectory α through a point $z \in \beta_i$ has, in both directions, a first intersection with γ_0. The corresponding subinterval $\hat{\alpha}$ of α is a cross cut of D_0 which separates ζ_1 and ζ_2. By a final marking of points on β we eliminate those trajectory intervals $\hat{\alpha}$ which meet a vertex of γ_0. We call the (open) subintervals of β generated by the markings β_i again. Let now S_i be the strip of trajectories orthogonal to β_i. Its top and bottom intervals are straight segments such that its F-image S_i' is a parallel strip cut by two rectilinear intervals. At least one of the subarcs of γ in S_i crosses S_i, and therefore the length of this subarc in the $|f(z)|^\lambda|dz|$-metric is at least $|\beta_i|$. As the S_i are disjoint, we get, by summing up, $|\gamma| \geqslant \sum |\beta_i| = |\beta|$.

Let equality hold. Thus γ must cross every S_i orthogonally in a single arc γ_i, and there can be no other arcs on γ except the γ_i. Let γ have a point in common with β, and assume, for convenience, that it is the initial point z_1. Then, the first interval β_1 must coincide with γ_1, and, by induction, we get $\gamma = \beta$. We have thus reproved the theorem, that a geodesic arc is the unique shortest connection of its endpoints.

If $\gamma \cap \beta = \emptyset$, the two arcs β and γ together with the intervals $[z_1, \zeta_1] \subset \alpha_1$ and $[z_2, \zeta_2] \subset \alpha_2$ bound a rectangle with respect to the metric $|f(z)|^\lambda|dz|$ which does not contain any zeroes of f. Its inner angles at the four corners (where two arcs come together) are $\pi/2(\lambda n+1)$, $n \geqslant 0$ being the order of the respective zero, while at all the other zeroes the inner angle is equal to $\pi/(\lambda n+1)$.

3.4. Using the divergence principle we can extend the corollary of Section 2.5 to geodesic cross cuts. *Let f be holomorphic in the disk D and let γ be a geodesic cross cut of D with endpoints z_1 and z_2 on ∂D. Assume that both points have neighborhoods U_i of finite area $|U_i|$.* (By Section 2.6 the two points z_i cannot coincide.) *Let $\tilde{\gamma}$ be any arc joining z_1 and z_2. Then $|\tilde{\gamma}| \geqslant |\gamma|$, and equality can only hold if $\tilde{\gamma} = \gamma$.*

Proof. By a reasoning similar to the one in Section 2.6 we can show that $|\tilde{\gamma}| \geqslant |\gamma|$. There are sequences of arbitrarily short circular arcs tending to z_1 and z_2 respectively and connecting $\tilde{\gamma}$ and γ. An application of 2.5 to the subintervals of γ and $\tilde{\gamma}$ and the corresponding circular arcs gives the inequality $|\tilde{\gamma}| \geqslant |\gamma|$.

Let equality hold. Then, $\tilde{\gamma}$ necessarily is a geodesic. Assume $\tilde{\gamma} \neq \gamma$ and let z_0 be a regular point on γ which is not a point of $\tilde{\gamma}$. Let β be a sufficiently short straight arc orthogonal to γ through z_0 which does not meet $\tilde{\gamma}$. As γ and $\tilde{\gamma}$ are cross sections of D, they subdivide it into domains $A, B, \tilde{A}, \tilde{B}$ respectively, where the notation is such that A and $\tilde{A}$ have a common boundary on ∂D. Let $\beta \subset \tilde{A}$, and let z be that endpoint of β which lies in B. Without restricting the generality, we can assume that the trajectory α orthogonal to β through z has well defined endpoints ζ_1 and ζ_2 on ∂D. It follows from the divergence principle that $z_i \neq \zeta_i$ for $i = 1, 2$. (The length of the segment $[z_0, z]$ on β is a lower bound for the length of any arc connecting α and γ.) Therefore $\tilde{\gamma}$ must cut α at interior points $\tilde{z}_1$ and $\tilde{z}_2$ which contradicts the uniqueness of geodesic connections.

4. Convergence of the geodesic rays

4.1. In this chapter, f is supposed to be holomorphic in the unit disk D and of finite $L_{2\lambda}$-norm, $0 < \lambda < \infty$. In our terminology this means that the area of D with respect to the metric $|f(z)|^\lambda|dz|$ is finite:

$$|D| = \iint_D |f(z)|^{2\lambda} dx\,dy < \infty .$$

In [3] I proved that for $\lambda = 1/2$ (metric associated with a quadratic differential) every non critical trajectory ray converges to a well determined boundary point of D. As was pointed out to me by R. Fehlmann, the proof contains an error. Here, that part of the proof is replaced by a different argument which in fact works for arbitrary λ, and for any geodesic ray.

For arbitrary points $z_1, z_2 \in D$ we define the distance

$$d(z_1, z_2) = \inf_{\{\gamma\}} \int_\gamma |f(z)|^\lambda |dz|,$$

where $\{\gamma\}$ ranges over all rectifiable arcs connecting z_1 and z_2 in D. It satisfies the triangle inequality

$$d(z_1, z_2) + d(z_2, z_3) \geqslant d(z_1, z_3).$$

For, let γ_{12} and γ_{23} be arcs joining z_1 to z_2 and z_2 to z_3 respectively. Then γ_{13}

$= \gamma_{12}+\gamma_{23}$ is an arc joining z_1 to z_3 and for their lengths with respect to the metric $|f(z)|^\lambda|dz|$ we have $|\gamma_{12}|+|\gamma_{23}| = |\gamma_{13}|$. Therefore $|\gamma_{12}|+|\gamma_{23}| \geqslant d(z_1, z_3)$, and as γ_{12} and γ_{23} are arbitrary, this is true for the respective infima.

If $|\zeta_2| = 1$, $z_1 \in D$ we define

$$d(z_1, \zeta_2) = \lim_{z\to\zeta_2} d(z_1, z) \equiv \lim_{r\to 0} \inf\{d(z_1, z); z \in D, |z-\zeta_2| < r\}.$$

As the quantity under the limit sign increases with decreasing r, the limit $d(z_1, \zeta_2)$ exists and $0 < d(z_1, \zeta_2) \leqslant \infty$. The set of points ζ_2 for which $d(z_1, \zeta_2) = \infty$ is independent of the point z_1. It follows from the Schwarz inequality that a.a. points of the circumference are at finite distance. Let $l_\varrho(\theta)$ be the length of the radial segment of the annulus $0 < \varrho < |z| < 1$ with angle θ. We have

$$l_\varrho(\theta) = \int_\varrho^1 |f(re^{i\theta})|^\lambda dr \leqslant \frac{1}{\varrho}\int_\varrho^1 |f(re^{i\theta})|^\lambda r\,dr,$$

hence

$$\int_0^{2\pi} l_\varrho(\theta)d\theta \leqslant \frac{1}{\varrho}\int_0^{2\pi}\int_\varrho^1 |f(re^{i\theta})|^\lambda r\,dr\,d\theta,$$

and by Schwarz' inequality

$$\Big(\int_0^{2\pi} l_\varrho(\theta)d\theta\Big)^2 \leqslant \pi\frac{1-\varrho^2}{\varrho^2}\int_0^{2\pi}\int_\varrho^1 |f(re^{i\theta})|^{2\lambda} r\,dr\,d\theta < \infty.$$

Similarly we define, for two boundary points ζ_1 and ζ_2,

$$d(\zeta_1, \zeta_2) = \liminf_{r\to 0}\{d(z_1, z_2); z_1, z_2 \in D, |z_1-\zeta_1| < r, |z_2-\zeta_2| < r\}.$$

We call this quantity the distance (in the $|f(z)|^\lambda|dz|$-metric) of the two boundary points ζ_1 and ζ_2. Of course, $d(\zeta_1, \zeta_2) = \infty$ is possible.

4.2. LEMMA. *Let $f \not\equiv 0$ be holomorphic in the disk D: $|z| < 1$ and let $d(\zeta_1, \zeta_2)$ be the distance of the boundary points ζ_1 and ζ_2, as defined above. Then, for $\zeta_1 \neq \zeta_2$, $d(\zeta_1, \zeta_2) > 0$.*

Proof. Let there be two points ζ_1 and $\zeta_2 \neq \zeta_1$, $|\zeta_1| = |\zeta_2| = 1$, such that $d(\zeta_1, \zeta_2) = 0$. Then, there is a sequence of arcs γ_n with initial points $z_n^{(1)} \to \zeta_1$ and endpoints $z_n^{(2)} \to \zeta_2$, the lengths $|\gamma_n| = \int_{\gamma_n} |f(z)|^\lambda|dz|$ of which tend to zero. As the lengths of the curves γ crossing a fixed annulus $0 < r_1 < |z| < r_2 < 1$ have a positive lower bound, the curves γ_n must tend to $|z| = 1$ and hence span one of the subarcs of ∂D determined by the two points ζ_1 and ζ_2. Let this arc be denoted by $[\zeta_1, \zeta_2]$. Choose the radii β_i: $\arg z = \theta_i$, $i = 1, 2$ of finite length with endpoints $\tilde{\zeta}_i$ in the interior of this interval. For n large enough, the arc γ_n cuts both β_1 and β_2. We find an increasing sequence of subdomains S_n of the sector S of D which

is bounded by the radii β_i and the subinterval $[\tilde{\zeta}_1, \tilde{\zeta}_2]$ of $[\zeta_1, \zeta_2]$. The length of the boundary of these exhausting domains S_n is uniformly bounded, and therefore f belongs to the Hardy class H_λ of this sector (for the definition see [1]). We conclude that f has a radial limit $f(e^{i\theta})$ almost everywhere on $[\tilde{\zeta}_1, \tilde{\zeta}_2]$. Let $f(e^{i\theta}) \neq 0$ on a set of positive measure. Then, by Egoroff's theorem, there is a closed set E of positive measure $m(E)$, a number $a > 0$ and a number $0 < \varrho < 1$, such that for $\varrho < r < 1$, $\theta \in E$, $|f(re^{i\theta})| > a$. Therefore $|\gamma_n| = \int_{\gamma_n} |f(z)|^\lambda|dz| \geqslant a^\lambda\cdot\varrho\cdot m(E)$ for all sufficiently large n, contradicting $|\gamma_n| \to 0$. We conclude that the radial limit of f is zero a.e. on $[\zeta_1, \zeta_2]$, and hence $f \equiv 0$.(1)

4.3. We are now ready to prove the convergence of every geodesic ray for all holomorphic functions of finite $L_{2\lambda}$-norm in the disk. The proof is partly the same as in [3], partly based on the lemma of the preceding section.

THEOREM. *Let f be holomorphic in the disk $D: |z| < 1$ and let $|D| = \iint |f(z)|^{2\lambda}dxdy < \infty$, $0 < \lambda < \infty$. Then, every geodesic ray with respect to the metric $|f(z)|^\lambda|dz|$ converges to a point on ∂D.*

Proof. As a special case, let γ be a (non critical) trajectory ray with initial point z_0 and suppose that ζ_1 and $\zeta_2 \neq \zeta_1$ are points of its cluster set on ∂D. Then, this set contains at least one of the subintervals of ∂D, $[\zeta_1, \zeta_2]$ say, with ζ_1 and ζ_2 as endpoints. Let β_0 be the orthogonal trajectory through z_0. By an arbitrarily short shift of z_0 along γ we can assume that β_0 is a convergent cross cut of D. As proved in Section 2.6, its two endpoints are necessarily distinct. It subdivides D into two Jordan domains. The curve γ and all its orthogonal trajectories β are contained in a fixed one of them. Moreover, no subray of a trajectory β can converge to an interior point of the interval $[\zeta_1, \zeta_2]$. Therefore there exists a positive number d such that every β (except possibly denumerably many critical ones) has Euclidean length at least d.

Multiplying f, if necessary, by a factor $e^{i\theta}$, we can assume that γ is the image, by F^{-1}, of the interval $u_0 \leqslant u < u_\infty$, $w = u+iv$, $u-u_0$ being the arc length on γ. The total length of γ is $u_\infty - u_0$. We now look at the orthogonal strip S of γ and its Euclidean image S' by F. Using the notations of Section 3.2, we get (for all but denumerably many u)

$$d \leqslant |\beta_u|_E \equiv l_E(u) = \int_{\beta_u} |dz| = \int_{\beta'_u} \left|\frac{dz}{dw}\right| dv.$$

Integrating over the finite subinterval $[u_0, u_1] \subset [u_0, u_\infty)$ we find

$$d(u_1-u_0) = \int_{u_0}^{u_1}\int_{\beta'_u} \left|\frac{dz}{dw}\right| dv\,du.$$

(1) It was remarked by Li Zhong that this argument works as soon as there is a dense set of radii of linite length.

The Schwarz inequality then yields

$$d^2(u_1-u_0)^2 \leqslant \int_{u_0}^{u_1}\int_{\beta'_u} dv\,du \cdot \int_{u_0}^{u_1}\int_{\beta'_u} \left|\frac{dz}{dw}\right|^2 dv\,du \leqslant |S'|_E \cdot |S|_E = |S| \cdot |S|_E \leqslant |D| \cdot \pi.$$

Letting $u_1 \to u_\infty$, we get

$$d^2(u_\infty - u_0)^2 \leqslant \pi \cdot |D|,$$

hence $u_\infty < \infty$: The trajectory γ necessarily has finite length in the metric $|f(z)|^{1/2}|dz|$.

In the general case, if γ is an arbitrary geodesic ray, we subdivide it into its straight open arcs γ_i, bounded by z_0 and the zeroes of f on γ. The orthogonal strips S_i of the arcs γ_i do not overlap, and we can use the above argument for every S_i. We get, by taking the square root,

$$d(u_i - u_{i-1}) \leqslant \sqrt{|S_i|} \cdot \sqrt{|S_i|_E},$$

hence

$$d(u_n - u_0) = d\sum_{i=1}^{n}(u_i - u_{i-1}) \leqslant \sum_{i=1}^{n}\sqrt{|S_i|}\cdot\sqrt{|S_i|_E}.$$

Applying again the Schwarz inequality yields

$$d^2(u_n-u_0)^2 \leqslant \sum_{i=1}^{n}|S_i| \cdot \sum_{i=1}^{n}|S_i|_E \leqslant |D| \cdot \pi.$$

The bound is independent of n, therefore

$$d^2(u_\infty - u_0)^2 \leqslant \pi \cdot |D|,$$

as before.

To finish the proof we remark that there exists a sequence of disjoint subarcs γ_n of γ with endpoints $z_n^{(1)} \to \zeta_1$, $z_n^{(2)} \to \zeta_2$, as both ζ_1 and ζ_2 belong to the cluster set of γ. The lengths of these arcs satisfy $\sum|\gamma_n| \leqslant |\gamma| < \infty$. We conclude $d(\zeta_1, \zeta_2) \leqslant \lim_{n\to\infty}|\gamma_n| = 0$, contradicting $d(\zeta_1, \zeta_2) > 0$.

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RELATIONS BETWEEN THE BOUNDARY VALUES AND PERIODS FOR GENERALIZED ANALYTIC FUNCTIONS IN R^n

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The basic idea of the concept of generalized analytic functions is the following: the class of holomorphic functions is to be replaced by a wider class in such a way that fundamental properties of holomorphic functions should retain their validity. It is possible to define generalized analytic functions of that kind either in C, C^n or in R^n.

In this paper the following approach to the concept of generalized analytic functions is applied: Instead of the real part u and the imaginary part v of a holomorphic function we regard a pair (u, ω_u), where u is a real-valued function and ω_u is a closed differential form depending on u and its first order derivatives. The aim of the paper is to discuss the interrelation between the periods of ω_u and the boundary values of u. This theory generalizes the well-known facts about the dependence of periods of a holomorphic function on the boundary values of its real part. We remark that the associated partial differential equations of second order are not necessarily self-adjoint.

1. A concept of generalized analytic functions in R^n

In order to define holomorphic functions or their generalizations there are three ways of approach. Firstly, one can define holomorphic and generalized analytic functions as vectors $(u_1, \ldots, u_m)$ with components u_i fulfilling a given first order partial differential system of elliptic type (this type of definition includes e.g. the notion of pseudo-analytic functions in L. Bers' sense). The second way is to define harmonic differential forms. This idea consists in replacing the real and imaginary parts of a holomorphic function by a differential form and its dual. The basic idea of the third generalization is to replace the imaginary part only by a differential form. The precise definition is the following:

Let ω_u be a differential form of degree $n-1$ depending on a real-valued func-