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TOPICS FROM CLASSICAL THEORY OF CONFORMAL MAPPING

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Topics prepared for this lecture consist of four parts which are almost independent of each other. However, three parts among them are reported here, since the remaining one was published in the Proceedings of 7th Conference on Analytic Functions in Kozubnik-Porąbka.

1. Conformal mapping onto polygonal domains

1.1. Schwarz–Christoffel formula. It is well known that the function f which maps $E = \{|z| < 1\}$ conformally onto a rectilinear polygonal domain P with interior angles $\alpha_\mu \pi$ $(\mu = 1, \ldots, m)$ at vertices $f(e^{i\varphi_\mu})$ is represented by the classical Schwarz–Christoffel formula ([52], [10], [11], [12])

$$f(z) = A\int_0^z \prod_{\mu=1}^{m} (e^{i\varphi_\mu}-z)^{\alpha_\mu-1}dz + B,$$

where the constants $A(\neq 0)$ and B depending only on the magnitude and the position of P are given by

$$A = f'(0)\exp\Big(i\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu\Big), \quad B = f(0).$$

As a generalization to the doubly-connected case, we can derive a similar formula for the function which maps an annulus onto a ring domain bounded by two rectilinear polygons. The mapping function is really represented explicitly in terms of elliptic functions. In order to show the actual procedure of its derivation, we first illustrate it for the Schwarz–Christoffel formula itself; the method will serve as an alternative proof of this formula; cf. [35].

Now, let Φ be holomorphic in E and the quantity ϱ defined by

$$\varrho(r, \varphi) = \int_0^\varphi \mathrm{Re}\,\Phi(re^{i\theta})d\theta$$

be of bounded variation in $\varphi \in (0, 2\pi)$ uniformly for $r \in (0, 1)$. Then Φ admits the Herglotz representation ([31])

$$\Phi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\varrho(\varphi) + i\,\mathrm{Im}\,\Phi(0),$$

where $\varrho(\varphi) = \lim \varrho(r_k, \varphi)$ for a suitable increasing sequence $\{r_k\}$ tending to 1. If, in particular, Φ is holomorphic throughout $\overline{E}$, then

$$\varrho(\varphi) = \int_0^{\varphi} \mathrm{Re}\,\Phi(e^{i\theta})\, d\theta.$$

The mapping function f of E onto P is piecewise holomorphic on $\overline{E}$ and the saltus of $\arg f'$ along ∂E at $e^{i\varphi_\mu}$ is equal to $(1-\alpha_\mu)\pi$. Hence the function Φ defined by

$$\Phi(z) = z\frac{d}{dz}\log\Big(f'(z)\prod_{\mu=1}^{m}(z-e^{i\varphi_\mu})^{1-\alpha_\mu}\Big) = z\frac{f''(z)}{f'(z)} + \sum_{\mu=1}^{m}\frac{(1-\alpha_\mu)z}{z-e^{i\varphi_\mu}}$$

behaves holomorphically throughout $\overline{E}$. By applying the Herglotz representation, we get

$$z\frac{f''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\varrho(\varphi) - \sum_{\mu=1}^{m}\frac{(1-\alpha_\mu)z}{z-e^{i\varphi_\mu}};$$

$$\varrho(\varphi) = \int_0^{\varphi}\mathrm{Re}\,\Phi(e^{i\theta})\,d\theta = \arg\frac{f'(e^{i\varphi})}{f'(1)} + \sum_{\mu=1}^{m}(1-\alpha_\mu)\left(\frac{\varphi}{2} - \pi\varepsilon_\mu(\varphi)\right),$$

ε_μ being the characteristic function of the interval $(\varphi_\mu, 2\pi]$. Thus we have

$$z\frac{f''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\arg f'(e^{i\varphi}),$$

and, after subtracting the relation obtained by putting $z = 0$ (non-vanishing of f' in E!) and by taking $\arg(df(e^{i\varphi})/d\varphi) = \pi/2 + \arg f'(e^{i\varphi})$ into account,

$$\frac{f''(z)}{f'(z)} = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{e^{i\varphi}-z}\, d\arg df(e^{i\varphi}).$$

Consequently, by remembering that $\arg df$ is piecewise constant along ∂E and jumps at $e^{i\varphi_\mu}$ by $(1-\alpha_\mu)\pi$, we obtain

$$\frac{f''(z)}{f'(z)} = \sum_{\mu=1}^{m}\frac{1-\alpha_\mu}{e^{i\varphi_\mu}-z}.$$

Successive integration yields readily the Schwarz–Christoffel formula.

1.2. Villat–Stieltjes representation. We consider the annulus $R \equiv R_q = \{q < |z| < 1\}$ with a fixed $q > 0$ and a function Φ holomorphic in R and continuous on $\overline{R}$. The formula corresponding to the Poisson representation in E is provided by the Villat representation; cf. [55], [14], [13]. It states

$$\Phi(z) = \frac{\omega_1}{\pi^2 i}\int_0^{2\pi}\mathrm{Re}\,\Phi(e^{i\varphi})\left(\zeta\left(\frac{\omega_1}{\pi}i\log\frac{z}{e^{i\varphi}}\right) - \left(\frac{1}{2\omega_3} - \frac{\eta_1}{\pi i}\right)\log z\right)d\varphi -$$
$$- \frac{\omega_1}{\pi^2 i}\int_0^{2\pi}\mathrm{Re}\,\Phi(qe^{i\varphi})\left(\zeta_3\left(\frac{\omega_1}{\pi}i\log\frac{z}{e^{i\varphi}}\right) - \left(\frac{1}{2\omega_3} - \frac{\eta_1}{\pi i}\right)\log z\right)d\varphi + iC,$$

where C is a real constant and the symbols from Weierstrassian theory of elliptic functions are concerned with those of primitive periods $2\omega_1$ and $2\omega_3$ which are real and purely imaginary, respectively, and satisfy

$$\log\frac{1}{q} = \frac{\pi\omega_3}{i\omega_1}.$$

The condition that Φ is single-valued in R is given by the monodromy relation

$$\int_0^{2\pi}\mathrm{Re}\,\Phi(e^{i\varphi})\,d\varphi = \int_0^{2\pi}\mathrm{Re}\,\Phi(qe^{i\varphi})\,d\varphi.$$

Since only the value of ratio $\omega_3/\omega_1 = -i\log q/\pi$ is essential, we may take for the sake of brevity

$$\omega_1 = \pi, \quad \omega_3 = -i\log q.$$

In this simplified case the representation for a single-valued Φ becomes

$$\Phi(z) = \frac{1}{\pi i}\int_0^{2\pi}\mathrm{Re}\,\Phi(e^{i\varphi})\,\zeta\left(i\log\frac{z}{e^{i\varphi}}\right)d\varphi - \frac{1}{\pi i}\int_0^{2\pi}\mathrm{Re}\,\Phi(qe^{i\varphi})\,\zeta_3\left(i\log\frac{z}{e^{i\varphi}}\right)d\varphi + iC.$$

Now, we put, as before,

$$\varrho(r, \varphi) = \int_0^{\varphi}\mathrm{Re}\,\Phi(re^{i\theta})\,d\theta \qquad (0 \leqslant \varphi \leqslant 2\pi,\ q < r < 1).$$

If it is of bounded variation in φ uniformly for r, then Φ admits the Villat–Stieltjes representation of Herglotz type

$$\Phi(z) = \frac{1}{\pi i}\int_0^{2\pi}\zeta\left(i\log\frac{z}{e^{i\varphi}}\right)d\varrho(\varphi) - \frac{1}{\pi i}\int_0^{2\pi}\zeta_3\left(i\log\frac{z}{e^{i\varphi}}\right)d\tau(\varphi) + ic,$$

where c is a real constant depending on Φ and $\varrho(\varphi) = \lim \varrho(r_k, \varphi)$, $\tau(\varphi) = \lim \varrho(t_k, \varphi)$ for suitable monotone sequences $\{r_k\}, \{t_k\}$ tending to 1, q, respectively. If, in particular, Φ is holomorphic and single-valued throughout $\overline{R}$, then

$$\varrho(\varphi) = \int_0^{\varphi}\mathrm{Re}\,\Phi(e^{i\theta})\,d\theta, \qquad \tau(\varphi) = \int_0^{\varphi}\mathrm{Re}\,\Phi(qe^{i\theta})\,d\theta.$$

1.3. Formula for mapping function onto polygonal ring domain. We mention the aimed result ([37], [39]):

THEOREM 1.1. *Let a finite polygonal ring domain P with (logarithmic) modulus $\log q^{-1}$ be given and its outer and inner boundary components be m-gon and n-gon, respectively. Let f map $R = \{q < |z| < 1\}$ conformally onto P such that $\{|z| = 1\}$ corresponds to the outer component. Let further the interior angles at vertices $f(e^{i\varphi_\mu})$ $(\mu = 1, \ldots, m)$ and $f(qe^{i\psi_\nu})$ $(\nu = 1, \ldots, n)$ be $\alpha_\mu\pi$ and $\beta_\nu\pi$, respectively, where $\beta_\nu\pi$ are exterior angles with respect to P. Then f is represented in the form*

$$f(z) = A\int^{z} z^{ic^*-1}\frac{\prod_{\mu=1}^{m}\sigma\left(i\log\frac{z}{e^{i\varphi_\mu}}\right)^{\alpha_\mu-1}}{\prod_{\nu=1}^{n}\sigma_3\left(i\log\frac{z}{e^{i\psi_\nu}}\right)^{\beta_\nu-1}}\,dz+B,$$

where c^* is a real constant defined by

$$c^* = \frac{\eta_1}{\pi}\left(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu-\sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu\right)$$

and $A(\neq 0)$ and B are constants depending only on the magnitude and the position of P.

Proof. Similarly as in the simply-connected case, the function Φ defined by

$$\Phi(z) = z\frac{d}{dz}\log\left(f'(z)\prod_{\mu=1}^{m}(z-e^{i\varphi_\mu})^{1-\alpha_\mu}\Big/\prod_{\nu=1}^{n}(z-qe^{i\psi_\nu})^{1-\beta_\nu}\right)$$

is holomorphic and single-valued on $\overline{R}$. By applying the Villat–Stieltjes representation, we get after some calculation

$$z\frac{f''(z)}{f'(z)} = \frac{1}{\pi i}\int_0^{2\pi}\zeta\left(i\log\frac{z}{e^{i\varphi}}\right)d\arg f'(e^{i\varphi})-\frac{1}{\pi i}\int_0^{2\pi}\zeta_3\left(i\log\frac{z}{e^{i\varphi}}\right)d\arg f'(qe^{i\varphi})+ic^*.$$

By adding the relation which is obtained either by applying the Villat formula to the particular function 1 or by using the residue calculus, we get

$$1+z\frac{f''(z)}{f'(z)} = \frac{1}{\pi i}\int_0^{2\pi}\zeta\left(i\log\frac{z}{e^{i\varphi}}\right)d\arg df(e^{i\varphi})-\frac{1}{\pi i}\int_0^{2\pi}\zeta_3\left(i\log\frac{z}{e^{i\varphi}}\right)d\arg df(qe^{i\varphi})+ic^*.$$

Since $\arg df$ jumps at $e^{i\varphi_\mu}$ and $qe^{i\psi_\nu}$ by $(1-\alpha_\mu)\pi$ and $(1-\beta_\nu)\pi$, respectively, we have

$$1+z\frac{f''(z)}{f'(z)} = -i\sum_{\mu=1}^{m}(1-\alpha_\mu)\zeta\left(i\log\frac{z}{e^{i\varphi_\mu}}\right)+i\sum_{\nu=1}^{n}(1-\beta_\nu)\zeta_3\left(i\log\frac{z}{e^{i\psi_\nu}}\right)+ic^*.$$

The integration with respect to $\log z$ yields

$$z\frac{f'(z)}{f'(1)} = z^{ic^*}\prod_{\mu=1}^{m}\left(\frac{\sigma\left(i\log(z/e^{i\varphi_\mu})\right)}{\sigma(\varphi_\mu)}\right)^{\alpha_\mu-1}\Big/\prod_{\nu=1}^{n}\left(\frac{\sigma_3\left(i\log(z/e^{i\psi_\nu})\right)}{\sigma_3(\psi_\nu)}\right)^{\beta_\nu-1}.$$

Finally, in order to determine the value of real constant c^*, we carry out the transformation $\log z|\log z+2\pi i$. Then, while the left member of the last relation remains unchanged, the right member gains a factor e^{Ω} with

$$\Omega = -2\pi c^*+\sum_{\mu=1}^{m}(\alpha_\mu-1)\left(\pi i-2\eta_1\left(i\log\frac{z}{e^{i\varphi_\mu}}-\pi\right)\right)+ + \sum_{\nu=1}^{n}(\beta_\nu-1)2\eta_1\left(i\log\frac{z}{e^{i\psi_\nu}}-\pi\right).$$

By taking the relation $\sum_{\mu=1}^{m}(\alpha_\mu-1) = \sum_{\nu=1}^{n}(\beta_\nu-1) = -2$ into account, we have

$$\Omega = -2\pi c^*-2\pi i+2\eta_1\left(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu-\sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu\right).$$

Since Ω must be an integral multiple of $2\pi i$, this leads to the desired expression of c^*.

It is noted that it is quite inessential to take the point 1 as the lower limit of the integral with respect to z. If this point were eventually the inverse image of a vertex of P, it is only necessary to take any ordinary point.

A formula in terms of theta functions which is equivalent to that stated in Theorem 1.1 is found in [44].

1.4. Polygonal ring domain containing the point at infinity. We have hitherto supposed that P does not contain the point at infinity in its interior. The formula must be modified when this is the case. For that purpose, we have only to replace $f'(z)$ by $f'(z)(z-p)^2$ in the defining equation of Φ, where p denotes the inverse image of ∞ by the mapping. Similar argument as above will then lead to the desired result.

The result obtained in the simply-connected case becomes

$$f(z) = A\int^{z}\prod_{\mu=1}^{m}(e^{i\varphi_\mu}-z)^{\alpha_\mu-1}\frac{dz}{(z-p)^2(1-\bar{p}z)^2}+B,$$

where $\alpha_\mu\pi$ denotes the interior angle at $f(e^{i\varphi_\mu})$ with respect to P; especially $\sum_{\mu=1}^{m}(\alpha_\mu-1) = 2$. Moreover, in view of the single-valuedness of f, the residue at p of the integrand in this expression must vanish, whence follows

$$\sum_{\mu=1}^{m}\frac{\alpha_\mu-1}{e^{i\varphi_\mu}-p} = \frac{2\bar{p}}{1-|p|^2}.$$

The corresponding result in the doubly-connected case will be stated in the following theorem ([41]):

THEOREM 1.2. *The function f which maps the annulus $R = \{q < |z| < 1\}$ onto a polygonal ring domain P containing ∞ is represented in the form*

$$f(z) = A\int^{z} z^{ic^*-1} \frac{\prod_{\mu=1}^{m} \sigma\left(i\log\frac{z}{e^{i\varphi_\mu}}\right)^{\alpha_\mu-1}}{\prod_{\nu=1}^{n} \sigma_3\left(i\log\frac{z}{e^{i\psi_\nu}}\right)^{\beta_\nu-1} \cdot \sigma\left(i\log\frac{z}{p}\right)^2 \sigma(i\log(\bar{p}z))^2}\, dz + B;$$

$$c^* = \frac{\eta_1}{\pi}\Big(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu - \sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu + 4\arg p\Big),$$

where $\alpha_\mu\pi$ denotes the interior angle at $f(e^{i\varphi_\mu})$ and $\beta_\nu\pi$ denotes the exterior angle at $f(qe^{i\psi_\nu})$, both with respect to P, while p denotes the inverse image of ∞ by f. Moreover, in view of the single-valuedness of f, the monodromy conditions

$$\sum_{\mu=1}^{m}(1-\alpha_\mu)\left(\zeta\left(i\log\frac{p}{e^{i\varphi_\mu}}\right) - \frac{\eta_1\varphi_\mu}{\pi}\right) - \sum_{\nu=1}^{n}(1-\beta_\nu)\left(\zeta_3\left(i\log\frac{p}{e^{i\psi_\nu}}\right) - \frac{\eta_1\psi_\nu}{\pi}\right) +$$
$$+2\left(\zeta(2i\log|p|) - \frac{2\eta_1}{\pi}\arg p\right) = 0,$$

$$\int_0^{2\pi} e^{-c^*\varphi} \frac{\prod_{\mu=1}^{m}\sigma(\varphi_\mu-\varphi)^{\alpha_\mu-1}}{\prod_{\nu=1}^{n}\sigma_3(\psi_\nu-\varphi)^{\beta_\nu-1}} \frac{d\varphi}{|\sigma(i\log p+\varphi)|^4} = 0$$

must hold among several parameters involved.

Proof. The expression of f is derived by the way explained above. In order to derive the first monodromy condition, we write the integrand of the expression for f in the form

$$\frac{f'(z)}{A} = \omega(z) \Big/ \sigma\left(i\log\frac{z}{p}\right)^2.$$

Then ω is holomorphic at p and the principal part of $\sigma(i\log(z/p))^{-2}$ at the pole of the second order is given by $-p^2(z-p)^{-2} - p(z-p)^{-1}$. Hence the residue of $f'(z)/A$ at p is equal to

$$-p\omega(p) - p^2\omega'(p) = -p[(z\omega(z))']^{z=p}.$$

Since this residue must vanish, we obtain

$$0 = \frac{1}{i}\left[\frac{(z\omega(z))'}{\omega(z)}\right]^{z=p} = \frac{1}{i}\left[\frac{d}{d\log z}\log(z\omega(z))\right]^{z=p}$$
$$= c^* + \sum_{\mu=1}^{m}(\alpha_\mu-1)\zeta\left(i\log\frac{p}{e^{i\varphi_\mu}}\right) - \sum_{\nu=1}^{n}(\beta\ -1)\zeta_3\left(i\log\frac{p}{e^{i\psi_\nu}}\right) - 2\zeta(i\log|p|^2),$$

whence follows by substituting the value of c^* the desired condition. The last condition is an immediate consequence of $\int_0^{2\pi} df(e^{i\varphi}) = 0$.

1.5. Remarks. In conclusion, we supplement some remarks.

First, it is noted that any result on the doubly-connected case of an annulus R_q reduces to the corresponding one on the simply-connected case of the unit disk when q tends to 0.

A polygonal domain is determined, provided, for instance, the coordinates of its vertices are assigned. But the preimages of vertices by the mapping are related rigidly so that for a given polygonal domain there are few parameters among $\{\varphi_\mu, \psi_\nu\}$ in the expression of mapping function which are at our disposal. In fact, the number of such real parameters is only three or one in the simply- or doubly-connected case, respectively. It will be a difficult and perhaps troublesome but important problem to determine a complete system of relations for the whole set of parameters in an available form; cf. [2], [21], [34], etc.

It is possible to generalize the formulas derived above to the case of Riemann surfaces. In fact, the polygonal domain may be admitted to ramify at some branch points and to contain some points lying over ∞; cf. [41].

On the other hand, the discussion explicitly made in simply- and doubly-connected cases will be suitably generalized to multiply-connected case. There are several possibilities in choosing the type of canonical domains D instead of E or R. We may take, for instance, a circular slit annulus or a fully circular domain. The way of deriving the above results has been essentially based on the complex form of Green formula. It gives the integral representation for an analytic function holomorphic on $\bar{D}$ in terms of its real part as the weight function in which the kernel is a definite domain function. Hence the method will apply similarly as above for deriving a formula for polygonal mapping function.

Finally, we can deal with polygonal domains bounded by circular arcs. The simply-connected case is classical; [52], [53]. A differential equation of the third order related to the Schwarzian derivative is derived for the mapping function also in the doubly-connected case; [39], [40].

2. Isoperimetric inequalities and related variational problems

2.1. The isoperimetric problem. The most simple isoperimetric problem formulated and solved by Steiner [54] is to maximize the area of a figure bounded by a curve with assigned length. Beside a purely geometrical proof due to Edler [16], an elegant analytical proof was given by Hurwitz [32], [33]. But the latter assumed the piecewise smoothness of boundary curves in order to assure the termwise differentiability of Fourier series used in his proof. Various proofs and generalizations have been subsequently published by Brunn [8], Minkowski [48], [49], Carathéodory–Study [9] and others, as fully explained in a book of Bonnesen–Fenchel [7].

The present purpose is to show that a function-theoretic proof based on conformal mapping is possible; cf. [38].

THEOREM 2.1. *Let F denote the area of a finite region bounded by a rectifiable Jordan curve with length L. Then the isoperimetric inequality $4\pi F \leqslant L^2$ holds and the equality sign appears iff the boundary curve is a circle.*

Proof. Let the given region E be a continuum with outer mapping radius (i.e., transfinite diameter or capacity) ϱ. The complement of E laid on the w-plane is then regarded as the image of $\{|z| \geqslant \varrho\}$ by a univalent function of the form

$$w = g(z) \equiv z + \sum_{k=0}^{\infty} b_k z^{-k} \quad (|z| > \varrho).$$

Let $F(r)$ $(r > \varrho)$ denote the area of the finite region bounded by the curve $\Gamma_r = g(\{|z| = r\})$. Then, by means of a way in the proof of Bieberbach's area theorem [5], we get

$$F(r) = \pi\Big(r^2 - \sum_{k=1}^{\infty} k|b_k|^2 r^{-2k}\Big);$$

$$F = F(\varrho+0) = \pi\Big(\varrho^2 - \sum_{k=1}^{\infty} k|b_k|^2 \varrho^{-2k}\Big) \leqslant \pi\varrho^2.$$

The equality sign in the last inequality holds iff $g(z) = z + b_0$. On the other hand, the length $L(r)$ of Γ_r is given by

$$L(r) = \int_{|z|=r} |g'(z)|\,|dz| = \int_0^{2\pi} |g'(re^{i\theta})|\,r\,d\theta.$$

In view of the univalency of g, its derivative does not vanish throughout $\{|z| > \varrho\}$. Hence we can take a single-valued branch of $g'^{1/2}$:

$$g'(z)^{1/2} = \Big(1 - \sum_{k=1}^{\infty} k b_k z^{-k-1}\Big)^{1/2} = 1 + \sum_{k=2}^{\infty} c_k z^{-k},$$

for which we have

$$\int_0^{2\pi} |g'(re^{i\theta})|\,d\theta = \int_0^{2\pi} |g'(re^{i\theta})^{1/2}|^2 d\theta = 2\pi\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 r^{-2k}\Big).$$

The image by g being a Jordan domain, g is continuous for $\{\varrho \leqslant |z| < \infty\}$. Hence the real-valued function defined by

$$G_m(z) = \sum_{\mu=1}^{m} \left|\frac{1}{z}\big(g(\omega_m^{\mu} z) - g(\omega_m^{\mu-1} z)\big)\right|, \quad \omega_m = e^{2\pi i/m},$$

is continuous and subharmonic for $\{|z| \geqslant \varrho\}$ so that it attains its maximum at a point $\hat{z}_m$ with $|\hat{z}_m| = \varrho$. Thus we get $G_m(z) \leqslant G_m(\hat{z}_m) \leqslant L/\varrho$ and, for $r > \varrho$,

$$L \geqslant \varrho \lim G_m(r) = \varrho \int_0^{2\pi} |g'(re^{i\theta})|\,d\theta = 2\pi\varrho\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 r^{-2k}\Big).$$

Consequently, we obtain for $r \to \varrho+0$ the evaluation

$$L \geqslant 2\pi\varrho\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 \varrho^{-2k}\Big) \geqslant 2\pi\varrho,$$

which, together with $F \leqslant \pi\varrho^2$, leads to the desired result.

Remark. Since the length functional is, in general, lower semicontinuous with respect to the strong convergence (in the sense of Fréchet's écart), we have

$$L \leqslant \varliminf_{r\to\varrho+0} L(r) = \lim_{r\to\varrho+0} L(r) = 2\pi\varrho\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 \varrho^{-2k}\Big),$$

whence follows the exact equality

$$L = 2\pi\varrho\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 \varrho^{-2k}\Big).$$

2.2. A related extremal problem. On the other hand, Bieberbach [3], [6] showed a related extremal property of circle, for which a proof similar to above applies.

THEOREM 2.2. *Let F denote the outer measure of a plane set with diameter D. Then the inequality $4F \leqslant \pi D^2$ holds and the equality sign appears iff the set is a circular disk.*

Proof. The given set may be supposed to be a continuum (and, moreover, a convex region). Retaining the previous notations in the proof of Theorem 2.1, put $h(z) = g(z) - g(-z)$. Then, denoting by $D(r)$ the diameter of Γ_r, we have

$$2 = h'(\infty) = \frac{1}{2\pi i}\int_{|z|=r>\varrho} \frac{h(z)}{z^2}dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{h(re^{i\theta})}{re^{i\theta}}d\theta,$$

$$2 \leqslant \frac{1}{2\pi r}\int_0^{2\pi} |g(re^{i\theta}) - g(-re^{i\theta})|\,d\theta \leqslant \frac{D(r)}{r},$$

whence follows, after $r \to \varrho+0$, $2\varrho \leqslant D$. This, combined with $F \leqslant \pi\varrho^2$, leads to $4F \leqslant \pi D^2$. The extremum assertion also follows readily.

Remark. The above inequality $2\varrho \leqslant D$ is essentially a classical result due to Landau–Toeplitz [45]; cf. also [4]. The equality sign here appears also iff E is a circular disk. In fact, if it is the case, we must have $g(z) - g(-z) = 2z$ so that $g(z) - z$ is an even function and ∂E is an oval of constant breadth 2ϱ. Thus, the length of ∂E being $2\pi\varrho$, we have in turn

$$2\pi\varrho = \int_0^{2\pi} |g'(e^{i\theta})|\,\varrho\,d\theta = 2\pi\varrho\Big(1 + \sum_{k=2}^{\infty} |c_k|^2 \varrho^{-2k}\Big);$$

$$c_k = 0 \ (k \geqslant 2), \quad g'(z) = 1; \quad g(z) = z + b_0.$$

2.3. Rengel's lemmas. Here we change our topic. In relation to the isoperimetric problem, we shall deal with some variational problems of Grötzsch [28], [29] which stand in close connection with conformal mapping of multiply-connected domains. Original proofs by Grötzsch were based on lemmas concerning modulus evaluation by the strip method which had been founded by himself [26], [27]. These lemmas have been subsequently proved by Rengel [50] in other ways. In the following lines, we shall give alternative proofs [42] for some theorems of Grötzsch by making use of Rengel's theorems which have been used by Rengel [51] for the existence proof of canonical mappings.

For the sake of convenience, we refer here to Rengel's theorems [51] without proof, which will play the role of lemmas in the following lines.

LEMMA 2.1. *Let the basic domain G be a circular slit disk, that is, a disk $\{|z| < R\}$ cut along finite number of concentric circular arcs. Let $w = F(z)$ with $F(0) = 0$ and $|F'(0)| = 1$ map G conformally in such a manner that the image of $\{|z| = R\}$ separates 0 and ∞. Then the distance M_R of the farthest point on this image from the origin satisfies $M_R \geqslant R$ and the equality sign appears iff F is a rotation: $F(z) = F'(0)z$.*

LEMMA 2.2. *Let the basic domain G be a radial slit disk, that is, a disk $\{|z| < R\}$ cut along finite number of rays centred at the origin. Let $w = F(z)$ with $F(0) = 0$ and $|F'(0)| = 1$ map G conformally in such a manner that the image of $\{|z| = R\}$ separates 0 and ∞. Then the distances M_R and m_R of the farthest and the nearest points on this image from the origin satisfy $m_R^2 \leqslant M_R R$ and the equality sign appears iff F is a rotation: $F(z) = F'(0)z$.*

2.4. Grötzsch's variational problems. Now, let B be a domain given on the z-plane which contains ∞ and is bounded by n boundary components R_ν $(\nu = 1, \ldots, n)$. Let $\mathscr{F}$ denote the class of analytic functions f, which are univalent in B and normalized by $f(\infty) = \infty$, $f'(\infty) = 1$. Let further $\Gamma_\nu = \Gamma_\nu[f]$ denote the boundary component of $f(B)$ corresponding to R_ν. Suppose that R_ν with an assigned suffix ν is a continuum. The diameter $D_\nu[f]$ of Γ_ν is regarded as a positive functional defined on $\mathscr{F}$. The variational problems proposed and solved by Grötzsch [28], [29] are to maximize and to minimize this functional.

THEOREM 2.3. *The maximum of $D_\nu[f]$ in $\mathscr{F}$ is attained iff Γ_ν is a rectilinear slit and the remaining Γ_μ $(\mu \neq \nu)$ are slits on confocal ellipses with both endpoints of Γ_ν as foci.*

Proof. Let $Z = \varphi(z)$ with $\varphi(\infty) = 0$ map B conformally onto a concentric slit disk such that R_ν corresponds to $\{|Z| = 1\}$. Then the absolute value of $\sigma = [z\varphi(z)]^{z=\infty}$ is uniquely determined and

$$w = p(z) \equiv \sigma\big(\varphi(z)+\varphi(z)^{-1}\big), \quad p(\infty) = \infty,\ p'(\infty) = 1,$$

maps B onto an extremal domain of the type mentioned in the theorem, of which the endpoints of $\Gamma_\nu[p]$ lie at $\pm 2\sigma$ and hence $D_\nu[p] = 4|\sigma|$. Every other extremal domain arises from $p(B)$ by a motion, and φ itself is determined up to a rotation about the origin. Now, for any $f \in \mathscr{F}$, the unbounded simply-connected domain bounded by Γ_ν alone is regarded as the image of $\{|\omega| > \varrho\}$ by a mapping of the form

$$w = g(\omega) \equiv \omega + \sum_{k=0}^{\infty} b_k \omega^{-k} \quad (|\omega| > \varrho).$$

Since $G(\omega) = (g(\varrho\omega)-w')/\varrho$ with any fixed $w' \in \Gamma_\nu$ vanishes nowhere throughout $\{|\omega| > 1\}$, the branch defined by

$$G(\omega^2)^{1/2} = \omega\left(\frac{G(\omega^2)}{2}\right)^{1/2} = \omega + \frac{b_0-w'}{2\varrho\omega} + \ldots$$

is univalent in $\{|\omega| > 1\}$. In view of Bieberbach's area theorem [5], we get $|b_0-w'| \leqslant 2\varrho$. The composite function

$$W = F(Z) \equiv \frac{\alpha\sigma}{g^{-1}\circ f \circ \varphi^{-1}(Z)}, \quad F(0) = 0,\ F'(0) = 1\ (|\alpha| = 1),$$

maps the circular slit disk onto a subdomain of $\{|W| < |\sigma|/\varrho\}$ such that the outer boundary circles correspond each other. By means of Lemma 2.1, we get $|\sigma| \geqslant \varrho$ and hence, for any $w,\ w' \in \Gamma_\nu$,

$$|w-w'| \leqslant |b_0-w|+|b_0-w'| \leqslant 4\varrho \leqslant 4|\sigma|,$$

whence follows the desired estimation $D_\nu[f] \leqslant 4|\sigma| = D_\nu[p]$. If the equality sign in this relation holds, then we have $|\sigma| = \varrho$ and, again in view of Lemma 2.1, $F(Z) = Z$, i.e. $\omega = \alpha\sigma/\varphi(z)$. On the other hand, the equality sign in the inequality concerning area theorem appears only when $b_0 - w' = 2\varepsilon\varrho$ $(|\varepsilon| = 1)$, $G(\omega^2)^{1/2} = \omega + \varepsilon/\omega$, $g(\omega) = \omega + b_0 + \varepsilon^2\varrho^2/\omega$. Thus we have

$$\begin{aligned} f(z) = g(\omega) &= \alpha\sigma\varphi(z)^{-1} + b_0 + \varepsilon^2|\sigma|^2\big(\alpha\sigma\varphi(z)^{-1}\big)^{-1} \\ &= \eta\sigma\big(\gamma\varphi(z) + (\gamma\varphi(z))^{-1}\big) + b_0, \end{aligned}$$

where $\eta = \varepsilon|\sigma|/\sigma$, $\gamma = \varepsilon|\sigma|/\alpha\sigma$ and hence $|\eta| = |\gamma| = 1$. Consequently, it is verified that the maximum of $D_\nu[f]$ in $\mathscr{F}$ is attained only by p which is obtained from $\gamma\varphi(z)$ instead of $\varphi(z)$, followed by a motion.

THEOREM 2.4. *The minimum of $D_\nu[f]$ in $\mathscr{F}$ is attained iff Γ_ν is a whole circumference and the remaining Γ_μ $(\mu \neq \nu)$ are slits on rays centred at the center of Γ_ν.*

Proof. Let $Z = \psi(z)$ with $\psi(\infty) = 0$ map B conformally onto a radial slit disk such that R_ν corresponds to $\{|Z| = 1\}$. Then the absolute value of $\tau = [z\psi(z)]^{z=\infty}$ is uniquely determined and

$$w = q(z) \equiv \tau\psi(z)^{-1}, \quad q(\infty) = \infty, \quad q'(\infty) = 1,$$

maps B onto an extremal domain of the type mentioned in the theorem, for which the radius of Γ_ν is equal to $|\tau|$ and hence $D_\nu[q] = 2|\tau|$. Every other extremal domain arises from $q(B)$ by a motion, and ψ itself is determined up to a rotation about the origin. Now, for any $f \in \mathscr{F}$, the unbounded simply-connected domain bounded by Γ_ν alone is regarded as the image of $\{|\omega| > \varrho\}$ by a mapping of the form

$$w = g(\omega) \equiv \omega + \sum_{k=0}^{\infty} b_k \omega^{-k} \quad (|\omega| > \varrho).$$

As shown in the proof of Theorem 2.2, we have $D_\nu[f] \geqslant 2\varrho$. The composite function

$$W = F(Z) \equiv \frac{\alpha\tau}{g^{-1}\circ f\circ \psi^{-1}(Z)}, \quad F(0) = 0,\ F'(0) = 1\ (|\alpha| = 1),$$

maps the radial slit unit disk onto a subdomain of $\{|W| < |\tau|/\varrho\}$ such that the outer boundary components correspond each other. By means of Lemma 2.2 we get $|\tau| \leqslant \varrho$ and hence $D_\nu[f] \geqslant 2\varrho \geqslant 2|\tau| = D_\nu[q]$. If the equality sign holds here, then we have $|\tau| = \varrho$ and, again in view of Lemma 2.2, $F(Z) = Z$, i.e. $\omega = \alpha\tau/\psi(z)$. On the other hand, since the diameter $d_\nu(r)$ $(r > \varrho)$ of the image curve of $\{|\omega| = r\}$ by $w = g(\omega)$ tends to $D_\nu[f]$ as $r \to \varrho+0$, there exists for any assigned $\delta > 0$ an r sufficiently near ϱ such that $d_\nu(r) < D_\nu[f]+2\delta = 2(\varrho+\delta)$. Hence we get for $h(\omega) = g(\omega)-g(-\omega)$ the relation

$$4(\varrho+\delta)^2 > d_\nu(r)^2 \geqslant \frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2 d\theta = 4\Big(r^2 + \sum_{j=1}^{\infty} |b_{2j-1}|^2 r^{-2(2j-1)}\Big).$$

Since $\delta > 0$ may be chosen arbitrarily near zero, we get $b_{2j-1} = 0$ $(j \geqslant 1)$ and hence $g(\omega)-g(-\omega) = h(\omega) = 2\omega$. This, combined with $D_\nu[f] = 2\varrho$, implies that $\Gamma_\nu[f]$ is an oval of constant breadth 2ϱ. Thus, as remarked in succession to the proof of Theorem 2.2, we have $g(\omega) = \omega + b_0$ and hence

$$f(z) = g(\omega) = \alpha\tau\psi(z)^{-1} + b_0.$$

This shows the extremum assertion of the theorem.

2.5. A related variational problem. Finally, we supplement another variational problem also dealt with by Grötzsch [29]. Suppose that the boundary component Γ_ν of the image domain $f(B)$ is a rectifiable Jordan curve with length $L_\nu[f]$. The problem is to minimize this functional.

THEOREM 2.5. *The minimum of $L_\nu[f]$ in $\mathcal{F}$ is attained iff f is an extremal mapping function mentioned in Theorem 2.4.*

Proof. We begin with the case of $n = 1$. While Grötzsch has referred to a result of Bieberbach [6], we proceed here in a slightly modified way. Let ϱ denote the outer mapping radius of the complement of B. Then $f(B)$ is regarded as the image of $\{|\omega| > \varrho\}$ by a univalent mapping $w = g(\omega)$ with $g(\infty) = \infty$ and $g'(\infty) = 1$. As shown in the proof of Theorem 2.1, we get $L_1[f] \geqslant 2\pi\varrho$ where the equality sign appears iff $g(\omega) = \omega + b_0$. The case $n > 1$ may be dealt with by following the argument of Grötzsch. In fact, if Γ_ν is not a whole circumference, we map the unbounded simply-connected domain bounded by Γ_ν alone onto the exterior of a circle by a function normalized at ∞. Then, as shown above, the length of the assigned boundary component will decrease. Next, if Γ_ν is a whole circumference but another Γ_μ is not a radial slit, we map the doubly-connected domain bounded by Γ_μ and Γ_ν onto a radial slit disk such that the mapping is normalized at ∞ and Γ_ν corresponds to the circumference. Then, the radius of the assigned boundary circle will again decrease. This establishes the desired result.

There will be some problems of similar nature which may be dealt with by the method used here.

3. Iteration method for conformal mapping of a ring domain onto an annulus

3.1. Iteration procedure. Let a non-degenerate ring domain be given. Then, it can be univalently mapped onto a conformally equivalent annulus. In order to construct such a canonical mapping effectively, there are several methods as illustrated in detail in a book of Gaier [21]. Among them we shall explain in the following lines an alternating iteration method, which reduces the problem to a sequence of mappings on simply-connected domains. The purpose in the original paper [36] aimed mainly at an existence proof of the canonical mapping. But Gaier [17]–[23] has subsequently derived error estimations quantitatively and shown that the method is useful also from practical standpoint.

We begin with explaining the iteration procedure. Let a non-degenerate ring domain D be given on the z-plane. Based on Riemann's mapping theorem on simply-connected domains, we may suppose that its outer boundary component is $\{|z| = 1\}$ and its inner one is a regular analytic curve γ surrounding the origin. The sequence of ring domains $\{D_n\}_{n=0}^{\infty}$ laid on the respective z_n-plane and of corresponding univalent functions $\{F_n\}_{n=0}^{\infty}$ with $z_n = F_n(z)\colon D \to D_n$ are defined inductively as follows:

1. $z_0 = z$, $D_0 = D$, $F_0(z) = z$ and $\gamma_0 = \gamma$;
2. D_n is a ring domain laid on the z_n-plane of the same character as D, and the distances of the farthest and nearest points from the origin on its inner boundary component γ_n are denoted by m_n^* and m_{*n}, respectively;
3. The exterior of γ_n is mapped onto $\{|t_n| > m_n^*\}$ by a function $t_n = h_n(z_n)$ with $h_n(\infty) = \infty$ (the determination of radius m_n^* serves only for fixing the dilatation of mapping at ∞), and the distances of the farthest and nearest points from the origin on the outer boundary component $\Gamma_n = h_n(\{|z_n| = 1\})$ of $\Delta_n = h_n(D_n)$ are denoted by M_n^* and M_{*n}, respectively;
4. The interior of Γ_n is mapped onto $\{|z_{n+1}| < 1\}$ by a function $z_{n+1} = g_n(t_n)$ with $g_n(0) = 0$, whence arise $D_{n+1} = g_n(\Delta_n)$ and $\gamma_{n+1} = g_n(\{|t_n| = m_n^*\})$.
5. Two mappings h_n and g_n are composed to a mapping $f_n = g_n \circ h_n$ of D_n onto D_{n+1} which is for every n uniquely determined under the normalization $f_n(1) = 1$;
6. The mapping $F_n\colon D \to D_n$ is uniquely defined by $F_n = f_{n-1}\circ F_{n-2}$, F_{-1} being regarded as the identity, i.e. $F_n = f_{n-1}\circ f_{n-2}\circ \cdots \circ f_0$.

Every step of constructing D_n from D_{n-1} will be denoted by $\mathfrak{A}$; in particular, we may write $D_n = \mathfrak{A}^n D$. In view of the inversion principle, we have $f_n(\bar{z}_n^{-1}) = \overline{f_n(z_n)}^{-1}$ $(z_n \in D_n)$ and $F_n(\bar{z}^{-1}) = \overline{F_n(z)}^{-1}$ $(z \in D)$. The last-mentioned property has been later used by Gaier [21] for deriving effective estimations.

3.2. Convergence proof accompanied by existence theorem. Now, the original theorem [36] was formulated in the following form:

THEOREM 3.1. *The sequence $\{F_n\}$ defined above converges uniformly in the wider sense in D, and really in the ring domain bounded by γ and its inverse image with respect to $\{|z| = 1\}$. The limit function $w = F(z) \equiv \lim F_n(z)$ with $F(1) = 1$ maps D onto an annulus $R_q = \{q < |w| < 1\}$.*

Proof. Suppose that D itself is not an annulus. We make repeated use of Schwarz lemma. First, since $h_n^{-1}(\tau^{-1})^{-1}$ satisfies the assumption of this lemma in $\{|\tau| < m_n^{*-1}\}$ with the bound m_{*n}^{-1}, we get $|h_n^{-1}(\tau^{-1})^{-1}| \leqslant (m_{*n}^{-1}/m_n^{*-1})|\tau|$. Putting $\tau = h_n(z_n)^{-1}$ and observing a point on $\{|z_n| = 1\}$ with $|h_n(z_n)| = M_n^*$, we obtain $m_{*n} < m_n^*/M_n^*$. Similarly, by considering $g_n^{-1}(z_{n+1})$ in $\{|z_{n+1}| < 1\}$, $h_n(\tau^{-1})^{-1}$ in $\{|\tau| < m_n^{*-1}\}$ and $g_n(t_n)$ in $\{|t_n| < 1\}$, we obtain $m_n^*/M_n^* < m_{*n+1}$, $M_{*n} > 1$ and $m_{n+1}^* < m_n^*$, respectively. Thus, $\{m_{*n}\}$ increases and $\{m_n^*\}$ decreases, both strictly, so that we can put $0 < p = \lim m_{*n} \leqslant \lim m_n^* = q < 1$ and $\lim M_n^* = q/p$. Now, $\{h_n^{-1}(m_n^*\omega/q)\}$ and $\{g_n^{-1}(\hat{w})\}$ form normal families in $\{|\omega| > q\}$ and $\{|\hat{w}| < 1\}$, respectively, and hence there exist subsequences convergent in the wider sense in respective domains. Let their limit functions corresponding to a subsequence $\{\nu_n\}$ of indices be denoted by $K(\omega)$ and $G^{-1}(\hat{w})$, respectively. They satisfy $p/q \leqslant |K'(\infty)| \leqslant 1$ and $1 \leqslant |G^{-1\prime}(0)| \leqslant q/p$. Let Δ_∞ and $\hat{D}_\infty$ denote the ring domains bounded by $\{|\omega| = q\}, \Gamma_\infty = K^{-1}(\{|w| = 1\})$ and by $\hat{\gamma}_\infty = G(\{|\omega| = q\})$, $\{|\hat{w}| = 1\}$, respectively. Then $K(\Delta_\infty)$ is the domain kernel D_∞ of the sequence $\{D_{\nu_n}\}$, that is, it is the intersection of $\{|w| < 1\}$ and the kernel of the sequence which consists of the exteriors of γ_{ν_n}, while $G^{-1}(\hat{D}_\infty)$ is the kernel of $\{\Delta_{\nu_n}\}$ which is equal to Δ_∞. In particular, D_∞ contains R_q and $\hat{D}_\infty$ coincides with $\mathfrak{A}D_\infty$. Since $\{D_{\nu_n}\}$ converges to D_∞, $\{F_{\nu_n}\}$ converges uniformly in the wider sense in D to a function F which maps D onto D_∞. Hence F is analytically continuable across $\{|z| = 1\}$. Similarly, $\{F_{\nu_n+1}\}$ converges uniformly in the wider sense in D to a function $\hat{F}$ which maps D onto $\hat{D}_\infty$. Consequently, $\hat{w} = F \circ \hat{F}^{-1}(w)$ maps D_∞ onto $\hat{D}_\infty$ and satisfies $\hat{F} \circ F^{-1}(1) = 1$. If R_q were a proper subdomain of D_∞, $\{|\hat{w}| = q\}$ would be contained in $\mathfrak{A}D_\infty$. This contradicts the fact that $\hat{D}_\infty = \mathfrak{A}D_\infty$ has a boundary point on $\{|\hat{w}| = q\}$. Hence we conclude $D_\infty = R_q$, so that $\hat{D}_\infty = D_\infty$ and $\hat{F} = F$. In view of the above arguments, every convergent subsequence of $\{F_n\}$ converges to a function which maps D onto R_q. Since the uniqueness of mapping function is readily established, it follows that the whole sequence $\{F_n\}$ converges to the desired function.

3.3. A distortion theorem. As noticed above, the sequences $\{m_{*n}\}$ and $\{m_n^*\}$ are strictly increasing and decreasing, respectively, unless D is an annulus. Gaier [19], [21] has shown that, for any initial domain D, the boundary curves γ_{n+1}

and Γ_n with $n \geqslant 0$ are all starlike with respect to the origin. In fact, we see, for instance, that g_n satisfies an inequality due to Grunsky [30]

$$\left| \frac{t_n g_n'(t_n)}{g_n(t_n)} - \frac{1}{1-r^2} \right| < \frac{r}{1-r^2} \qquad (|t_n| = r < 1),$$

whence follows more precisely $\operatorname{Re}(t_n g_n'(t_n)/g_n(t_n)) > 1/(1+m_n^*)$ for $t_n \in \Delta_n$ and hence $\gamma_{n+1} = g_n(\{|t_n| = m_n^*\})$ is surely starlike; cf. also Lavrentieff–Chepeleff [47]. It is further shown that the curves γ_n are convex provided n is large enough.

On the other hand, Gaier [17], [18] has shown that the inner boundary component γ_n becomes gradually near a circle in the sense that $m_n^* - m_{*n}$ and $\varkappa_n^*$ are both of order $O(\varrho^{2n})$ as $n \to \infty$ where $\varrho = (4/\pi)\arctan q$ and $\varkappa_n^*$ denotes the maximal angle between radius vector and the normal of γ_n, and further by means of a theorem on conformal mapping of nearly annular domain that $|F_n(z) - F(z)| = O(\varrho^{2n})$ holds uniformly in D. However, he has later shown that the last estimation can be made more precise. For that purpose, a distortion theorem on functions univalent in an annulus is referred to, which is in itself of interesting nature.

THEOREM 3.2. *Let $\mathscr{F}$ be the class of analytic functions f which are univalent in a fixed annulus $R_q = \{q < |w| < 1\}$ and satisfy $|f(w)| = 1$ on $\{|w| = 1\}$, $0 < |f(w)| < 1$ in R_q and are normalized by $f(1) = 1$. Then any $f \in \mathscr{F}$ satisfies*

$$|f(w) - w| < 8q \quad \text{in } R_q$$

where the factor 8 cannot be replaced by any smaller number. The conclusion may be stated in slightly precise manner that $\limsup\limits_{|w| \to q+0} |f(w) - w| \leqslant 5q$ and $\max\limits_{|w|=1} |f(w) - w| \leqslant 8q$ hold where the factors 5 and 8 cannot be diminished.

This theorem was proved by Duren–Schiffer [15] and Gaier–Huckemann [24] and a little later independently by Gehring–af Hällström [25]. By the way, it was also proved that the factor 8 is replaced by 3 when the image $f(R_q)$ is symmetric with respect to the origin.

3.4. Approximation order of iteration. Based on the distortion Theorem 3.2, the approximation order of F_n to F in Theorem 3.1 can be given in explicit manner [21]; cf. also [19].

THEOREM 3.3. *The sequence considered in Theorem 3.1 tends to the limit function in such a manner that*

$$|F_n(z) - F(z)| < 8q^{2n+1} \qquad (n = 1, 2, \ldots)$$

holds throughout D.

Proof. We retain the notations in the proof of Theorem 3.1 and denote further, in general, by $\tilde{X}[\xi]$ the union of a ring domain X, its either boundary component ξ and the inverse image of X with respect to ξ, provided the inversion is possible. Then, in view of the first step of iteration process, we see that $z = F^{-1}(w)$ is analytically continuable across $\{|w| = 1\}$ by inversion and hence it maps $\{q$

$< |w| < q^{-1}\}$ onto $\tilde{D}[\{|z| = 1\}]$. Consequently, $t = h \circ F^{-1}(w)$ maps $\{q < |w| < q^{-1}\}$ onto $E = \Delta[\Gamma]$ and hence it is analytically continuable across $\{|w| = q\}$ by inversion so that it maps $\{q^3 < |w| < q^{-1}\}$ onto $\tilde{E}[\{|t| = m^*\}]$. In particular, $h \circ F^{-1}$ is analytic in $\{q^3 < |w| < 1\}$ so that $F_1 \circ F^{-1} = g \circ h \circ F^{-1}$ also so is. In other words, $F_1 \circ F^{-1}$ is analytically continuable inward by successive two times inversions across $\{|w| = q\}$. It is verified by induction that $z_n = F_n \circ F^{-1}(w)$ is analytically continuable inward by successive $2n$ times inversions across $\{|w| = q\}$ and hence it maps $\{q^{2n+1} < |w| < 1\}$ onto a subdomain of $\{0 < |z_n| < 1\}$ such that $\{|w| = 1\}$ corresponds to $\{|z_n| = 1\}$ and $F_n \circ F^{-1}(1) = 1$. Thus it follows from Theorem 3.2 that we have $|F_n \circ F^{-1}(w) - w| < 8q^{2n+1}$ in $R_q \subset \{q^{2n+1} < |w| < 1\}$, that is, $|F_n(z) - F(z)| < 8q^{2n+1}$ in D.

3.5. Some remarks. It is noted that the validity of convergence of $\{F_n\}$ on the closure $\bar{D}$ is especially assured by Theorem 3.3, since F_n and F are continuous on $\bar{D}$. On the other hand, Gaier [19] states a supplement to Theorem 3.3 that the factor 8 involved cannot be replaced by a number smaller than 4. To see this, we observe, for instance, an eccentric circular ring $D = l(R_q; a)$ obtained from R_q by a linear transformation

$$z = l(w; a) = \frac{1-\bar{a}}{1-a} \frac{w-a}{1-a\bar{w}},$$

where a is a fixed point satisfying $|a| < q$. It is readily verified that every D_n is also an eccentric circular ring and is really defined by $D_n = l(R_q; aq^{2n})$. The maximum error is then given by

$$\delta_n = \max_{z \in \bar{D}} |F_n(z) - F(z)| = \max_{w \in \bar{R}_q} |l(w; aq^{2n}) - w| = \max_{|w|=1} |l(w; aq^{2n}) - w|,$$

whence follows in particular case of $a = i|a|$, after direct calculation, $\delta_n = 4|a|q^{2n} + O(q^{4n})$ as $n \to \infty$. Since the parameter a is restricted only by the condition $|a| < q$, this leads to the desired result.

Finally, we remember that every step in the iteration process explained above consists of a mapping of a simply-connected domain onto a circle, which is based on Riemann's mapping theorem. It has been shown by Albrecht [1] that the step may be modified by making use of any osculating mapping (Schmiegungsabbildung) instead of circular mapping. On the other hand, some variants of the abovementioned iteration process have been obtained subsequently by O. Hübner (cf. Gaier [19]) and Landau [46]. The point of these methods is to insert an inversion in every step of the process. It is verified especially that the approximation is then considerably accelerated.

The iteration method can be generalized to constructing a mapping function of a multiply-connected domain onto a circular domain, that is, a domain bounded by disjoint whole circles. The detail of such a mapping (Vollkreisabbildung) is found in Gaier's book [21].

4. A one-parameter family of operators defined on analytic functions in a circle

As mentioned at the beginning of this report, the contents of this part appeared as an original paper [43] in the Conference Proceedings. So suffice it to state its title.

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