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ON BOUNDARY BEHAVIOUR OF BERGMAN KERNEL FUNCTION FOR PLANE DOMAINS AND THEIR CARTESIAN PRODUCTS

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In [7] two conditions were stated concerning the Bergman kernel function $K_D(z, t)$ of a bounded domain $D \in C^n$, $n \geqslant 1$, and describing a good boundary behaviour of $K_D(z, t)$:

(A_k) The Bergman kernel function $K_D(z, t)$ can be extended to a C^k-function on $\overline{D} \times D$. This means that every derivative of $K_D(z, t)$ up to order k can be extended to $\overline{D} \times D$ to a continuous function $(1 \leqslant k \leqslant \infty)$.

(B) For every $z_0 \in \overline{D}$ there exist $n+1$ points $t_0, \ldots, t_n \in D$ such that

$$\det \begin{bmatrix} K_D(z_0, t_j) \\ \dfrac{\partial K_D}{\partial t_i}(z_0, t_j) \end{bmatrix}_{\substack{j=0,\ldots,n,\\ i=1,\ldots,n}} \neq 0.$$

We shall also need the following

Definition. We say that the boundary ∂D of a bounded domain D *satisfies minimal regularity conditions* iff it is locally the graph of a Lipschitzian function from R^{2n-1} into R. This means that for every $z \in \partial D$ there exist an open neighborhood U of z, a coordinate system $x_1, \ldots, x_{2n}$ in $C^n = R^{2n}$ and a Lipschitzian function φ: $R^{2n-1} \to R$ such that

$$U \cap D = U \cap \{x \in R^{2n}\colon x_{2n} > \varphi(x_1, \ldots, x_{2n-1})\}.$$

Conditions (A_k) and (B) are important in the theory of biholomorphic mappings because of the following fact, proved in [7].

Theorem I. *Let D and G be bounded domains in C^n, whose boundaries ∂D and ∂G satisfy minimal regularity conditions. If the Bergman kernel functions $K_D(z, t)$ and $K_G(w, s)$ satisfy conditions* (A_k) *and* (B), *then every biholomorphic mapping between D and G extends to a diffeomorphism of class C^k between some open neighborhoods of $\overline{D}$ and $\overline{G}$.*

It should be mentioned that a preliminary version of condition (B) and Theorem I (less general than the above) was obtained independently in [6] and [8].

We shall also need the following fact proved in [7], Remark on Bell's result, with use of Bell's "density lemma" [1].

THEOREM II. *Let D be a bounded domain in C^n, $n \geqslant 1$, with boundary of class C^∞. Denote by $W^s(D)$ the usual Sobolev space and by $H^s(D)$ its subspace consisting of holomorphic functions. Assume that there exist $s > 2n+1$ and $M \geqslant 0$ such that the orthogonal projector K from $L^2(D)$ onto the space $L^2H(D)$ of square integrable holomorphic functions (the so called Bergman projector) is a bounded operator from $W^{s+M}(D)$ into $H^s(D)$. Then*

1. *Condition* (A_k), $k = s-2n-1$, *holds for* $K_D(z, t)$.
2. *Condition* (B) *is also satisfied.*

The aim of this note is to prove the following

THEOREM. *Let $D \subset C$ be a bounded plane domain with boundary ∂D satisfying minimal regularity conditions.*

Conditions (A_∞) and (B) are both valid for $K_D(z, t)$ if and only if the boundary ∂D is of class C^∞.

Proof. (a) Suppose that the boundary ∂D is of class C^∞. Let K denote, as above, the orthogonal projection from $L^2(D)$ onto $L^2H(D)$. For every $f \in L^2(D)$ the function $h = f-K(f)$ is orthogonal to the space $L^2H(D)$. It was proved by Burghea [2] that the space $(L^2H(D))^\perp \subset L^2(D)$ is identical to the space $\left\{\frac{\partial u}{\partial z} : u \in \mathring{W}^1(D)\right\}$. $\mathring{W}^1(D)$ denotes here, as usual, the closure of $C_0^\infty(D)$ in the Sobolev space $W^1(D)$.

If $f \in W^s(D)$, $s \geqslant 1$, then

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial^2 u}{\partial \bar{z} \partial z} = \frac{1}{4} \Delta u = \frac{\partial f}{\partial \bar{z}}.$$

Thus the function u is a solution of the nonhomogeneous Dirichlet problem $\Delta u = 4\frac{\partial f}{\partial \bar{z}}$ with the boundary condition $u = 0$ on ∂D.

Since the Dirichlet form for $-\Delta$ is strictly coercive over $\mathring{W}_1(D)$ and the domain D has a C^∞-boundary, it follows from Theorems 7.14, 7.20 and 7.32 of [3] that:

(i) If $g \in W^s(D)$ then the equation $\Delta u = g$ has a unique solution $u \in W^{s+2} \cap \mathring{W}_1$.

(ii) There exists a constant $c_s > 0$ such that

$$||u||_{s+2} \leqslant c_s ||g||_s.$$

This means that there exists a continuous operator S from W^s into $W^{s+2} \cap \mathring{W}^1$ which solves the nonhomogeneous Dirichlet problem. We have $K(f) = f - \frac{\partial}{\partial z}\left(S\left(\frac{\partial f}{\partial \bar{z}}\right)\right)$ for every $f \in W^s(D)$. Consequently the Bergman projector K is a continuous operator from $W^s(D)$ into $W^s(D)$ for every $s \geqslant 0$. Thus, by Theorem II, conditions (A_∞) and (B) are valid for $K_D(z, t)$.

(b) Let us now assume, that conditions (A_∞) and (B) hold for $K_D(z, t)$. Since ∂D satisfies the minimal regularity conditions, D must be finitely connected. By the Koebe theorem (see [5], Chapter V, § 6, Theorem 2) the domain D is biholomorphically equivalent to certain circled domain G (a circled domain is a domain whose boundary consist of a finite number of disjoint circumferences). Since the boundary of G is real analytic, we see, by part (a) of this proof and Theorem I, that the biholomorphic mapping from D onto G can be extended to a C^∞-diffeomorphism between some neighbourhoods of $\bar{D}$ and $\bar{G}$. This implies that ∂D is of class C^∞.

As a corollary we obtain the following well known fact:

COROLLARY. *If D and G are two bounded plane domains with C^∞-boundaries then every biholomorphic mapping from D onto G can be extended to a C^∞-diffeomorphism between $\bar{D}$ and $\bar{G}$.*

Remark 1. The first part of the proof of the Theorem together with the Corollary can serve as an instructive example of connection between the theory of boundary behaviour of biholomorphic mappings and partial differential equations.

Note that this proof follows the scheme of the proof of Fefferman's theorem given in [7]. The one-dimensional situation, however, is much simpler because we use the classical nonhomogeneous Dirichlet problem instead of $\bar{\partial}$-Neumann problem. The $\bar{\partial}$-Neumann problem is much more difficult to handle.

Remark 2. It follows from part (b) of the proof of the Theorem that if D is a bounded plane domain whose boundary satisfies minimal regularity conditions and is not of class C^k, conditions (A_k) and (B) cannot hold simultaneously for $K_D(z, t)$. Therefore the operator S solving the nonhomogeneous Dirichlet problem cannot be a continuous operator from $W^r(D)$ into $W^s(D)$ for any $r, s > k+3$.

One might expect that if the boundary ∂D is of class C^k then conditions (A_k) and (B) hold for $K_D(z, t)$. However, for $k = 1$ this is not true in view of Webster's example [8].

DEFINITION. We say that the Bergman kernel function $K_D(z, t)$ *satisfies condition* (A_ω) if it satisfies condition (A_1) and for every $t \in D$, $K_D(z, t)$ can be extended to a function analytic in some open neighbourhood U_t of $\bar{D}$.

It was proved in [7] that if D and G are bounded domains in C^n, the boundaries ∂D and ∂G satisfy minimal regularity conditions and conditions (A_ω) and (B) hold for $K_D(z, t)$ and $K_G(w, s)$, then every biholomorphic mapping from D onto G can be extended to a biholomorphic mapping between some open neighbourhoods of $\bar{D}$ and $\bar{G}$.

PROPOSITION 1. *Let $D \subset C$ be a bounded domain. Suppose that the boundary*

∂D satisfies minimal regularity conditions. Conditions (A_ω) *and* (B) *are valid for* $K_D(z, t)$ *if and only if the boundary ∂D is real analytic.*

Proof. Suppose that ∂D is real analytic. It follows from the Theorem that conditions (A_ω) and (B) are valid for $K_D(z, t)$. Fix $t \in D$. There exists a function $\varphi(z) \in C_0^\infty(D)$ such that for every $f \in L^2H(D)$

$$f(t) = \int_D f(z)\overline{\varphi(z)}\,d\lambda(z)$$

(see for example [1] or the proof of Theorem 3 in [7]). The function $K_D(z, t) - \varphi(z)$ is orthogonal to the space $L^2H(D)$, and so, as in the proof of the Theorem, there exists $u \in C^\infty(\overline{D})$, $u = 0$ on ∂D, such that $\partial u/\partial z = K_D(z, t) - \varphi(z)$.

Since φ has a compact support in D, u is harmonic near ∂D. The function u is equal to zero on ∂D and ∂D is real analytic, and so, by the symmetry rule, u can be extended to a harmonic function on some open neighbourhood of ∂D. Since a harmonic function is real analytic and $\dfrac{\partial u}{\partial z} = K_D(z, t)$ on $D \setminus \operatorname{supp}\varphi$, $K_D(z, t)$ can be extended to a holomorphic function on some open neighbourhood of $\overline{D}$. Hence condition (A_ω) is satisfied.

Let D be a domain such that $K_D(z, t)$ satisfies (A_ω) and (B). We can find, as in part (b) of the proof of the Theorem, a circled domain G biholomorphically equivalent to D. The boundary of G is real analytic, so the biholomorphic mapping between D and G extends to a biholomorphic mapping between some open neighbourhoods of $\overline{D}$ and $\overline{G}$. Thus the boundary of D must be real analytic.

We shall now state more precisely the invariance of conditions (A_k) and (B) under Cartesian products, which was mentioned in Remark 1 of [7].

PROPOSITION 2. *Let D_1 be a bounded domain in C^n and let D_2 be a bounded domain in C^m. Suppose that the boundaries ∂D_1 and ∂D_2 satisfy minimal regularity conditions. The function $K_{D_1\times D_2}(z, t)$ has properties* (A_k) ((A_ω)) *and* (B) *if and only if $K_{D_1}(z_1, t_1)$ and $K_{D_2}(z_2, t_2)$ have these properties.*

Proof. It follows from Bremermann's theorem that

$$K_{D_1\times D_2}(z, t) = K_{D_1}(z_1, t_1)\cdot K_{D_2}(z_2, t_2), \quad z = (z_1, z_2), \quad t = (t_1, t_2).$$

Hence it is obvious that $K_{D_1\times D_2}$ has property (A_k) ((A_ω)) if and only if K_{D_1} and K_{D_2} have this property.

Thus we only need consider property (B). This property is equivalent to the following: For every $z_0 \in \overline{D} \subset C^r$ there exist $r+1$ points $t_0, \ldots, t_r \in D$ such that $K_D(z_0, t_0) \neq 0$ and

$$\det\left[\frac{\partial u_j}{\partial z_i}(z_0)\right]_{i,j=1,\ldots,r} \neq 0, \quad \text{where} \quad u_j(z) = \frac{K_D(z, t_j)}{K_D(z, t_0)}.$$

Suppose that K_{D_1} and K_{D_2} have property (B). Let $z_0 = (z_0^1, z_0^2) \in D_1 \times D_2$, let $t_0^1, \ldots, t_n^1$ be points of D_1 such that $K_{D_1}(z_0^1, t_0^1) \neq 0$ and $\left[\det \dfrac{\partial u_j^1}{\partial z_i^1}(z_0^1)\right] \neq 0$ and let $t_0^2, \ldots, t_m^2$ be points of D_2 such that $K_{D_2}(z_0^2, t_0^2) \neq 0$ and $\det\left[\dfrac{\partial u_j^2}{\partial z_i^2}(z_0^2)\right] \neq 0$.

Put $t_0 = (t_0^1, t_0^2)$, $t_l = (t_l^1, t_0^2)$ for $l \leqslant n$ and $t_l = (t_0^1, t_{l-n})$ for $n < l \leqslant n+m$. Then by Bremermann's theorem, we have $u_l(z) = u_l^1(z^1)$ for $l \leqslant n$ and $u_l(z) = u_{l-n}^2(z^2)$ for $n < l \leqslant n+m$. Thus $\det\left[\dfrac{\partial u_l}{\partial z_i}(z_0)\right] = \det\left[\dfrac{\partial u_j^1}{\partial z_i^1}(z_0^1)\right]\cdot\det\left[\dfrac{\partial u_j^2}{\partial z_i^2}(z_0^2)\right] \neq 0$ and property (B) holds for $K_{D_1\times D_2}(z, t)$.

Now assume that property (B) is valid for $K_{D_1\times D_2}$. It was proved in Remark 1 of [7] that $\partial(D_1\times D_2)$ satisfies minimal regularity conditions. Thus, by property (B), for every $z_0 = (z_0^1, z_0^2) \in D_1 \times D_2$ there exist points $t_l = (t_l^1, t_l^2)$, $l = 0, \ldots, n+m$, such that $K_{D_1\times D_2}(z_0, t_0) \neq 0$ and the mapping $h(z) = (u_l(z))$ can be extended to a diffeomorphism from a neighbourhood of z_0 into C^{n+m}. If we restrict this mapping to the set of points $z = (z^1, z_0^2)$, we obtain a diffeomorphic imbedding of a neighbourhood of z_0^1 in C^n into C^{n+m}. Hence the rank of the Jacobi matrix of this restriction at z_0^1 must be equal to n. This means that there exist n points among all t_l, say $t_1, \ldots, t_n$, such that $\det\left[\dfrac{\partial u_j^1}{\partial z_i^1}(z_0^1)\cdot u_j^2(z_0^2)\right]_{\substack{j=1,\ldots,n\\ i=1,\ldots,n}} \neq 0$. This implies that $\det\left[\dfrac{\partial u_j^1}{\partial z_i^1}(z_0^1)\right]_{\substack{j=1,\ldots,n\\ i=1,\ldots,n}} = 0$, and so property (B) is valid for K_{D_1}. In the same manner we can prove that this property holds also for K_{D_2}. From the Theorem, Proposition 1 and Proposition 2 we obtain immediately

PROPOSITION 3. *Let $D = D_1 \times \ldots \times D_n \subset C^n$, where each D_i is a bounded plane domain with boundary satisfying minimal regularity conditions. Then*:

1. *The Bergman kernel function $K_D(z, t)$ satisfies conditions* (A_∞) *and* (B) *if and only if each D_i has a boundary of class C^∞.*

2. *The Bergman kernel function $K_D(z, t)$ satisfies conditions* (A_ω) *and* (B) *if and only if each D_i has a real analytic boundary.*

Remark 3. The Theorem and Proposition 1 yield one more significant difference between the boundary behaviour of the Bergman kernel function in the one-dimensional case and in the case of several complex variables. In the latter case product domains and complete circular strictly starlike domains provide a large class of bounded domains, whose Bergman functions satisfy conditions (A_∞) (or even (A_ω)) and (B) and whose boundaries satisfy minimal regularity conditions and are not of class C^1. This phenomenon is characteristic for the multidimensional case.

Remark 4. Let $D = \{z \in C^n: |\exp z_1| < 1, |z_i| < 1, i = 1, \ldots, n\}$. Note that D is a Weil analytic polyhedron in C^n. The domain D is the Cartesian product $D_1 \times \ldots \times D_n$, where D_1 is the left half of the unit disc and D_i is the unit disc for $i > 1$. It follows from Proposition 3 that properties (A_k) and (B) cannot be simultaneously valid for $K_D(z, t)$ for any $k \geqslant 1$. However, we do not know whether properties (A_k) and (B) are valid for certain special classes of Weil analytic polyhedra, e.g. for polyhedra in C^n whose skeleton is totally real or for polyhedra in C^n which are defined by n holomorphic functions.

CONCLUSION. *If we join together the results of* [6], [8], [7] *and this paper we obtain the following picture:*

Conditions (A_∞) *and* (B) *hold for Bergman kernel functions of the following bounded domains*: 1. *Plane domains with C^∞-boundaries*; 2. *Strictly pseudoconvex domains with C^∞-boundaries*; 3. *Pseudoconvex domains with real analytic boundaries*; 4. *Complete circular strictly starlike domains*; 5. *Cartesian products of domains belonging to the union of classes* 1–4.

Thus our approach to the problem of the smooth extension of biholomorphic mappings, based on conditions (A_∞) and (B), seems to be quite universal. However, there exist at least three important classes of domains, for which we have no information about the boundary behaviour of their Bergman kernel functions: pseudoconvex domains with C^∞-boundaries, strictly pseudoconvex domains with boundaries of class C^k, $2 \leqslant k < \infty$, and analytic polyhedra (see Remark 4). It is a difficult and important problem to study the boundary behaviour of the Bergman kernel function in these cases.

Added in proof. In a paper: E. Ligocka, *The Hölder continuity of the Bergman projection and proper holomorphic mappings* (Studia Math., to appear) it was proved that if D is a strictly pseudoconvex domain with a boundary of class C^{K+4} then conditions (A_K) and (B) are valid for D.

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ANALYTIC FUNCTIONALS ON THE SPHERE AND THEIR FOURIER–BOREL TRANSFORMATIONS

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0. Introduction

Let S^{n-1} be the unit sphere in $\boldsymbol{R}^n$. The space $\mathscr{B}(S^{n-1})$ of hyperfunctions on S^{n-1} is, by definition, the dual space of the space $\mathscr{A}(S^{n-1})$ of real-analytic functions on S^{n-1}. For a hyperfunction $T \in \mathscr{B}(S^{n-1})$, Hashizume–Kowata–Minemura–Okamoto [2] defined the transformation

$$\mathscr{P}_\lambda\colon\ T \in \mathscr{B}(S^{n-1}) \mapsto \mathscr{P}_\lambda T(\xi) = \big\langle T_\omega, \exp(i\lambda\langle \xi, \omega\rangle)\big\rangle, \tag{0.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \boldsymbol{R}^n$ and $\lambda \neq 0$ is a fixed complex number. They showed that the image of $\mathscr{B}(S^{n-1})$ under the transformation $\mathscr{P}_\lambda$ is strictly contained in the space $C_\lambda^\infty(\boldsymbol{R}^n)$ of C^∞-functions f on $\boldsymbol{R}^n$ which satisfy the differential equation

$$(\Delta_\xi + \lambda^2) f(\xi) = 0, \tag{0.2}$$

where $\Delta_\xi = \dfrac{\partial^2}{\partial \xi_1^2} + \dfrac{\partial^2}{\partial \xi_2^2} + \ldots + \dfrac{\partial^2}{\partial \xi_n^2}$ is the Laplacian. They constructed a space $\tilde{\mathscr{B}}(S^{n-1})$ which contains strictly $\mathscr{B}(S^{n-1})$, using the sequence of spherical harmonic functions and claimed the transformation $\mathscr{P}_\lambda$ maps $\tilde{\mathscr{B}}(S^{n-1})$ onto $C_\lambda^\infty(\boldsymbol{R}^n)$. But the meaning of $\tilde{\mathscr{B}}(S^{n-1})$ was obscure for us. In the case $n = 2$, Helgason [4] showed that $\tilde{\mathscr{B}}(S^{n-1})$ is the space of "entire functionals". (See our previous paper [9] for the details of the case $n = 2$.) Our aim in this paper is to extend Helgason's result in the case of general n. The space $\tilde{\mathscr{B}}(S^{n-1})$ turns out to be the dual space of the space $\mathrm{Exp}(\tilde{S}^{n-1})$ of the holomorphic functions of exponential type on the complex sphere $\tilde{S}^{n-1} = \{z \in \boldsymbol{C}^n;\ z_1^2 + z_2^2 + \ldots + z_n^2 = 1\}$.

We will consider the following spaces of functions or functionals on the sphere S^{n-1}: $L^2(S^{n-1})$ is the space of L^2 functions on S^{n-1}, $C^\infty(S^{n-1})$ is the space of C^∞-functions on S^{n-1} and $\mathscr{A}(S^{n-1})$ is the space of real-analytic functions on S^{n-1}. $\mathscr{O}(\tilde{S}^{n-1})$ is the space of holomorphic functions on the complex sphere $\tilde{S}^{n-1}$ and $\mathrm{Exp}(\tilde{S}^{n-1})$ is the subspace of $\mathscr{O}(\tilde{S}^{n-1})$ of holomorphic functions of exponential type. By the restriction of variables, the spaces $\mathscr{O}(\tilde{S}^{n-1})$ and $\mathrm{Exp}(\tilde{S}^{n-1})$ are con-