COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

SCHROEDER'S FUNCTIONS

HENRYKA ŚMIAŁEK

Institute of Mathematics, Łódź Technical University, Poland

Introduction and summary

The purpose of the present note is to give a characterization of the complex univalent functions which are solutions of well-known functional Schroeder's equation ([3], [4], Chapt. VI).

According to the result given by E. Peschl ([8], p. 69), for any bounded univalent function

(0.1) $$b(z) = b_1 z + \ldots, \quad b_1 > 0, |b(z)| < 1, z \in K(0, 1),$$

(comp. § 1, Point 1) there exists exactly one univalent, not necessarily bounded, function of the form

(0.2) $$G(z) = z + G_2 z^2 + \ldots, \quad z \in K(0, 1),$$

which satisfies with b the above-mentioned functional equation

(0.3) $$G\big(b(z)\big) = b_1 G(z);$$

the function G will be called Schroeder's univalent function corresponding to the function b, shortly Schroeder's univalent function.

On the other hand, we know that every univalent function of the form

(0.4) $$A(z) = z + A_2 z^2 + \ldots, \quad z \in K(0, 1),$$

has the parametric representation

(0.5) $$A(z) = \lim_{t \to \infty} e^t f(z, t), \quad z \in K(0, 1),$$

where

(0.6) $$w = f(z, t)$$

is the simple characteristic function (§ 1, Point 7) of the corresponding generalized Loewner's equation of the type

(0.7 $$\frac{dw}{dt} = -wc(w, t), \quad w \in K(0, 1),\ t \in [0, \infty[$$

with the initial condition $w = z$ for $t = 0$; the function on the right-hand side of (0.7)

$$c(w, t), \quad w \in K(0, 1), \ t \in [0, \infty[, \tag{0.8}$$

is, for any fixed t, holomorphic with respect to the complex variable w, and for any fixed w, measurable in the sense of Lebesgue with respect to the real variable t, moreover, $c(0, t) = 1$, $\operatorname{Re} c(w, t) > 0$ (§ 1, Point 10). In other words, for every function (0.4), there exists a function (0.8) such that, for the corresponding simple characteristic function (0.6) of (0.7), the representation (0.5) of (0.4) holds. There may exist more than one function $c(w, t)$ of the type (0.8) for a function (0.4).

In view of (0.3), (0.5) and (0.7), the set of all Schroeder's univalent functions constitutes a subclass of all univalent functions generated, in accordance with the above, by a subclass of all functions of the type (0.8). Therefore, it is natural to ask: For which function (0.8) will the suitable function (0.6) give, by (0.5), a Schroeder univalent function and what property does the Schroeder domain, i.e., the image of the unit disc under the mapping by a Schroeder univalent function, possess? The answer to these questions is given in this paper. Namely, any Schroeder univalent function may be characterized by the representation (0.5), where $w = f(z, t)$ is the simple characteristic function of generalized Loewner's equation (0.7) in which the function (0.8) is, for any fixed w, a periodic function with respect to the variable t (Theorem 2.2). The Schroeder domain is characterized by pseudo-starlikeness (comp. § 1, Point 2 and Theorem 3.1). Both the above theorems were announced without proofs in [11], p. 948.

In this paper there is also given another proof of Theorem 2.1 due to E. Peschl, [8], p. 69. The presented proof has been based on a power series representation in the whole domain of existence of a given function without making use of Koenigs' result from [3], as it was done by Peschl.

1. Preliminaries

In this part some notations, definitions and various results, which we shall find of importance in the sequel, have been collected together.

1. The symbols $]\alpha, \beta[$, $[\alpha, \beta]$, $[\alpha, \beta[$ denote, respectively, an open interval, a closed one and a left-hand side closed interval of the real line with the end points α and β.

An open disc with centre ζ_0 and radius r, $r > 0$, will be denoted by $K(\zeta_0, r)$.

We are allowed to denote a function by a letter or, if it is suitable, by writing its value for any argument.

Let E be a set in the complex plane. A function q is called *univalent in* E if $q(\zeta_1) \neq q(\zeta_2)$ for different points ζ_1 and ζ_2 of E.

A sequence of functions q_j, $j = 1, 2, \ldots$, uniformly convergent in every bounded and closed subset of E, is said to be *almost uniformly*, or in other words, *locally uniformly convergent in* E (cf. [2], p. 16; [9], p. 27).

2. A set E of the complex plane is said to be *pseudo-starlike with respect to a fixed point* if there exists a homothetic mapping with centre at that point and ratio λ, $0 < \lambda < 1$, transforming E onto its subset. A set which is pseudo-starlike with respect to the origin will be called a *pseudo-starlike set*.

It is easy to verify that every bounded simply connected domain containing a point ζ_0 is pseudo-starlike with respect to that point. It is also obvious that, if a domain is starlike with respect to the origin, then it is pseudo-starlike. There exist, however, domains which are not pseudo-starlike, for example, the complex plane from which a half-straight line $p = \{\zeta\colon \zeta = a+ti, a > 0, t \in [0, \infty[\}$ has been removed.

3. We denote by S the set of all holomorphic and univalent functions A in the unit disc $K(0, 1)$ which have the expansion

$$A(z) = z + A_2 z^2 + \ldots, \quad z \in K(0, 1). \tag{1.1}$$

4. Every holomorphic and univalent function b in the unit disc, which has the expansion

$$b(z) = b_1 z + \ldots, \quad 0 < b_1 < 1, \ z \in K(0, 1), \tag{1.2}$$

and satisfies the boundedness condition

$$|b(z)| < 1, \quad z \in K(0, 1), \tag{1.3}$$

is called the *bounded in Loewner sense*, shortly, *bounded function* ([7]). The set of all bounded in Loewner sense functions will be denoted by $S_{(1)}$. The condition (1.3) can be written as the inclusion

$$b\big(K(0, 1)\big) \subset K(0, 1). \tag{1.4}$$

5. For any function $b \in S_{(1)}$ the sequence of iterates $b^n(z)$ may be defined by the formulae

$$b^0(z) = z, \quad b^n(z) = b\big(b^{n-1}(z)\big), \quad z \in K(0, 1), \ n = 1, 2, \ldots \tag{1.5}$$

b^n is called the n-multiple iterate of b or, the iterate of b of the multiplicity n. (Following the notation used in [4], upper indices at the sign of a function will denote the iterates. Exponents of a power of a function will be written after a bracket containing the whole expression for the function.) We can state without difficulty that, for any n

$$b\big(b^{n-1}(z)\big) = b^{n-1}\big(b(z)\big) = b_1^n z + \ldots \tag{1.6}$$

Let us put

$$A^{(n)}(z) = \frac{b^n(z)}{b_1^n}, \quad z \in K(0, 1), n = 1, 2, \ldots \tag{1.7}$$

where $b^n(z)$ is defined by (1.5). In view of (1.6) and (1.7) we have that

$$A^{(n)}(z) = z + \ldots \tag{1.8}$$

Therefore, the function (1.7) belongs to the class S and, by the distortion theorem ([9], p. 21), the inequality

$$|A^{(n)}(z)| \leqslant \frac{r}{(1-r)^2}, \quad z \in \overline{K(0, r)}, \tag{1.9}$$

holds for $r \in]0, 1[$.

6. Let us take the set

$$\Delta = K(0, 1) \times [0, \infty[\tag{1.10}$$

of the points (w, t), and denote by $\mathfrak{C}$ a family of all complex functions c defined on Δ, such that: (i) for any fixed t, $c(w, t)$ is a Carathéodory function, which means it is holomorphic with respect to the complex variable w and has a positive real part, besides, $c(0, t) = 1$; (ii) for any fixed w, $c(w, t)$ is measurable in the sense of Lebesgue with respect to variable t.

It will be said that a function $d \in \mathfrak{C}$ is *periodic with respect to the variable* t, shortly *periodic*, if there exists a positive number $\mathfrak{T}$, called the period of c, such that for any $w \in K(0, 1)$ and $t \in [0, \infty[$, the equality $d(w, t+\mathfrak{T}) = d(w, t)$ holds.

7. Let us consider a generalized Loewner's equation, i.e., the equation

$$\frac{dw}{dt} = -wc(w, t), \quad (w, t) \in \Delta, \tag{1.11}$$

where c is a function of the class $\mathfrak{C}$ ([5], [9], [12]). Every equation (1.11) is known to have a general solution $\varphi(z, \tau, t)$, $(z, \tau) \in \Delta$, $t \in [\tau, \infty[$, otherwise called the characteristic function. This means that, for any fixed z, τ, the function $\omega(t) = \varphi(z, \tau, t)$ is absolutely continuous and satisfies almost everywhere on the interval $[\tau, \infty[$ the generalized Loewner's equality

$$\frac{d}{dt}\omega(t) = -\omega(t)c(\omega(t), t) \tag{1.12}$$

and the initial condition

$$\omega(\tau) = z, \tag{1.13}$$

i.e.

$$\varphi(z, \tau, \tau) = z. \tag{1.14}$$

Besides, the function described above is the only one which satisfies (1.12) and (1.13) on the interval $[\tau, \infty[$ and, as it is seen, also on a lesser interval $[\tau, T]$.

Moreover, for any $(z, \tau) \in \Delta$ and $\tau \leqslant \mathfrak{T} \leqslant t < \infty$ the equality

$$\varphi(\varphi(z, \tau, \mathfrak{T}), \mathfrak{T}, t) = \varphi(z, \tau, t) \tag{1.15}$$

holds.

Further, for any fixed τ and t, a function $a(z) = \varphi(z, \tau, t)$ is holomorphic and univalent in the unit disc $K(0, 1)$ and satisfies the conditions

$$|\varphi(z, \tau, t)| < 1, \quad \varphi(0, \tau, t) = 0, \quad \varphi_z'(0, \tau, t) = e^{-(t-\tau)}. \tag{1.16}$$

Hence, in particular, every function of the form

$$f(z, t) = \varphi(z, 0, t) \tag{1.17}$$

is, for any fixed t, holomorphic and univalent with respect to the complex variable $z \in K(0, 1)$, and

$$|f(z, t)| < 1, \quad f(0, t) = 0, \quad f_z'(0, t) = e^{-t} \tag{1.18}$$

(comp. [6]; [9], Chapt. VI, Section 6.1; [12], §§ 3, 5).

In the sequel the function (1.17) will be called the simple characteristic function of (1.11) ([12], § 3).

8. According to the above, we state that every function $a(z) = \varphi(z, \tau, t)$ belongs to the class $S_{(1)}$. Hereby, $\varphi(z, \tau, t)$ is the characteristic function of an equation (1.11), τ, t are arbitrarily fixed numbers from the intervals $[0, \infty[$, $[\tau, \infty[$, respectively, z is the variable from the unit disc, and $a'(0) = e^{-(t-\tau)}$. In particular, every function

$$b(z) = f(z, T), \tag{1.19}$$

where $f(z, t)$ is the simple characteristic function of (1.11), belongs to the class $S_{(1)}$ and

$$b'(0) = e^{-T}. \tag{1.20}$$

T is a fixed number from the interval $[0, \infty[$.

9. Conversely, it has been stated that every function $b \in S_{(1)}$ has the representation (1.19), where $T = -\log b'(0)$, and $f(z, t)$ is the simple characteristic function of (1.11). More exactly, for every function $b \in S_{(1)}$, there exists a function $c \in \mathfrak{C}$ such that (1.19) holds, where $f(z, t)$ is the simple characteristic function of (1.11) with the above-mentioned c (cf. [1], p. 2; [10]; [12], § 5).

In connection with the above, for any given function $b \in S_{(1)}$ and the corresponding function c described above, c is said to be the function of representation for b or, in other words, b has the function of representation c ([12], § 5).

It is remarkable that, in general, there may exist many functions of representation for a given b.

10. It has been stated that the limit

$$A(z) = \lim_{t\to\infty} e^t f(z, t) \tag{1.21}$$

exists almost uniformly in the unit disc $K(0, 1)$ and belongs to the class S for any simple characteristic function $f(z, t)$ of (1.11) (comp. [2], p. 95; [9], Th. 6.3; [12], § 5).

11. Conversely, any function of the class S has the representation (1.21), where $f(z, t)$ is the simple characteristic function of (1.11). More exactly, for every function $A \in S$, there exists a function $c \in \mathfrak{C}$ such that, for the simple characteristic function $f(z, t)$ of (1.11) with the above-mentioned c, the representation (1.21) is valid (cf. [2], p. 95; [9], Th. 6.3; [12], § 5).

Similarly to Point 9, for any given function $A \in S$ and the corresponding function c described above, c is said to be the function of representation for A or, in other words, b has the function of representation c ([12], § 5).

12. Now, we shall show a specific property of a characteristic function of some generalized Loewner's equation, which will play an important role in the proof of the main result.

LEMMA 1.1. *Suppose that a function $d \in \mathfrak{C}$ has a positive period $\mathfrak{T}$, and $\psi(z, \tau, t)$ is the characteristic function of the equation*

$$\frac{dw}{dt} = -wd(w, t), \quad (w, t) \in \Delta. \tag{1.22}$$

Then, for $(z, \tau) \in \Delta$, $t \in [\tau, \infty[$, $k = 1, 2, \ldots$, the equalities

$$\psi(z, \tau + k\mathfrak{T}, t + k\mathfrak{T}) = \psi(z, \tau, t), \tag{1.23}$$

and

$$\psi(\psi(z, \tau, \tau + k\mathfrak{T}), \tau, t) = \psi(z, \tau, t + k\mathfrak{T}) \tag{1.24}$$

are satisfied.

Proof. Let us consider, for fixed z, τ and k, two functions:

(i) $$\omega(t) = \psi(z, \tau, t)$$

and

(ii) $$\omega_k(t) = \psi(z, \tau + k\mathfrak{T}, t + k\mathfrak{T}),$$

where $t \in [\tau, \infty[$. It results from Point 7 that these functions satisfy almost everywhere the following equalities and the initial conditions:

(iii) $$\frac{d}{dt}\omega(t) = -\omega(t)d(\omega(t), t), \quad \omega(\tau) = z$$

and

(iv) $$\frac{d}{dt}\omega_k(t) = -\omega_k(t)d(\omega_k(t), t + k\mathfrak{T}), \quad \omega_k(\tau) = z.$$

In view of the periodicity of d we have that

$$d(\omega_k(t), t + k\mathfrak{T}) = d(\omega_k(t), t)$$

for any t (comp. Point 6). Hence and from (iv) it follows that

(v) $$\frac{d}{dt}\omega_k(t) = -\omega_k(t)d(\omega_k(t), t).$$

Therefore, in view of (iii)–(v) and the uniqueness of the function satisfying (iii) (comp. Point 7) we infer that the function (ii) is identical with (i). This shows that (1.23) is true.

Now, if we replace z by $\psi(z, \tau, \tau + k\mathfrak{T})$ in (1.23) and then apply the basic identity (1.15) we immediately get (1.24).

In particular, if we put

$$g(z, t) = \psi(z, 0, t) \tag{1.17'}$$

and

$$b(z) = g(z, \mathfrak{T}), \tag{1.19'}$$

then we have that

$$b^n(z) = g^n(z, \mathfrak{T}) = \psi^n(z, 0, \mathfrak{T}) = \psi(z, 0, n\mathfrak{T}) = g(z, n\mathfrak{T}) = e^{-n\mathfrak{T}}z + \ldots, \tag{1.25}$$

where b^n is the n-multiple iterate of b from (1.19').

In fact, this follows at once by making use of Point 5, Point 8, (1.16), (1.20), (1.17'), (1.19'), (1.24) and reasoning by induction.

2. Properties of Schroeder's univalent functions

At the begining, we shall recall a well known result due to Peschl, [8], p. 69, and prove it in a somewhat different way.

THEOREM 2.1. *For any function $b \in S_{(1)}$ of the form* (1.2) *there exists a function G, holomorphic in $K(0, 1)$, vanishing at the zero point and satisfying with b the Schroeder functional equation, i.e.,*

$$G(b(z)) = b_1 G(z), \quad z \in K(0, 1). \tag{2.1}$$

Basides, G is univalent and belongs to the class S.

Proof. Let us consider a function $b \in S_{(1)}$ of the form (1.2) and a normalized sequence of univalent iterates $A^{(n)}$, generated by b of the form (1.7), (1.8). First, we prove that the sequence $(A^{(n)})$ is almost uniformly convergent in the disc $K(0, 1)$ to a function of the class S. To this aim, it is sufficient to show that the series

$$A^{(1)}(z) + \sum_{n=1}^{\infty}\left(A^{(n+1)}(z) - A^{(n)}(z)\right), \quad z \in K(0, 1), \tag{2.2}$$

is uniformly convergent in $\overline{K(0, r)}$ for any $r \in]0, 1[$. In fact, on account of (1.2), (1.6) and (1.7), we get

$$\begin{aligned} A^{(n+1)}(z) - A^{(n)}(z) &= \frac{b^{n+1}(z)}{b_1^{n+1}} - \frac{b^n(z)}{b_1^n} = \frac{1}{b_1^{n+1}}\left(b(b^n(z)) - b_1 b^n(z)\right) \\ &= \frac{1}{b_1^{n+1}}\left(b_1 b^n(z) + b_2(b^n(z))^2 + \ldots - b_1 b^n(z)\right) \\ &= \frac{1}{b_1^{n+1}}\sum_{k=2}^{\infty} b_k(b^n(z))^k = \frac{1}{b_1^{n+1}}\sum_{k=2}^{\infty} b_k(b_1^n)^k\left(\frac{b^n(z)}{b_1^n}\right)^k \\ &= \frac{1}{b_1^{n+1}}\sum_{k=2}^{\infty} b_k(b_1^n)^k(A^{(n)}(z))^k, \quad z \in K(0, 1),\ n = 1, 2, \ldots \end{aligned} \tag{2.3}$$

Let us remark that, for any fixed $r \in]0, 1[$ and for a given $b_1 \in]0, 1[$, there exists a number N such that

$$b_1^n \frac{r}{(1-r)^2} < \frac{1}{2}, \quad n > N. \tag{2.4}$$

By making use of (1.9), (2.3) and (2.4) we obtain the inequality

$$(2.5)\quad |A^{(n+1)}(z)-A^{(n)}(z)| \leqslant \frac{1}{b_1^{n+1}}\sum_{k=2}^{\infty}\left(b_1^n\frac{r}{(1-r)^2}\right)^k = \frac{1}{b_1^{n+1}}\frac{\left(b_1^n\frac{r}{(1-r)^2}\right)^2}{1-b_1^n\frac{r}{(1-r)^2}}$$

$$\leqslant 2b_1^{n-1}\left(\frac{r}{(1-r)^2}\right)^2, \quad z\in\overline{K(0,r)},\ n>N.$$

Hence, recalling the condition $0<b_1<1$, we conclude that the series (2.2) is convergent in every closed disc $\overline{K(0,r)}$, $r\in\,]0,1[$. This implies almost uniform convergence of the sequence $(A^{(n)})$ in the unit disc $K(0,1)$. In that situation let us assume that

$$(2.6)\quad G(z) = \lim_{n\to\infty} A^{(n)}(z).$$

Then, in view of (1.5), (1.6) and (1.7), we can write

$$(2.7)\quad A^{(n+1)}(z) = \frac{b^{n+1}(z)}{b_1^{n+1}} = \frac{1}{b_1}\frac{b^n(b(z))}{b_1^n} = \frac{1}{b_1}A^{(n)}(b(z)),$$

$$z\in K(0,1),\ n=1,2,\ldots$$

If we let n tend to infinity in (2.7), we obtain, by (2.6), the equality of the form (2.1). This completes the proof.

Let us remark that (2.6) is the only holomorphic function in the unit disc which satisfies the equation (2.1).

Really, it follows immediately from Koenigs' result ([3], p. 31; [4], p. 140) according to which the uniqueness of the holomorphic solution of (2.1) in a neighbourhood of the zero point takes place.

Henceforth, the solution (2.6), described above, of the Schroeder functional equation (2.1) will be called the *Schroeder univalent function*. Because of the above we see the set of all Schroeder functions, if b runs within the class $S_{(1)}$, is a subclass of S.

Now, we prove the main result.

THEOREM 2.2. *A function of the class S is a Schroeder function if and only if it has a periodic function of representation.*

Proof. At first, assume that a function G is a Schroeder function. This means that it satisfies the equality

$$(2.8)\quad G(b(z)) = b_1 G(z), \quad z\in K(0,1)$$

with a function b of the form (1.2). Let us put

$$(2.9)\quad \mathfrak{T} = -\log b'(0), \quad \mathfrak{T}>0.$$

Next, take the representation

$$(2.10)\quad b(z) = f(z,\mathfrak{T}) = e^{-\mathfrak{T}}z+\ldots, \quad z\in K(0,1),$$

where $f(z,t)$, $(z,t)\in K(0,1)\times[0,\infty[$ is the simple characteristic function of (1.11) with a suitable function of representation c (see § 1, Point 9). Next, take a periodic function d with period $\mathfrak{T}$ (comp. § 1, Point 6), for example, the function defined by the formulae

$$(2.11)\quad \begin{cases} d(w,t+\mathfrak{T}) = d(w,t), & t\in[0,\infty[,\\ d(w,t) = c(w,t), & t\in[0,\mathfrak{T}],\end{cases} \quad w\in K(0,1).$$

Further, consider the simple characteristic function $g(z,t)$, $(z,t)\in K(0,1)\times[0,\infty[$ of equation (1.22) with d defined by (2.11). Then, it is easily justified that, from the identity $c(w,t)$ and $d(w,t)$ for $t\in[0,\mathfrak{T}]$, the identity

$$(2.12)\quad g(z,t) = f(z,t), \quad t\in[0,\mathfrak{T}]$$

follows (comp. § 1, Point 7). Now, using (1.25), (2.9), (2.10) and (2.12), we get that, for any n,

$$(2.13)\quad b^n(z) = f^n(z,\mathfrak{T}) = g^n(z,\mathfrak{T}) = g(z,n\mathfrak{T}) = e^{-n\mathfrak{T}}z+\ldots$$

Therefore, from (1.7), (2.6), (2.8), (2.10), (2.12), and (2.13) it follows that

$$(2.14)\quad G(z) = \lim_{n\to\infty} e^{n\mathfrak{T}}b^n(z) = \lim_{n\to\infty} e^{n\mathfrak{T}}g(z,n\mathfrak{T})$$

where the convergence is almost uniform in $K(0,1)$.

On the other hand, we know (§ 1, Point 10) that there exists a limit of $e^t g(z,t)$ almost uniformly in $K(0,1)$, as t tends to infinity. Moreover, the limit belongs to the class S. Hence, in view of (2.14) and the uniqueness of the limit, we conclude that

$$(2.15)\quad G(z) = \lim_{t\to\infty} e^t g(z,t), \quad z\in K(0,1).$$

This means that the Schroeder function G has a periodic function of representation, namely, the one defined in (2.11) (comp. § 1, Point 11).

Conversely, assume that a function $G\in S$ has the representation (2.15), where $g(z,t)$ is the simple characteristic function of (1.22) with a function of representation d having a period $\mathfrak{T}$.

Let us take the function b defined by the formula

$$(2.16)\quad b(z) = g(z,\mathfrak{T}) = e^{-\mathfrak{T}}z+\ldots, \quad z\in K(0,1).$$

It belongs to the class $S_{(1)}$ (see § 1, Point 8). Now, if we use (1.6), (2.13), (2.15), (2.16), putting $t=n$ in (2.15), then we obtain successively that

$$G(z) = \lim_{n\to\infty} e^{n\mathfrak{T}}g(z,n\mathfrak{T}) = \lim_{n\to\infty} e^{n\mathfrak{T}}b^n(z) = e^{\mathfrak{T}}\lim_{n\to\infty} e^{(n-1)\mathfrak{T}}b^{n-1}(b(z)) = e^{\mathfrak{T}}G(b(z)).$$

This proves that G is the Schroeder function corresponding to the given function (2.16).

3. Schroeder's domains

The image of the unit disc $K(0,1)$ under the mapping by a Schroeder univalent function is said to be *Schroeder's domain.*

Besides, let us utilize the definitions of a pseudo-starlike domain (§ 1, Point 2) and the conformal radius (inner mapping radius) of a domain with respect to a point (see [2], p. 32; [9], p. 11).

In what follows we shall give a geometrical characterization of Schroeder univalent functions.

THEOREM 3.1. *A domain D of the open complex plane is a Schroeder domain if and only if the following characteristic conditions are satisfied*: (i) *D is simply connected and other than the whole plane,* (ii) *D contains the origin,* (iii) *D has the conformal radius with respect to the origin, equal to one,* (iv) *D is pseudo-starlike.*

Proof. Let us assume that D is a Schroeder domain. This means that

$$D = G(K(0,1)), \tag{3.1}$$

where G is a Schroeder function which satisfies the equality (2.1) with a suitable function $b \in S_{(1)}$. Then, it is easy to verify that D satisfies the conditions (i)–(iii) (cf. (1.1), Th. 2.1, (3.1) and [9], Corollary 1.4, p. 22). Further, if we take $h(\zeta) = b_1\zeta$, where b_1 is defined in (1.2), then, using (1.4), (2.1) and (3.1), we get the inclusion

$$D = G(K(0,1)) \supset G(b(K(0,1))) = h(G(K(0,1))) = h(D).$$

Thus D satisfies also the condition (iv) (cf. § 1, Point 2).

Conversely, suppose that a domain D satisfies the conditions (i)–(iv). According to (ii) and (iv), there exists a homothetic transformation $h(\zeta) = \lambda\zeta$, $0 < \lambda < 1$, such that

$$h(D) \subset D \tag{3.2}$$

(cf. § 1, Point 2). Moreover, according to (i), (ii), (iii), there exists a uniquely determined function G of the class S, such that (3.1) holds (cf. [2], p. 32). Hence, in view of (3.2), the inclusion

$$h(G(K(0,1))) \subset G(K(0,1)) \tag{3.3}$$

holds. Next, basing ourself on (3.3), we define the function

$$b(z) = G^{-1}(h(G(z))) = G^{-1}(\lambda G(z)) = b_1 z + \ldots, \quad z \in K(0,1), \tag{3.4}$$

where G^{-1} denotes the inverse function of G. Using (3.4), it can easily be verified that $b_1 = \lambda$ and $G(b(z)) = b_1 G(z)$ for $z \in K(0,1)$. This means that G is a Schroeder function. Thus, in view of (3.1), D is a Schroeder domain.

References

[1] I. Dziubiński, *Équation genérale de Loewner*, Bull. Soc. Sci. Lettres Łódź, XI (4) (1960), 1–8.

[2] Г. М. Голузин, *Геометрическая теория функций комплексного переменного*, Москва 1966.
[3] M. G. Koenigs, *Recherches sur les intégralés des certaines équations fonctionnelles*, Ann. Sci. École Norm. Sup. 31 (1884), Supplément, 3–41.
[4] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.
[5] П. П. Куфарев, *Об однопараметрических семействах аналитических функций*, Матем. Сб. 13 (55): 1 (1943), 87–118.
[6] —, *Теорема о решениях одного дифференциального уравнения*, Учен. Зап. Томск. ин-та 5 (1947), 20–21.
[7] K. Loewner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. 89 (1923), 103–121.
[8] E. Peschl, *Zur Theorie der schlichte Funktionen*, J. Reine Angew. Math. 176 (1937), 61–94.
[9] Ch. Pommerenke, *Univalent functions*, Vandenhoeck und Ruprecht, Goettingen 1975.
[10] A. C. Schaeffer and D. C. Spencer, *The coefficients of schlicht functions, II*, Duke Math. J. 12 (1945), 107–125.
[11] H. Śmiałkówna, *On some properties of Schroeder's univalent functions*, Bull. Acad. Polon. Sci. 28 (9) (1975), 947–949.
[12] H. Śmiałek, *Study of generalized Loewner's equation* (to appear).

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979