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ON THE EQUIVALENCE BETWEEN LOCALLY POLAR AND GLOBALLY POLAR SETS IN C^n

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B. Josefson [3] has recently proved that every locally C^n-polar set is a globally C^n-polar set. Using the method of the proof developed by Josefson we prove that every locally C^n-polar set E is an L-polar set (i.e. there exists a function W plurisubharmonic in C^n such that $W = -\infty$ on E and $W(x) \leqslant \beta+\log^+|x|$ for all $x \in C^n$, where β is a real constant).

1. Introduction

Given an open subset D of C^n we denote by PSH(D) the family of all functions plurisubharmonic in D. We denote by L the class of all functions U plurisubharmonic in C^n such that

$$U(x) \leqslant \beta+\log^+|x|, \quad x \in C^n,$$

where β is a real constant depending on U and $|x| := \max_{1\leqslant i\leqslant n} |x_i|$.

The aim of this paper is to prove the following

THEOREM. *Given any subset E of C^n the following conditions are equivalent:*

(a) *E is locally C^n-polar, i.e. for every point $a \in E$ there exist a neighbourhood U_a of a and a function $W \in \mathrm{PSH}(U_a)$ such that $W = -\infty$ on $E \cap U_a$;*

(b) *E is L-polar, i.e. there exists a function W of the class L such that $W = -\infty$ on E;*

(c) *E is globally C^n-polar, i.e. there exists $W \in \mathrm{PSH}(C^n)$ such that $W = -\infty$ on E.*

The implication (a) $\Rightarrow$ (c) was a question posed by P. Lelong [4] which has been recently solved by B. Josefson [3]. The main tool of the proof given by Josefson is an "elementary" Lemma on systems of homogeneous linear equations. The same lemma will be basic for the proof of our theorem.

The equivalence of locally polar and globally polar sets in R^n with respect to subharmonic functions is well known. However in the case of C^n-polar sets we cannot apply the methods of the classical potential theory.

Functions of the class L found some interesting applications in the theory of analytic functions of several complex variables, in particular in theory of interpolation and approximation by polynomials of several complex variables.

The class L permits to extend the classical notion of the Green function with pole at infinity to the case of C^n. It also permits to generalize the notion of the classical logarithmic capacity to the case of C^n in such a way that the C^n-capacity of a subset E of C^n is equal zero if and only if the set E is C^n-polar (see [6], [7], [8]).

The class L is a narrow and a very special subclass of the class $\mathrm{PSH}(C^n)$. Nevertheless it follows from the Theorem that if U is any plurisubharmonic function in an open set $D \subset C^n$ then there exists a function $W \in L$ such that

$$\{x \in D\colon\ U(x) = -\infty\} \subset \{x \in C^n\colon\ W(x) = -\infty\}.$$

In fact the Theorem says more, namely, if E is any subset of C^n such that E is locally of the form $\{x \in D\colon\ U(x) = -\infty\}$ with $U \in \mathrm{PSH}(D)$, then $E \subset \{x \in C^n\colon W(x) = -\infty\}$ with $W \in L$. As a direct consequence we get that every locally analytic set $E \subset C^n$ (i.e. for every point $a \in E$ there exist functions $f_1, \ldots, f_s$ holomorphic in a ball $B(a, R)$ such that $E \cap B(a, R) = \{x \in B(a, R)\colon |f_1(x)| + \ldots + |f_s(x)| = 0\}$) is L-polar.

2. Some known results

We shall recall some known results which will be used in the proof of the Theorem.

2.1. THEOREM (Bremermann [1]). *If D is a domain of holomorphy in C^n and if $U \in \mathrm{PSH}(D)$ then Hartogs domain*

$$H := \{(x, y) \in D \times C\colon\ |y| e^{U(x)} < 1\}$$

is a domain of holomorphy.

If F is a holomorphic function on H not continuable analytically beyond D and if

$$F(x, y) = \sum_{j=0}^{\infty} f_j(x) y^j, \quad (x, y) \in H,$$

is its development into the Hartogs series then

$$U(x) = V^*(x) := \limsup_{x' \to x} V(x'), \quad x \in D$$

where

$$V(x) := \limsup_{j \to \infty} (1/j) \log |f_j(x)|, \quad x \in D.$$

2.2. HARTOGS LEMMA, [2]. *Let $\{U_k\}$ be a sequence of plurisubharmonic functions in $D \subset C^n$ which are uniformly bounded from above on every compact subset of D. If $\limsup_{k\to\infty} U_k(x) \leqslant m$ in D, then for every $\varepsilon > 0$ and for every compact set $K \subset D$ one can find k_0 so that $U_k(x) \leqslant m + \varepsilon$, $x \in K$, $k \geqslant k_0$.*

2.3. *Let $\{U_k\}$ be a sequence of plurisubharmonic functions in a domain $D \subset C^n$ hich are uniformly bounded from above on every compact subset of D. Put $U(x) = \limsup_{k\to\infty} U_k(x)$ and $U^*(x) := \limsup_{x'\to x} U(x')$ for $x \in D$. Then either $U \equiv -\infty$ or $r^* \in \mathrm{PSH}(D)$. Moreover the set $N := \{x \in D\colon\ U(x) < U^*(x)\}$ is of $(2n)$-dimensional ebesgue measure zero. In particular for every point $x_0 \in D$ there exists a sequence $x_k\}$ such that $\lim_{k\to\infty} x_k = x_0$ and $U^*(x_0) = \lim_{k\to\infty} U(x_k)$.*

2.4. *For every function U of the class L and for every $r > 0$*

$$U(x) \leqslant \sup\{U(y)\colon\ |y| \leqslant r\} + \log^+(|x|/r), \quad x \in C^n.$$

In particular, if p is a polynomial of n complex variables then

$$|p(x)| \leqslant \|p\|_r (|x|/r)^{\deg p}, \quad |x| \geqslant r,$$

where

$$\|p\|_r := \sup\{|p(x)|\colon\ |x| \leqslant r\} \quad \text{(see [6], [7]).}$$

3. A lemma on systems of homogeneous linear equations

Let n, c, j, k, p and α denote fixed positive integers satisfying the following conditions

$$c > 1, \quad j \geqslant p = k^n > (2n+2)!, \quad \alpha = \alpha_j := 4cjk^{n-1}. \tag{3.1}$$

Put

$$M := \{r \in Z_+^n\colon\ r_s < pj,\ s = 1, \ldots, n\},$$
$$M^\alpha := \{r \in Z_+^n\colon\ r_s < \alpha,\ s = 1, \ldots, n\}.$$

ELEMENTARY LEMMA, [3]. *Let a_r, $r \in Z_+^n$ be an n-fold sequence of complex numbers satisfying the following conditions*

$$|a_0| > e^{-cj}, \quad |a_r| \leqslant 1 \quad \textit{for all } r \in Z_+^n. \tag{3.2}$$

Then there exists a solution $(U_t)_{t\in M}$ of the following system (S) *of the homogeneous linear equations*

$$\sum_{t\in M} a_{r-t} x_t = 0, \quad r \in M \setminus M^\alpha, \tag{S}$$

such that $\max_{t\in M} |U_t| = 1$ and $\max_{r\in M} |d_r| > e^{-pj/2}$, where

$$d_r := \sum_{t\in M} a_{r-t} U_t \quad \textit{and} \quad a_{r-t} = 0 \quad \textit{if} \ \min_{1\leqslant s\leqslant n} (r_s - t_s) < 0.$$

Proof. Put $M_0 = N_0 := M$ and $M_s := M \setminus \{r^1, \ldots, r^s\}$, where r^s, $s = 1, \ldots, \alpha^n$ denote all the points of M^α.

The system of homogeneous linear equations

$$(S_1)\qquad \sum_{t\in N_0} a_{r-t}x_t = 0, \quad r\in M_1$$

has $(pj)^n-1$ equations and $(pj)^n$ unknowns. Let $(U_t^1)_{t\in N_0}$ be a solution of (S_1) such that $\max|U_t^1| = 1$, and let t^1 be a point of N_0 such that $|U_{t^1}^1| = 1$.

Put $N_1 := N_0\setminus\{t^1\}$ and let $(U_t^2)_{t\in N_1}$ be a solution of the system

$$(S_2)\qquad \sum_{t\in N_1} a_{r-t}x_t = 0, \quad r\in M_2$$

such that $\max|U_t^2| = 1$, and let t^2 be a point of N_1 with $|U_{t^2}^2| = 1$.

Continuing this procedure we may define for every $i = 1, \ldots, \alpha^n$ a set $N_i = M\setminus\{t^1, \ldots, t^i\}$, where $t^1, \ldots, t^{\alpha^n}$ are different points of M^α, and a solution $(U_t^i)_{t\in N_{i-1}}$ of the system

$$(S_i)\qquad \sum_{t\in N_{i-1}} a_{r-t}x_t = 0, \quad r\in M_i,$$

such that $\max_t|U_t^i| = |U_{t^{i+1}}^{i}| = 1$.

Put $U_t^i = 0$ for $t\in M\setminus N_{i-1}$ and define

$$d_r^i = \sum_{t\in M} a_{r-t}U_t^i, \quad r\in M, \quad i = 1, \ldots, \alpha^n.$$

Then $d_r^i = 0$ for $r\in M\setminus M^\alpha$, because $d_r^i = 0$ for $r\in M_i$ and $M_i\supset M\setminus M^\alpha$.

In order to prove the lemma it is enough to show that

$$(3.3)\qquad \max\{|d_r^i|\colon r\in M,\ i = 1, \ldots, \alpha^n\} > e^{-pj/2},$$

because $(U_t^i)_{t\in M}$ is a solution of (S) with $\max_t|U_t^i| = 1$ for $i = 1, \ldots, \alpha^n$.

Given two subsets K and L of M with the same number of elements, $\#K = \#L = m$, we put

$$D(K, L) := \det[a_{r-t}]_{\substack{r\in K\\ t\in L}} := \det[a_{r^i-t^l}], \quad i, l = 1, \ldots, m,$$

where $K = \{r^1, \ldots, r^m\}$, $L = \{t^1, \ldots, t^m\}$ and $r^i \prec r^{i+1}$, $t^i\prec t^{i+1}$, $\prec$ denoting the lexicographical order in Z_+^n.

Suppose (3.3) to be false, i.e. suppose that

$$(3.4)\qquad |d_r^i| \leqslant e^{-pj/2}, \quad r\in M, \quad i = 1, \ldots, \alpha^n.$$

We claim that these inequalities imply the following inequalities

$$(3.5)\qquad |D(M_i, N_i)| > e^{-cj(jp)^n+ipj/2}, \quad i = 1, \ldots, \alpha^n.$$

Indeed, if $i = 0$ we have

$$|D(M_0, N_0)| = |D(M, M)| = |a_0|^{(pj)^n} > e^{-cj(jp)^n}$$

because $D(M, M)$ is diagonal with all the diagonal elements equal to a_0. So (3.5) is true for $i = 0$.

Suppose (3.5) to be true for i, where $0\leqslant i < \alpha^n$ and put $b_{rt} := a_{r-t}$ when $(r, t)\in M_i\times N_i$ and $(r, t)\neq (r^{i+1}, t^{i+1})$. When $(r, t) = (r^{i+1}, t^{i+1})$ we put

$$b_{rt} := a_{r^{i+1}-t^{i+1}} - d_{r^{i+1}}^{i+1}/U_{t^{i+1}}^{i+1}.$$

Then $(U_t^{i+1})_{t\in N_i}$ is a nontrivial solution of the system

$$(3.6)\qquad \sum_{t\in N_i} b_{rt}x_t = 0, \quad r\in M_i.$$

Indeed, if $r\neq r^{i+1}$ then

$$\sum_{t\in N_i} b_{rt}U_t^{i+1} = \sum_{t\in N_i} a_{r-t}U_t^{i+1} = 0.$$

If $r = r^{i+1}$,

$$\sum_{t\in N_i} b_{rt}U_t^{i+1} = \sum_{t\in N_i} a_{r^{i+1}-t}U_t^{i+1} - d_{r^{i+1}}^{i+1} = 0.$$

Since the number of equations of the system (3.6) is equal to the number of its unknowns x_t, it follows that

$$0 = \det[b_{rt}] = D(M_i, N_i) - D(M_{i+1}, N_{i+1})d_{r^{i+1}}^{i+1}/U_{t^{i+1}}^{i+1}.$$

Hence, by virtue of (3.4) and by the induction assumption we get

$$|D(M_{i+1}, N_{i+1})| \geqslant |D(M_i, N_i)|e^{pj/2} > e^{-cj(jp)^n+(i+1)jp/2}.$$

By the induction principle this implies that (3.5) follows from (3.4).

Now observe that the inequalities $|a_{r-t}|\leqslant 1$, $r, t\in M$, imply the inequalities

$$|D(M_i, N_i)| \leqslant (pj)^{n(pj)^n}, \quad i = 1, \ldots, \alpha^n.$$

But $(pj)^n\leqslant j^{2n} < j^{2n+1}/(2n+1)! < e^j$, because $j\geqslant p > (2n+1)!$.

Therefore

$$(3.7)\qquad |D(M_i, N_i)| < e^{j(pj)^n}, \quad i = 1, \ldots, \alpha^n.$$

Now, if $i = \alpha^n$ the inequalities (3.5) and (3.7) imply

$$-cj(pj)^n + \alpha^n pj/2 < j(pj)^n,$$

whence

$$(4c)^n j(pj)^n < 2(c+1)j(pj)^n,$$

because $\alpha = 4cjp^{(n-1)/n}$. This contradicts the assumption that $c > 1$. Therefore (3.3) is true, and consequently the Lemma is true.

4. Two approximation lemmas

Again following the idea of Josefson we shall prove two approximation lemmas which say that for every function f analytic in a given ball there exists a polynomial g which is sufficiently "small" in the set where f is "small".

APPROXIMATION LEMMA 1, [3]. *Assume that c, j, p and α satisfy* (3.1), *and* *let* $\mathscr{F}_0(c,j)$ *denote the set of all functions* f *holomorphic in the ball* $B(0,3)$ *such that* $|a_0| = |f(0)| > e^{-cj}$ *and* $|a_r| \leqslant 1$ *for all* $r \in Z_+^n$, *where* $a_r := D^r f(0)/r!$ *denotes the* r-*th Taylor coefficient of* f *at* 0.

Then for every $f \in \mathscr{F}_0(c,j)$ *there exists a polynomial* g *of* n *complex variables of degree at most* $t_j := n\alpha$ *such that*

$$1 \leqslant ||g||_1 \leqslant e^{t_j}$$

and

$$|g(z)| < e^{-pj/12} \quad \text{when} \quad |z| \leqslant 1/2 \quad \text{and} \quad |f(z)| < e^{-pj},$$

where $||g||_R$ *denotes the supremum of the absolute value of* g *on the ball* $B(0,R)$ $:= \{|z| < R\}$.

Proof. Given $f \in \mathscr{F}_0(c,j)$ let $(U_t)_{t\in M}$ denote a solution of the system (S) given by the Elementary Lemma, where a_r denote the Taylor coefficients of f. Put

$$H(z) := \Big(\sum_{t\in M} U_t z^t\Big)\cdot f(z) = \sum_{|r|\geqslant 0} d_r z^r,$$

$$G(z) := \sum_{r\in M} d_r z^r, \qquad g(z) := G(z)/d,$$

where

$$d_r := \sum_{t\in M} a_{r-t} U_t \quad \text{and} \quad d := \max\{|d_r|: r \in M\}.$$

Then $\deg g \leqslant n\alpha = t_j$, $||g||_1 \geqslant 1$ by the Cauchy inequalities, and

$$||g||_1 \leqslant ||G||_1/d \leqslant \# M^\alpha = \alpha^n \leqslant e^{n\alpha} = e^{t_j}.$$

If $|z| \leqslant 1/2$ and $|f(z)| < e^{-pj}$ then

$$|G(z)| \leqslant |H(z)| + |H(z) - G(z)| < (pj)^n e^{-pj} + (pj)^n \sum_{r\in Z_+^n \setminus M} \Big(\frac{1}{2}\Big)^{|r|}$$
$$= (pj)^n[e^{-pj} + 2^n - 2^n(1-2^{-pj})^n],$$

because $|d_r| \leqslant (pj)^n$ for all $r \in Z_+^n$. Therefore, since by the Bernoulli inequality $(1-2^{-pj})^n \geqslant 1 - n\cdot 2^{-pj}$, we get

$$|g(z)| < (pj)^n e^{pj/2}[e^{-pj} + n\cdot 2^n \cdot 2^{-pj}]$$
$$= (pj)^n e^{-pj/6}[e^{-pj/3} + n\cdot 2^n\cdot (e^{2/3}/2)^{pj}] \leqslant (3pj)^n e^{-pj/6} \leqslant e^{-pj/12},$$

when $j \geqslant p > (2n+2)!$. The proof is concluded.

APPROXIMATION LEMMA 2, [3]. *Let* $\mathscr{F}(c,j)$ *denote the family of all functions* f *holomorphic in the ball* $B(0,4)$ *such that*

$$||f||_2 \leqslant 1 \quad \text{and} \quad ||f||_{1/4} > e^{-cj}.$$

Then for every function $f \in \mathscr{F}(c,j)$ *and for every* $p = k^n$ *with* $j \geqslant p > (2n+2)!$ *there exists a polynomial* g *of degree at most* $t_j := n\alpha$ *such that*

$$2^{-t_j} \leqslant ||g||_1 \leqslant 4^{t_j} \tag{4.1}$$

and

$$|g(z)| < e^{-pj/12} \quad \text{when} \quad |z| \leqslant 1/2 \quad \text{and} \quad |f(z)| < e^{-pj}. \tag{4.2}$$

Proof. Given $f \in \mathscr{F}(c,j)$ there exists $x \in B(0,1/4)$ such that $|f(x)| > e^{-cj}$. Put $F(z) := f(x+z)$. Then $F \in \mathscr{F}_0(c,j)$ and by the Approximation Lemma 1 there exists a polynomial G of degree at most t_j such that $1 \leqslant ||G||_1 \leqslant e^{t_j}$, and $|G(z)| < e^{-pj/12}$ when $|z| \leqslant 1/2$ and $|F(z)| < e^{-pj}$.

Put $g(z) := G(z-x)$. Then g is a polynomial of degree $\leqslant t_j$ and $||g||_1 \leqslant ||G||_{5/4}$. Hence by 2.4

$$||g||_1 \leqslant e^{t_j}(5/4)^{t_j} < 4^{t_j}.$$

Since by 2.4 we have $||G||_1 \leqslant ||G||_{3/4}(4/3)^{t_j}$, so $||g||_1 \geqslant ||G||_{3/4} \geqslant (3/4)^{t_j} > 2^{-t_j}$.

Let now $|z| \leqslant 1/4$ and $|f(z)| < e^{-pj}$. Then $|F(z-x)| < e^{-pj}$ and $|z-x| \leqslant 1/2$. Therefore $|g(z)| = |G(z-x)| < e^{-pj/12}$, when $|z| \leqslant 1/4$ and $|f(z)| < e^{-pj}$. The lemma is proved.

5. Main Lemma

Let U *be a plurisubharmonic function in a ball* $B(a,R)$. *Then the set* $E := \{z \in C^n: |z-a| \leqslant R/16,\ U(z) = -\infty\}$ *is* L-*polar*.

Proof. Without loss of generality we may assume that $a = 0$ and $R = 4$. We may also assume that $U(z) < -3$ in the ball $|z| \leqslant 2$.

Let F be a holomorphic function in the Hartogs domain

$$H := \{(z,w) \in C^{n+1}: |z| < 4,\ |w|e^{U(z)} < 1\}$$

such that F cannot be continued holomorphically beyond H and

$$|F(z,w)| < 1 \quad \text{when} \quad |z| \leqslant 2 \text{ and } |w| \leqslant e^2.$$

The function F may be written in the form

$$F(z,w) = \sum_{j=0}^{\infty} f_j(z) w^j, \qquad (w,z) \in H,$$

where f_j are holomorphic in $B(0,4)$. By Cauchy inequalities

$$|f_j(z)| \leqslant 1, \qquad |z| \leqslant 2,\ j \geqslant 0.$$

Since F is not continuable beyond H, we have by 2.1

$$U(z) = \limsup_{z'\to z} V(z') = V^*(z)$$

where

$$V(z) := \limsup_{j\to\infty} (1/j)\log|f_j(z)|, \qquad z \in B(0,4).$$

The set $E \cup \{z \in C^n: |z| \leqslant 1/4,\ V(z) < U(z)\}$ is of $(2n)$-dimensional Lebesgue measure zero. Therefore there exists a point x in $B(0,1/4)$ such that $V(x) = U(x) > -\infty$.

Hence there exists an infinite subset J of positive integers and an integer $c > 1$ such that

$$||f_j||_{1/4} > e^{-cj}, \quad j \in J.$$

Therefore $f_j \in \mathscr{F}(c, j)$ for every $j \in J$. So we may apply Approximation Lemma 2 to each function $f_j, j \in J$. Thus for every j and p such that $j \in J$ and $j \geqslant p = k^n > (2n+2)!$ we can find a polynomial g_{jp} of degree at most $t_j := n\alpha_j$ such that

$$2^{-t_j} < ||g_{jp}||_1 < 4^{t_j}$$

and

(5.0) $$|g_{jp}(z)| \leqslant e^{-pj/12} \text{ when } |z| \leqslant 1/4 \quad \text{and} \quad |f_j(z)| < e^{-pj}.$$

Put

$$g_p(z) := \limsup_{\substack{j\to\infty \\ (j\in J)}} |g_{jp}(z)|^{1/t_j}, \quad z \in C^n.$$

It follows from 2.4 that for all admissible j and p we have

$$|g_{jp}(z)|^{1/t_j} \leqslant \max\{1, 4|z|\}, \quad z \in C^n,$$

so the functions $|g_{jp}|^{1/t_j}$ are uniformly bounded from above on every compact subset of C^n, and moreover

(5.1) $$g_p(z) \leqslant \max\{1, 4|z|\} \text{ in } C^n$$

for all $p > (2n+2)!$.

We claim that

(5.2) $$\forall\, T < 0\ \exists\, p\ \forall\, x \in E \quad g_p^*(x) < e^{T+1},$$

where $g_p^*(z) := \limsup_{z' \to z} g_p(z')$. Suppose (5.2) to be false. Then there exists $T < 0$ such that for every p one can find $z \in E$ such that $g_p^*(z) \geqslant e^{T+1}$. Take p so large that $T > -\sqrt{p}/48cn$. Next take $z_0 \in E$ such that $g_p^*(z_0) \geqslant e^{T+1}$. By 2.3 there exists a sequence of points z_ν such that for every ν $g_p(z_\nu) = g_p^*(z_\nu)$ and $\lim_{\nu\to\infty} g_p(z_\nu) = g_p^*(z_0)$.

Therefore

$$g_p(z_\nu) > e^{T+1/2}, \quad \nu \geqslant \nu_0.$$

Hence for every $\nu > \nu_0$ there exists an infinite subset J_ν of J such that

$$|g_{jp}(z_\nu)| > e^{T\cdot t_j} > e^{-\frac{p^{1/n} 4cnjp^{(n-1)/n}}{48cn}} = e^{-pj/12}, \quad j \in J_\nu.$$

Therefore by (5.0)

$$|f_j(z_k)| \geqslant e^{-pj}, \quad j \in J_\nu, \quad \nu > \nu_0.$$

Hence

$$V(z_\nu) \geqslant -p, \quad \nu > \nu_0,$$

so

$$U(z_0) = -\infty \geqslant -p.$$

We have got a contradiction which shows that (5.2) is true.

By 2.3 the functions g_p^* are plurisubharmonic in C^n. We claim that there exists a point ξ in the unit ball $B(0, 1)$ such that

(5.3) $$\limsup_{p\to\infty} g_p^*(\xi) \geqslant 3/10.$$

Suppose that (5.3) is false. Then $\limsup_{p\to\infty} g_p^*(x) < 3/10$ in $B(0, 1)$. Therefore by Hartogs Lemma, given any ε with $0 < \varepsilon < 1$, we have

$$g_p(z) \leqslant g_p^*(z) \leqslant 3/9, \quad |z| \leqslant 1-\varepsilon, \quad p \geqslant p_0 = p_0(\varepsilon).$$

Hence

$$\limsup_{\substack{j\to\infty \\ (j\in J)}} |g_{jp}(z)|^{1/t_j} \leqslant 3/9, \quad |z| \leqslant 1-\varepsilon, \quad p \geqslant p_0.$$

Again by the Hartogs Lemma

$$|g_{jp}(z)| \leqslant (3/8)^{t_j}, \quad |z| \leqslant 1-2\varepsilon, \quad j > j_0(\varepsilon, p),\ p > p_0(\varepsilon).$$

Therefore by 2.4

$$||g_{jp}||_1^{1/t_j} \leqslant (3/8)(1-2\varepsilon) < 1/2, \quad j > j_0(\varepsilon, p),\ p > p_0(\varepsilon),$$

when $0 < \varepsilon \leqslant 1/8$.

But we know that $||g_{jp}||_1 \geqslant 2^{-t_j}$ for all j and p sufficiently large with $j \geqslant p$. We have obtained a contradiction which shows that (5.3) is true.

Fix any point ξ satisfying (5.3). Then by (5.2) we can find a sequence of positive integers $\{p_s\}$ such that

(5.4) $$g_{p_s}^*(x) \leqslant e^{-s^s} \quad \text{for all } x \in E \text{ and } s \geqslant 1,$$

and

(5.5) $$\lim_{s\to\infty} g_{p_s}^*(\xi) \geqslant 3/10.$$

We claim that the function W defined by

(5.6) $$W(x) := \sum_{s=1}^{\infty} 2^{-s} \log g_{p_s}^*(x), \quad x \in C^n$$

is a function of the class L such that $W = -\infty$ on E.

Indeed, by (5.4) we have $W = -\infty$ on E. So it is enough to show that W is a plurisubharmonic function of the class L.

First observe that W is uppersemicontinuous because for every $R > 1$ the partial sums of the series

$$W(x) = \sum_{s=1}^{\infty} 2^{-s} \log g_{p_s}^*(x)/4R + \log(4R)$$

constitute a decreasing sequence of plurisubharmonic functions in the ball $B(0, R)$. Since by (5.3) $W(x) \not\equiv -\infty$, the function W is plurisubharmonic in C^n. It is of the class L because of (5.1).

6. Proof of the theorem

If E is locally C^n-polar then for every $a \in E$ we can find a ball $B(a, R_a)$ and a function U_a plurisubharmonic in $B(a, R_a)$ such that $E \cap B(a, R_a/16) \subset E_a := \{z \in C^n: |z-a| \leqslant R_a/16,\ U_a(z) = -\infty\}$. By the Main Lemma the set E_a is L-polar. Since E may be covered by a sequence of balls $B(a_s, R_{a_s})$, E is contained in a countable union of L-polar sets. Hence it is sufficient to apply the following

PROPOSITION, [7]. *A countable union $E = \bigcup_{k=1}^{\infty} E_k$ of L-polar sets E_k is an L-polar set.*

Proof. Since a finite union of L-polar sets is L-polar, we may assume that $E_k \subset E_{k+1}$ (because $E = \bigcup_{k=1}^{\infty} F_k$, where $F_k := E_1 \cup \ldots \cup E_k$). Let W_k be a function of the class L such that $E_k \subset \{x: W_k(x) = -\infty\}$. We may assume that

$$\sup\{W_k(x): |x| \leqslant 1\} = 0, \quad k \geqslant 1. \tag{6.1}$$

By 2.4 we have

$$W_k(x) \leqslant \log^+|x| \quad \text{in } C^n \text{ for all } k \geqslant 1, \tag{6.2}$$

so that the functions W_k are uniformly bounded from above on every compact subset of C^n.

We claim that there exists a point $\xi \in C^n$ and a number $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \exp W_k(\xi) > \varepsilon. \tag{6.3}$$

Otherwise we would have

$$\limsup_{k\to\infty} \exp W_k(x) \leqslant 0 \quad \text{in } C^n.$$

Hence by Hartogs Lemma

$$\exp W_k(x) \leqslant 1/e, \quad |x| \leqslant 1,\ k > k_0.$$

Thus $W_k(x) \leqslant -1$, $|x| \leqslant 1$, $k \geqslant k_0$. This is impossible because of (6.1).

Let now $\{k_i\}$ be an increasing sequence of positive integers such that

$$\lim W_{k_i}(\xi) \geqslant \log\varepsilon > -\infty. \tag{6.4}$$

One can easily check that

$$W(x) := \sum_{i=1}^{\infty} 2^{-i} W_{k_i}(x), \quad x \in C^n,$$

is a function of the class L such that $W(x) = -\infty$ on E.

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