Литература

[1] Н. И. Ахиезер и И. М. Глазман, *Теория линейных операторов в гильбертовом пространстве*, Наука, Москва 1966.
[2] Ф. Аткинсон, *Дискретные и непрерывные граничные задачи*, Мир, Москва 1968.
[3] М. Г. Крейн и А. А. Нудельман, *Проблема моментов Маркова и экстремальные задачи*, Наука, Москва 1973.
[4] Н. И. Ахиезер и М. Г. Крейн, *О некоторых вопросах теории моментов*, Харьков 1938.
[5] Г. М. Голузин, *Геометрическая теория функций комплексного переменного*, Наука, Москва 1966.
[6] Г. Е. Шилов, *Математический анализ, Специальный курс*, Физматгиз, Москва 1960.
[7] И. А. Александров, *Параметрические продолжения в теории однолистных функций*, Наука, Москва 1976.
[8] П. П. Куфарев, В. В. Соболев, Л. В. Спорышева, *Об одном методе исследования экстремальных задач для функций, однолистных в полуплоскости*, Вопросы геометрической теории функций, вып. 5 (1968) (Тр. Томского ун-та 200), 142–164.
[9] В. В. Соболев, *Параметрические представления для некоторых классов функций, однолистных в полуплоскости*, Учен. зап. Кемеровского пед. ин-та 23 (1970), 30–41.
[10] С. Т. Александров, *Параметрическое представление функций однолистных в полуплоскости*, Вопросы геометрической теории функций, вып. 9 (находится в печати).
[11] В. Я. Гутлянский, *Параметрическое представление однолистных функций*, ДАН СССР 194 (1970), 750–753.
[12] Э. А. Коддингтон, Н. Левинсон, *Теория обыкновенных дифференциальных уравнений*, ИЛ, Москва 1958.
[13] Дж. Сансоне, *Обыкновенные дифференциальные уравнения*, ИЛ, Москва 1954.
[14] Н. А. Лебедев, *Некоторые оценки и задачи на экстремум в теории конформного отображения*, Диссертация, Ленинградский ун-т, 1951.
[15] В. Я. Гутлянский, *Параметрические представления и экстремальные задачи в теории однолистных функций*, Диссертация, Киев, Матем. ин-т АН УССР, 1972.
[16] Т. Н. Селляхова, *Исследование экстремальных свойств одного класса однолистных конформных отображений полуплоскости в себя*, Вопросы геометрической теории функций, вып. 8 (1977) (Тр. Томского ун-та), 55–70.

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ON THE DOMAINS OF EXISTENCE FOR PLURISUBHARMONIC FUNCTIONS

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1. Introduction

Let U be an open subset of C^n and denote by PSH(U) the plurisubharmonic functions on U. In Cegrell ([3], p. 322) it is proved that there is an open subset $\tilde{U}$ containing U such that

(1) $|_U\colon \mathrm{PSH}(\tilde{U}) \to \mathrm{PSH}(U)$ is a bijection.

(2) If $|_U\colon \mathrm{PSH}(V) \to \mathrm{PSH}(U)$ is a bijection then $V \subset \tilde{U}$.

Here $|_U$ denotes the restriction map.

The purpose of this paper is to study the situation where

$$|_U\colon \mathrm{PSH}(V) \to \mathrm{PSH}(U)$$

is a surjection, not necessarily a bijection. We wish to point out that we know of no example where the restriction map is surjective without being bijective.

2. Domains of existence

DEFINITION. Let U be an open connected subset of C^n. We say that U is a *domain of existence* for the plurisubharmonic functions on U if there is a plurisubharmonic function on U which cannot be extended as a plurisubharmonic function to any open connected set strictly containing U.

In the same way we may speak about domains of existence for analytic functions, pluriharmonic functions and so on.

EXAMPLE. Any pseudoconvex domain is an example of a domain of existence for the plurisubharmonic functions. The converse is not true. Cf. Bremermann [2] and Cegrell [3], p. 329.

THEOREM 2.1. *Let U be an open connected subset of C^n. Then there exists an*

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open connected set $\check{U}$ containing U such that the restriction map $|_U$: $\mathrm{PSH}(\check{U}) \to \mathrm{PSH}(U)$ is surjective and $\check{U}$ is a domain of existence for the plurisubharmonic functions.

Proof. Consider the class A of open connected subsets U' containing U such that $|_U$: $\mathrm{PSH}(U') \to \mathrm{PSH}(U)$ is a surjection. Partial order A by saying that $U' \prec U''$ if $U' \subset U''$ and $|_{U'}$: $\mathrm{PSH}(U'') \to \mathrm{PSH}(U')$ is a surjection. It is then clear that any totally ordered subset of A has an upper bound in A. So, by Zorn's lemma, A has a maximal element $\check{U}$ which means that

(1) $|_{\check{U}}$: $\mathrm{PSH}(\check{U}) \to \mathrm{PSH}(U)$ is a surjection.

(2) If $W \supset \check{U}$ and if $|_{\check{U}}$: $\mathrm{PSH}(W) \to \mathrm{PSH}(\check{U})$ is a surjection then $\check{U} = W$.

It follows from Proposition 2.2 that $\check{U}$ is a domain of existence.

PROPOSITION 2.2. *If U is not a domain of existence for the plurisubharmonic function, then there is an open connected set U' containing U such that $U \neq U'$ and $|_U$: $\mathrm{PSH}(U') \to \mathrm{PSH}(U)$ is a surjection.*

For the proof we need some preparation.

THEOREM 2.3. *If $U_1 \supsetneqq U_2$ and if $\mathrm{PSH}(U_1)|_{U_2}$ is non-meager in $\mathrm{PSH}(U_2)$ then there is an open set U_3, $U_2 \subsetneqq U_3 \subset U_1$ such that*

$$\mathrm{PSH}(U_3)|_{U_2} = \mathrm{PSH}(U_2).$$

(We consider $\mathrm{PSH}(U)$ as a complete metric space with topology induced by $L^1_{\mathrm{loc}}(U)$.)

Proof of Theorem 2.3. Choose $z_0 \in U_1 \cap \partial U_2$ and $r > 0$ such that $B(z_0, r)$ is relatively compact in U_1. Put

$$A_n = \{\varphi \in \mathrm{PSH}(U_2 \cup B(z_0, r));\ \varphi|_{B(z_0,r)} \leqslant n\}.$$

It is clear that $A_n|_{U_2}$ is closed in $\mathrm{PSH}(U_2)$. Furthermore, $\bigcup_{n=1}^{\infty} A_n|_{U_2} \supset \mathrm{PSH}(U_1)|_{U_2}$, hence there exists a number n such that $A_n|_{U_2}$ has an interior point in $\mathrm{PSH}(U_2)$. Let φ_0 be such a point. Then there is an $\varepsilon > 0$ and a compact subset k of U_2 such that if $\varphi \in \mathrm{PSH}(U_2)$ and $\int_k |\varphi - \varphi_0| dz < \varepsilon$ then $\varphi \in A_n|_{U_2}$. Choose N so that $\int_k \varphi_N - \varphi_0 < \varepsilon/3$ where $\varphi_N = \sup(\varphi_0, -N)$. Put

$$f_m = \sup\left(\varphi_0 + \frac{1}{m}\log|z - z_0|, -N\right)$$

and choose m so large that $\int_k |f_m - \varphi_n| < \varepsilon/3$. Given $\psi \in \mathrm{PSH}(U_2)$. Put

$$\theta = \frac{\varepsilon}{3} \cdot \frac{1}{\int_k |\psi| + 1}.$$

Then $\theta\psi + f_m \in \mathrm{PSH}(U_2)$ and $\int_k |\theta\psi + f_m - \varphi_0| < \varepsilon$. Hence, there is a $\widetilde{\theta\psi + f_m} \in A_n$ with

$$\widetilde{\theta\psi + f_m}|_{U_2} = \theta\psi + f_m.$$

Put $E = \left\{z \in U_2 \cup B(z_0, r);\ \varphi_0 + \frac{1}{m}\log|z - z_0| < -N\right\}$ and choose $r_1 < r$ such that $B(z_0, r_1)$ is relatively compact in E. It then follows that on $B(z_0, r_1) \cap U_2$ we have $f_m = -N$. Put

$$\widetilde{\psi(z)} = \begin{cases} \psi(z), & z \in U_2, \\ \dfrac{1}{\theta}(\widetilde{\theta\psi + f_m} + N), & z \in B(z_0, r_1). \end{cases}$$

It is clear that $\tilde{\psi}$ is a plurisubharmonic extension of ψ to $U_2 \cup B(z_0, r_1)$ so we can take U_3 to be $U_2 \cup B(z_0, r_1)$.

LEMMA 2.4. *The connected open set U is a domain of existence if and only if $\mathrm{PSH}(U')|_U$ is meager in $\mathrm{PSH}(U)$ for every connected $U' \supsetneqq U$.*

Proof. If $\mathrm{PSH}(U')|_U$ is meager in $\mathrm{PSH}(U)$ for every $U' \supsetneqq U$ we may use the same idea as in Lelong [4], p. 31, to see that U is a domain of existence.

Conversely, if there is a $U' \supsetneqq U$ such that $\mathrm{PSH}(U')|_U$ is non-meager in $\mathrm{PSH}(U)$ it follows from Theorem 2.3 that U cannot be a domain of existence.

Proof of Proposition 2.2. Assume that U does not satisfy the conclusion of the proposition. It follows from (2) and Theorem 2.3 that $\mathrm{PSH}(W)$ is meager in $\mathrm{PSH}(U)$ for every connected $W \supsetneqq U$. Hence by Lemma 2.4, U is a domain of existence.

Remark. Proposition 2.2 is also stated in Bedford and Burns [1]. Cf. Zentralblatt für Mathematik (1979), 403, 32011.

3. Some special cases

The following theorem contains the theorem in Bedford and Burns [1] as a special case.

THEOREM 3.5. *Let U be an open connected subset of $\mathbf{C}^n$ such that $\partial U \subset \overline{(\complement U)}^0$ and there is a dense set of points $(P_i)_1^\infty$ in ∂U such that for each point P_i there is a ball $B(P_i, r_i)$ and a complex hyperplane H of codimension 1 containing P_i such that either*

(i) $$B(P_i, r_i) \cap \bar{U} \cap H = P_i$$

or

(ii) $$B(P_i, r_i) \cap U \supset H \setminus (P_i).$$

Then U is a domain of existence for the plurisubharmonic functions.

Proof of Theorem 3.5. Let P be a point where (i) is satisfied. Put

$$c = \sup_{\bar{U} \cap \partial B(P, r/2)} -\log|H|$$

and

$$\psi(z) = \begin{cases} c, & z \in U \setminus B(P, r/2), \\ \sup\limits_{B(P, r/2)} (c, -\log|H|), & z \in U \cap B(P, r/2). \end{cases}$$

Then $\psi \in \mathrm{PSH}(U)$ and since $\lim\limits_{\substack{z' \to P \\ z' \in U}} \psi(z') = +\infty$ ψ has no plurisubharmonic extension over P.

Let now P be a point where (ii) is valid. We can assume that $P = 0$ and that H contains $\{(z_1, 0, \ldots, 0); z_1 \in \boldsymbol{C}\}$. There is a $\xi, 0 < \xi < r/2$, so that for each $x \in \boldsymbol{C}^n$ with $|x| < \xi$ we have $[\partial B(0, r/2) \cap H] + x \subset U$. Since $0 \in \partial U$ and since $\overline{(\complement U)^0} \supset \partial U$ by assumption we can choose $x^0 \in (\complement U)^0$ with $|x^0| < \xi$. There is an $\eta, 0 < \eta < r/2$, such that $B(x^0, \eta) \subset (\complement U)^0$.

Put

$$\varphi_1 = \inf\{-\log|z_1 - x_1^0|, -\log(\eta/2)\},$$

$$\varphi_2 = \log\left[\frac{4}{\eta^2}|(z_2 - x_2^0, z_n - x_n^0)|\right].$$

Now $\varphi = \sup(\varphi_1, \varphi_2)$ is plurisubharmonic outside $\overline{B(x^0, \eta)}$ since $|z_1 - x_1^0| > \eta/2$ gives $-\log(\eta/2) > -\log|z_1 - x_1^0|$ so

$$\varphi = \sup\left\{-\log|z_1 - x_1^0|, \log\left[\frac{4}{\eta^2}|(z_2 - x_2^0, \ldots, z_n - x_n^0)|\right]\right\}.$$

If $|z_1 - x_1^0| \leqslant \eta/2$ then $|(z_2 - x_2^0, \ldots, z_n - x_n^0)| > \eta/2$ so

$$\frac{4}{\eta^2}|(z_2 - x_2^0, \ldots, z_n - x_n^0)| > \frac{2}{\eta}$$

which means that

$$\log\frac{4}{\eta^2}|(z_2 - x_2^0, \ldots, z_n - x_n^0)| > -\log\frac{\eta}{2}.$$

Thus

$$\varphi = \log\frac{4}{\eta^2}|(z_2 - x_2^0, \ldots, z_n - x_n^0)|.$$

Now, we have $\varphi = -\log|z_1 - x_1^0|$ on

$$U \cap \{z_2 - x_2^0 = \ldots = z_n - x_n^0 = 0\}$$

so if φ has a plurisubharmonic extension $\tilde{\varphi}$ to a connected set containing U and $B(P, r)$ we have

$$\sup_{z_1 \in \partial B(x_1^0, r/2)} \tilde{\varphi}(z_1, x_2^0, \ldots, x_n^0) < \sup_{z_1 \in B(x_1^0, r/2)} \tilde{\varphi}(z_1, x_2^0, \ldots, x_n^0)$$

which contradicts the maximum principle.

It follows now from Proposition 2.2 that if there is a dense set $(P_i)_{i=1}^{\infty}$ where (i) or (ii) is satisfied, then U has to be a domain of existence for the plurisubharmonic functions.

PROPOSITION 3.6. *Assume that $U \subset V$. If $|_U$: $\mathrm{PSH}(V) \to \mathrm{PSH}(U)$ is a surjection and if $V \diagdown U$ is compact in V then $V \diagdown U$ has no interior points.*

Proof. The proof of this theorem is similar to the second part of the proof of Theorem 3.5 and will not be repeated.

THEOREM 3.7. *Let U be an open connected subset of $\boldsymbol{C}^n$. If one $\check{U}$ is pseudoconvex then every pluriharmonic function on U has a (unique) pluriharmonic extension to $\check{U}$.*

Proof. Put $W = \bigcap U'$; $U' \supset U$, U' pseudoconvex. It is clear that $|_U$: $\mathrm{PSH}(W) \to \mathrm{PSH}(U)$ is surjective. Given $\varphi \in \mathrm{PSH}(U)$. Then there is a $\varphi \in \mathrm{PSH}(W)$ which extends φ. Now, by Cegrell ([3], Theorem 6.2), $\complement \operatorname{supp} \Delta\varphi$ is pseudoconvex. Hence $\tilde{\varphi}$ is pluriharmonic on W. Thus, any pluriharmonic function on U extends to a pluriharmonic function on W.

Choose now a fundamental sequence $(K_n)_{n=1}^{\infty}$ of compacts in W. If there is a point z_0 in $\check{U} \cap \overline{W}$ take $z_n \to z_0$, $n \to \infty$; $z_n \in W \diagdown \hat{K}_n$, $z_n \in \hat{K}_{n+1}$. Then there exists f_n, analytic on W such that

$$\sup_{K_n} |f_n| < \frac{1}{n^2}, \quad \operatorname{Re} f_n(z_n) > n + \sum_{\nu=1}^{n-1} |f_\nu(z_n)|.$$

Put $h = \sum \operatorname{Re} f_n$. Then

$$h(z_n) = \lim_{\mu \to +\infty} \operatorname{Re} \sum_{\nu=1}^{\mu} f_\nu(z_n) \geqslant -1 + \operatorname{Re} f_n(z_n) - \left|\sum_{\nu=1}^{n-1} f_\nu(z_n)\right| \geqslant n - 1.$$

Now h is pluriharmonic on W so any plurisubharmonic extension of h to $\check{U}$ is equal to h on W. But $h(z_n) = n$ so $\varlimsup\limits_{W \ni z' \to z_0} h(z') = +\infty$ which proves that h has no plurisubharmonic extension to a neighborhood of z_0. Since z_0 was any element in $\check{U} \cap \overline{W}$ and since U and W are connected we have proved that $\check{U} = W$.

References

[1] E. Bedford and D. Burns, *Domains of existence for plurisubharmonic functions*, Math. Ann. 238 (1978).

[2] H. J. Bremermann, *On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions*, ibid. 131 (1956).

[3] U. Cegrell, *Removable singularities for plurisubharmonic functions and related problems*, Proc. London Math. Soc. 36 (1978).

[4] P. Lelong, *Fonctionelles analytiques et fonctions entières (n variables)*, Presses Univ. Montréal 1968.

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