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Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978

PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

ON EXISTENCE AND NONEXISTENCE RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Introduction

In this paper we shall speak about existence and nonexistence results for initial value problems for equations of the form

$$(1) \qquad iu_t + \Delta u + f(|u|^2)u = 0, \quad i^2 = -1, \; u_t = \frac{\partial u}{\partial t},$$

where Δ is the n-dimensional Laplacian and f is a continuous real function. In the special case $f(s) = qs$, $q = \bar{q} = \text{const.}$, (1) is the dimensionless standard form of the nonlinear Schrödinger equation which has been sometimes called Ginsburg–Landau equation or recently also Zakharov–Shabat equation. The latter notation is due to the fact that Zakharov and Shabat [18] were the first to see that Cauchy's problem for the spatially one-dimensional Schrödinger equation can be solved globally by means of the inverse scattering method. This famous method was discovered by Gardner, Greene, Kruskal and Miura [4] and firstly applied to Cauchy's problem for the Korteweg–de Vries equation. Unfortunately the approach of Zakharov–Shabat does not seem to generalize neither to higher space dimensions nor to other functions f than $f(s) = qs$. Since we are interested in more general cases we do not go into details of the inverse scattering method here.

In the last decade, existence and nonexistence results for initial value problems for (1) have been published by many authors. In this paper we take into account existence results of Shabat [13], Strauss [15], Baillon, Cacenave & Figueira [1] and Ginibre & Velo [5] as well as nonexistence results of Talanov [16], Shabat [13], Zakharov, Sobolev & Synach [19], Kudrashov [8] and Glassey [6].

All the papers mentioned are concerned with Cauchy's problem for (1). Only little is known about solutions to (1) satisfying boundary conditions. Especially we are not aware of any nonexistence result for initial-boundary value problems for the nonlinear Schrödinger equation. So we shall restrict ourself also substantially to Cauchy's problem. Only in the end of this paper we shall give some existence and nonexistence results for simple one-dimensional initial-boundary value problems which include the n-dimensional spherically symmetrical case.

The paper consists of five sections. In the first one we introduce some notations and prove a lemma motivating in some respect the notion of solution used in the paper. Section 2 is devoted to a local existence theorem for the n-dimensional Schrödinger equation. A global existence theorem for the case $n \leqslant 3$ is proved in Section 3. Section 4 contains a nonexistence result connecting the space dimension n and the growth of the function f. Our results concerning one-dimensional initial-boundary value problems are given in Section 5.

1. Preliminaries

For a complex number z we denote by $|z|$, $\bar{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$ modulus, conjugate complex number, real and imaginary part, respectively. The letter c stands for various constants. C_0, L^p, W^l_p $(H^l = W^l_2)$ are the usual spaces of complex-valued functions defined on R^n (cf. [9], [17]). The symbols $(\cdot,\cdot)$ and $\|\cdot\|$, $\|\cdot\|_p$, $\|\cdot\|_{l,p}$ denote scalar product in $H = H^0 = L^2$ and norms in H, L^p, W^l_p, respectively.

We shall use the embedding theorem [10]

(1.1) $$W^l_r \subset W^{[l-n/r+n/p-\varepsilon]}_p, \quad 1 \leqslant r \leqslant p \leqslant \infty, \ \varepsilon > 0,$$
$$[a] = \text{integer part of } a,$$

and the inequality [11]

(1.2) $$\|D^j v\|_p \leqslant c\|v\|_q^{1-a}\|D^l v\|_r^a, \quad \forall v \in W^l_r \cap L^q,$$
$$1 \leqslant q,\ r \leqslant \infty, \quad j/l \leqslant a < 1, \quad 1/p = j/n + a(1/r - l/n) + (1-a)/q,$$

where $\|D^j v\|_p$ denotes the maximum of the L^p-norm of all jth derivatives $D^j v$ of v.

For a Banach space B and a time interval $[0, T)$ we denote by $C^l(0, T; B)$ $(C^l_w(0, T; B))$ the space of all on $[0, T)$ l-times continuously (weakly continuously) differentiable B-valued functions $(C(0, T; B) = C^0(0, T; B))$ and by $L^2(0, T; B)$ the space of the quadratically Bochner-integrable functions $u \in ((0, T) \to B)$.

We consider initial value problems of the form

(1.3) $$iu_t + \Delta u + f(|u|^2)u = 0, \quad u(0) = \varphi,$$

and assume throughout the function f to be real-valued and continuous on R^1_+. We define the functions

$$F(s) = \int_0^s f(r)\,dr, \quad s \geqslant 0, \qquad g(z) = f(|z|^2)z, \quad z \in C.$$

We look for solutions u of (1.3) belonging to spaces of the form

$$X^l(T) = C(0, T; H^l \cap L^\infty \cap W) \cap C^1(0, T; H),$$

where $W = \{v \in H \mid \int |x|^2 |v|^2 dx < \infty\}$, $\int = \int_{R^n}$.

These spaces turn out to be suitable for formulating existence as well as nonexistence results. Indeed, one of the main tools for proving such results is the

LEMMA 1. *Let $u \in X^2(T)$, $T > 0$, be a solution of (1.3). Set*

$$I_1(u(t)) = \int |u(t)|^2 dx, \qquad I_2(u(t)) = \int \left(|\nabla u(t)|^2 - F(|u(t)|^2)\right) dx,$$
$$e(t) = \int |x|^2 |u(t)|^2 dx.$$

Then for $t \leqslant T$ the following identities are valid:

(1.4) $$I_1(u(t)) = I_1(\varphi),$$

(1.5) $$I_2(u(t)) = I_2(\varphi),$$

(1.6) $$\dot{e}(t) = 4\operatorname{Im}\left(\nabla u(t), xu(t)\right), \qquad \dot{e} = de/dt,$$

(1.7) $$\ddot{e}(t) = 4\left(2I_2(u(t)) + \int \left((2+n)F(|u(t)|^2) - nf(|u(t)|^2)\,|u(t)|^2\right) dx\right).$$

Proof. The identity (1.4) ((1.5)) follows by multiplying (1.3) scalarly by u (u_t), taking the imaginary (real) part and integrating with respect to t.

In order to prove the remaining identities we denote by h a real function with the following properties

$$h \in C_0^\infty, \quad 0 \leqslant h(x) \leqslant 1, \qquad h(x) = 0 \text{ if } |x| \geqslant 2, \qquad h(x) = 1 \text{ if } |x| \leqslant 1,$$

and set

$$h_j(x) = h(x/j), \qquad v_j(x) = |x|^2 h_j(x), \qquad w_j(x) = xh_j(x), \qquad j = 1, 2, \ldots$$

By Lebesgue's theorem on dominated convergence we have for each $a \in L^1$

(1.8) $$\lim_{j\to\infty} \int h_j a\,dx = \int a\,dx$$

and

$$\lim_{j\to\infty}\Big|\int x\cdot \nabla h_j a\,dx\Big| \leqslant \lim_{j\to\infty}\int |x|\,|\nabla h_j|\,|a|\,dx = \lim_{j\to\infty}\int_{j\leqslant x\leqslant 2j} |x|\,|\nabla h_j|\,|a|\,dx \tag{1.9}$$

$$\leqslant \lim_{j\to\infty}\Big(2\max|\nabla h|\int_{j\leqslant x}|a|\,dx\Big) = 0.$$

Now we have

$$\begin{aligned}(v_j, |u|^2)_t &= 2\operatorname{Re}(v_j u_t, u) = -2\operatorname{Im}\big(v_j(\Delta u + g(u)), u\big)\\ &= -2\operatorname{Im}(\Delta u, v_j u) = 2\operatorname{Im}(\nabla u, \nabla v_j u + v_j \nabla u)\\ &= 2\operatorname{Im}\big(\nabla u, (\nabla h_j|x|^2 + 2w_j)u\big)\end{aligned}$$

and by (1.8), (1.9) and Lebesgue's theorem

$$\begin{aligned}e(t)-e(0) &= \lim_{j\to\infty}[(v_j, |u|^2)]_0^t = \lim_{j\to\infty}\int_0^t 2\operatorname{Im}\big(\nabla u, (\nabla h_j|x|^2 + 2xh_j)u\big)\,ds\\ &= 4\operatorname{Im}\int_0^t(\nabla u, xu)\,ds.\end{aligned}$$

Hence (1.6) follows by differentiating.

Now we want to prove (1.7). Taking into account that $u_t \in C^1(0, T; H^{-1})$, we obtain

$$\begin{aligned}\operatorname{Im}(u, w_j u)_t &= \operatorname{Im}\big((\nabla u_t, w_j u) + (w_j\cdot\nabla u, u_t)\big)\\ &= -\operatorname{Im}\big((u_t, \nabla\cdot w_j u + w_j\cdot\nabla u) - (w_j\cdot\nabla u, u_t)\big)\\ &= \operatorname{Im}(2w_j\cdot\nabla u + \nabla\cdot w_j u, u_t)\\ &= -\operatorname{Re}\big(2w_j\cdot\nabla u + \nabla\cdot w_j u, \Delta u + g(u)\big)\\ &= \operatorname{Re}\big(\nabla(2w_j\cdot\nabla u + \nabla\cdot w_j u), \nabla u\big) +\\ &\quad + \big(\nabla\cdot w_j, F(|u|^2) - f(|u|^2)|u|^2\big) =: E_j.\end{aligned}$$

Thus, using (1.6), (1.8), (1.9) and Lebesgue's theorem, we get

$$\begin{aligned}&1/4\big(\dot e(t) - \dot e(0)\big)\\ &= \lim_{j\to\infty}[\operatorname{Im}(w_j\cdot\nabla u, u)]_0^t = \lim_{j\to\infty}\int_0^t E_j\,ds\\ &= \int_0^t\big[-(n, |\nabla u|^2) + 2\|\nabla u\|^2 + (n\nabla u, \nabla u) + \big(n, F(|u|^2)\big) - f(|u|^2)|u|^2\big]\,ds\\ &= \int_0^t\Big[2I_2(|u|^2) + \int\big((2+n)F(|u|^2) - nf(|u|^2)|u|^2\big)\,dx\Big]\,ds.\end{aligned}$$

Hence (1.7) follows by differentiating.

Remark 1.1. For smooth solutions which decrease sufficiently rapidly as $x\to\infty$, Lemma 1 has been proved by Glassey [6].

2. Local existence

In this section we prove a local existence theorem for the Cauchy problem (1.3). Our technique of proof is essentially due to Shabat [13] who proved the existence of a unique local solution $u\in C^1(0, T; \mathcal{S})$ to (1.3) for the case $f(s) = qs$, $q = \bar q = \text{const.}$, where $\mathcal{S}$ is the Schwartz space of smooth functions decreasing rapidly as $x\to\infty$.

THEOREM 1. *Let $\varphi\in H^l\cap W$, $l\geqslant [n/2]+1$. Suppose that the function g is l-times continuously differentiable such that*

$$\|g(v)\|_{l,2}\leqslant \varrho(\|v\|_{l,2}),\qquad \forall v\in H^l, \tag{2.1}$$

where ϱ is a real nondecreasing locally Lipschitzian function on R^1_+. Then the problem (1.3) has a unique solution $u\in X^l(T_0)$, where $[0, T_0)$ is the existence interval of the solution y to the ordinary differential equation

$$\dot y(t) = \varrho\big(y(t)\big),\qquad y(0) = \|\varphi\|_{l,2}.$$

Proof. We rewrite (1.3) as equivalent integral equation

$$u(t) = U(t)\varphi + i\int_0^t U(t-s)g\big(u(s)\big)\,ds, \tag{2.2}$$

where U is the group generated by the operator $i\Delta u$, that is

$$U(t)\colon \varphi\to u(t) = U(t)\varphi,\qquad u_t = i\Delta u,\qquad u(0) = \varphi.$$

It is easy to see that

$$\|U(t)\varphi\|_{l,2} = \|\varphi\|_{l,2}\quad\text{and}\quad \|U(t)\varphi\|_W\leqslant c(\|\varphi\|_W + t\|\varphi\|_{l,2}). \tag{2.3}$$

We consider the iteration sequence $(u^k)\in X^l(T_0)$ defined by

$$u^{k+1}(t) = U(t)\varphi + i\int_0^t U(t-s)g\big(u^k(s)\big)\,ds,\qquad u^0 = \varphi,\qquad k = 1, 2, \ldots \tag{2.4}$$

From (2.1) and (2.3) it follows

$$\|u^{k+1}(t)\|_{l,2}\leqslant\|\varphi\|_{l,2} + \int_0^t\varrho\big(\|u^k(s)\|_{l,2}\big)\,ds,\qquad \|u^0\|_{l,2} = \|\varphi\|_{l,2}.$$

Besides (u^k) we define a sequence (y^k) by

$$y^{k+1}(t) = \|\varphi\|_{l,2} + \int_0^t\varrho\big(y^k(s)\big)\,ds,\qquad y^0(t) = \|\varphi\|_{l,2}.$$

It is easy to see that $\big(y^k(t)\big)$ is monotoneously increasing and that $y^k(t)\to y(t)$ as $k\to\infty$, $t<T_0$. Moreover, it follows by induction that

$$\|u^k(t)\|_{l,2}\leqslant y^k(t)\leqslant y(t),\qquad k = 1, 2, \ldots, t<T_0,$$

and in view of (1.1)

$$\|u^k\|_{C(0,T;L^\infty)}\leqslant c\|u^k\|_{C(0,T;H^l)}\leqslant c(T),\qquad T<T_0. \tag{2.5}$$

Further, taking into account (2.3)–(2.5), we get

$$\|u^{k+1}(t)\|_W \leqslant \|U(t)\varphi\|_W + \int_0^t \big\|U(t-s)\, g\big(u^k(s)\big)\big\|_W ds$$

$$\leqslant c(T)\Big(1+\int_0^t \|u^k(s)\|_W ds\Big).$$

Hence, using similar arguments as in the proof of (2.5), we deduce

(2.6) $$\|u^{k+1}\|_{C(0,T;W)} \leqslant c(T).$$

Finally, rewriting (2.2) as

(2.7) $$iu_t^{k+1} + \Delta u^{k+1} + g(u^k) = 0, \qquad u^{k+1}(0) = \varphi,$$

and using (2.5), we get the a priori estimate

(2.8) $$\|u_t^{k+1}\|_{C(0,T;H^{l-2})} \leqslant c(T).$$

Since the embedding from $H^l \cap W$ into H is compact, we can pass to the limit $k \to \infty$ in (2.7) because of (2.5), (2.6), (2.8) and obtain the theorem by standard arguments (cf. [2], [9]).

Remark 2.1. One can show (cf. the proof of Theorem 2) that in the case $n = l = 1$ the local solution to (1.3) guaranteed by Theorem 1 belongs in fact to $X^2(T_0)$, provided $\varphi \in H^2 \cap W$.

Remark 2.2. It follows from (1.2) by setting $a = j/l$, $q = \infty$ that the functions $f(s^2) = qs^p$ for integers $p = 0, 2, 4, \ldots$ or reals $p \geqslant l-1 \geqslant [n/2]$ fulfill the hypotheses of Theorem 1.

COROLLARY 1. *Under the hypotheses of Theorem* 1 *either* (1.3) *has a unique solution* $u \in X^l(T)$ *for each* $T > 0$, *or there exists a finite time* T_0 *such that* $\|u(t)\|_{l,2} \to \infty$ *as* $t \to T_0^-$. *Moreover, if* $n \leqslant 3$ *and* $l = [n/2]+1$ *then even* $\|u(t)\|_\infty \to \infty$ *as* $t \to T_0^-$.

Proof. From the proof of Theorem 1 the first statement follows immediately. To prove the last statement it suffices to show that $\|u(t)\|_\infty \leqslant c$, $t \leqslant T_0$, implies $\|u(t)\|_{l,2} \leqslant c$ provided $n \leqslant 3$, $l = [n/2]+1$. Let $n = 1$. We deduce from (1.4), (1.5)

$$\|u_x(t)\|^2 = I_2(\varphi) + \int F\big(|u(t)|^2\big)dx = I_2(\varphi) + \int \int_0^{|u(t)|^2} f(s)\, ds dx$$

$$\leqslant I_2(\varphi) + cI_1\big(u(t)\big) = I_2(\varphi) + cI_1(\varphi) = c.$$

For $n = 2, 3$ it follows from (1.3) that

$$\|\Delta u(t)\|^2 = \|\Delta\varphi\|^2 - 2\mathrm{Re}\big[\big(g(u), \Delta u\big)\big]_0^t + 2\,\mathrm{Re}\int_0^t \Big(\big(g(u)\big)_t, \Delta u\Big) ds$$

$$\leqslant 1/2\, \|\Delta u(t)\|^2 + c\Big(1 + \int_0^t \|u_t\|\, \|\Delta u\| ds\Big)$$

$$\leqslant 1/2\, \|\Delta u(t)\|^2 + c\Big(1 + \int_0^t \|\Delta u\|^2 ds\Big).$$

Thus Gronwall's lemma implies $\|u(t)\|_{2,2} \leqslant c(T_0)$ and the corollary is proved.

3. Global existence

In this section we prove for $n \leqslant 3$ the existence of a unique global solution to the Cauchy problem (1.3). Recently, similar results have been published by Baillon, Cacenave & Figueira [1] for $f(s^2) = qs^p$, $1 \leqslant p < 4/n$ and more general for twice continuously differentiable functions g satisfying estimates like

$$g(z) = c(1 + |z|^{p_1})|z|, \qquad 0 \leqslant p_1 < 4/n,$$
$$|g'(z)| \leqslant c(|z|^p + |z|^{p_2}), \qquad 1 \leqslant p \leqslant p_2 < 4/(n-2),$$

by Ginibre & Velo [5]. Whereas in [1] and [5] the global results are based on local ones like Theorem 1, we shall use a parabolic regularization technique which allows us to replace the restriction $p \geqslant 1$ by $p > \max\Big(0, \dfrac{n-2}{n+2}\Big)$.

THEOREM 2. *Let* $n \leqslant 3$ *and* $\varphi \in H^2 \cap W$. *Suppose that*

(3.1) $$f(s^2) \leqslant c(1 + s^{p_1}), \qquad \forall s \geqslant 0,\ 0 \leqslant p_1 < 4/n$$

and that g *is continuously differentiable. If* $n \geqslant 2$, *suppose in addition that*

(3.2) $$|g'(z)| \leqslant c(|z|^{p_2} + |z|^{p_3}) \quad \forall z \in C, \qquad \frac{na}{2+n} < p_2 \leqslant p_3 < \frac{2an}{n-2},$$

with $(n-2)/n < a < 2/n$.

Then the problem (1.3) *for each* $T < \infty$ *has a unique solution* $u \in X^2(T)$.

Proof. Let (φ_ε), $0 < \varepsilon \leqslant 1/2$, be a set of functions such that

(3.3) $$\varphi_\varepsilon \in C_0^\infty, \quad \varphi_\varepsilon \to \varphi \quad \text{in } H^2 \cap W \text{ as } \varepsilon \to 0.$$

For $r > 0$ we set

$$g_r(z) = \{g(z) \text{ if } |z| \leqslant r,\ g(rz/|z|) \text{ if } |z| > r\}.$$

We consider the regularized problems

(3.4) $$iu_t + (1 - i\varepsilon)\Delta u + g_r(u) = 0, \qquad u(0) = \varphi_\varepsilon.$$

Because of

$$\|g_r(v)-g_r(w)\| \leqslant 2\max_{|z|\leqslant r}|g'(z)|\,\|v-w\|, \qquad \forall v, \forall w \in H$$

the operator $v \to g_r(v)$ is Lipschitzian in H. Thus from results on parabolic equations (cf. [3], [7]) it follows that for each $T < \infty$ (3.4) has a unique solution $u_{\varepsilon r} \in C(0, T; H^2) \cap C^1(0, T; H)$. Moreover, it holds $u_{\varepsilon rt} \in L^2(0, T; H^2)$.

Now we want to show that for sufficiently large r $u_{\varepsilon r}$ is the solution to

$$iu_t + (1-i\varepsilon)\Delta u + g(u) = 0, \qquad u(0) = \varphi_\varepsilon. \tag{3.5}$$

Evidently, for this it suffices to find an a priori estimate for $u_{\varepsilon r}$ in $C(0, T; L^\infty)$ independent of r. Firstly we have (cf. (1.4))

$$\|u_{\varepsilon r}(t)\|^2 + \varepsilon\int_0^t \|\nabla u_{\varepsilon r}\|^2 ds = \|\varphi_\varepsilon\|^2. \tag{3.6}$$

Next we find using (1.2), (3.1), (3.3), (3.4) and (3.6)

$$\begin{aligned}\|\nabla u(t)\|^2 + \varepsilon\int_0^t \|\Delta u\|^2 ds &= \|\nabla\varphi_\varepsilon\|^2 + 2\,\mathrm{Im}\int_0^t \big(g_r(u), \Delta u\big)\,ds\\
&\leqslant c + \int_0^t \Big(c(\varepsilon)\|u\|_{2(p_1+1)}^{2(p_1+1)} + \tfrac{1}{2}\varepsilon\|\Delta u\|^2\Big)ds\\
&\leqslant c + \int_0^t \Big(c(\varepsilon)\|\nabla u\|^{p_1 n} + \tfrac{1}{2}\varepsilon\|\Delta u\|^2\Big)ds\\
&\leqslant c + \int_0^t \Big(c(\varepsilon)\|\nabla u\|^4 + \tfrac{1}{2}\varepsilon\|\Delta u\|^2\Big)ds.\end{aligned}$$

Hence, using (3.6) and Gronwall's lemma, we get

$$\|u_{\varepsilon r}\|_{C(0,T;H^1)} + \|u_{\varepsilon r}\|_{L^2(0,T;H^2)} \leqslant c(\varepsilon, T). \tag{3.7}$$

For $n = 1$ this implies

$$\|u_{\varepsilon r}\|_{C(0,T;L^\infty)} \leqslant c(\varepsilon, T).$$

Let now $n \geqslant 2$. Then it follows from (3.2), (3.3) and (3.7) that

$$\begin{aligned}&(1+\varepsilon^2)\|\Delta u(t)\|^2\\
&= (1+\varepsilon^2)\|\Delta\varphi_\varepsilon\|^2 + \int_0^t \Big(2\,\mathrm{Re}\big(\nabla g_r(u), (1-i\varepsilon)\nabla u_t\big) - \varepsilon\|\nabla u_t\|^2\Big)ds\\
&\leqslant (1+\varepsilon^2)\|\Delta\varphi_\varepsilon\|^2 + \int_0^t \Big(c\,\|(|u|^{p_2}+|u|^{p_3})\,|\nabla u|\|\,\|\nabla u_t\| - \varepsilon\|\nabla u_t\|^2\Big)ds\end{aligned}$$

$$\begin{aligned}&\leqslant c(\varepsilon)\Big(1+\int_0^t \||u|^{p_3}|\nabla u|\|^2 ds\Big) \leqslant c(\varepsilon)\Big(1+\int_0^t \|u\|_\infty^{2p_3}\|\nabla u\|^2 ds\Big)\\
&\leqslant c(\varepsilon)\Big(1+\int_0^t \|u\|_\infty^{2p_3} ds\Big) \leqslant c(\varepsilon)\Big(1+\int_0^t \|u\|_{2,2}^4 ds\Big).\end{aligned}$$

The latter inequality is clear for $0 \leqslant p_3 \leqslant 2$. If $2 < p_3 < 2an/(n-2)$, it follows from (1.2) with $p = \infty$, $j = 0$, $l = 2$, $r = 2$, $a = 2/p_3$, $q = (p_3-2)n/(4-n)$ $(< 2n/(n-2))$. Hence, taking into account (3.7), we get by Gronwall's lemma $\|u_{\varepsilon r}\|_{(0,T;H^2)} \leqslant c(\varepsilon, T)$. Because of (1.1) this implies the desired a priori estimate

$$\|u_{\varepsilon r}\|_{C(0,T;L^\infty)} \leqslant c(\varepsilon, T).$$

Now we want to prove some a priori estimates for the solution u_ε to (3.5) uniform with respect to ε. We suppress the subscript ε wherever misunderstandings are excluded.

(i) Because of

$$\begin{aligned}0 &= -2\,\mathrm{Re}\big(iu_t + (1-i\varepsilon)\Delta u + g(u), (1-i\varepsilon)u_t\big)\\
&= 2\varepsilon\|u_t\|^2 + \Big((1+\varepsilon^2)\|\nabla u\|^2 - \int F(|u|^2)\,dx\Big)_t + 2\varepsilon\,\mathrm{Im}\big(g(u), u_t\big),\end{aligned}$$

we obtain by (1.2), (3.1) and (3.6)

$$\begin{aligned}(1+\varepsilon^2)\|\nabla u(t)\|^2 &\leqslant c + \int F\big(|u(t)|^2\big)dx + \varepsilon\int_0^t \|g(u)\|^2 ds\\
&\leqslant c\Big(1 + \|u(t)\|_{p_1+2}^{p_1+2} + \varepsilon\int_0^t \|u\|_{2(p_1+1)}^{2(p_1+1)} ds\Big)\\
&\leqslant c\Big(1 + \|\nabla u(t)\|^{np_1/2} + \varepsilon\int_0^t \|\nabla u\|^{np_1} ds\Big).\end{aligned}$$

Since $np_1 < 4$, we conclude from this and (3.6) by Gronwall's lemma that

$$\|u_\varepsilon\|_{C(0,T;H^1)} \leqslant c(T). \tag{3.8}$$

(ii) Let $h \in C_0^\infty$ be a real function as in the proof of Lemma 1. For $j = 1, 2, \ldots$ we define $w_j(x) = |x|\,h(x/j)$ and conclude from (3.5) and (3.8)

$$\begin{aligned}\|w_j u(t)\|^2 + 2\varepsilon\int_0^t \|w_j\nabla u\|^2 ds &= \|w_j\varphi_\varepsilon\|^2 + 2\,\mathrm{Im}(1-i\varepsilon)\int_0^t (\nabla u, 2w_j\nabla w_j u)\,ds\\
&\leqslant c\Big(1+\int_0^t (\|\nabla u\|^2 + \|w_j u\|^2)\,ds\Big)\\
&\leqslant c\Big(1+\int_0^t \|w_j u\|^2 ds\Big).\end{aligned}$$

Gronwall's lemma yields $\|w_j u_\varepsilon\|_{C(0,T;H)} \leqslant c(T)$. Hence by Fatou's lemma it follows

(3.9) $$\|u_\varepsilon\|_{C(0,T;W)} \leqslant \|u_\varepsilon\|_{C(0,T;H)} + \||x|\,u_\varepsilon\|_{C(0,T;H)} \leqslant c(T).$$

(iii) Now we wish to prove the estimate

(3.10) $$\|u_\varepsilon\|_{C(0,T;L^\infty)} \leqslant c(T).$$

For $n = 1$ (3.10) follows from (3.8) and (1.2). In order to verify (3.10) for $n = 2, 3$ we introduce the operators

$$U_\varepsilon(t)\colon\ v \to U_\varepsilon(t)v = u_\varepsilon(t), \qquad u_{\varepsilon t} = i(1-i\varepsilon)\,\Delta u, \qquad u(0) = v.$$

It is easy to see that

(3.11) $$\|U_\varepsilon(t)v\|_{l,2} \leqslant \|v\|_{l,2} \qquad \forall v \in H^l,\ t \geqslant 0.$$

Moreover, we have the representation (cf. [12])

$$\big(U_\varepsilon(t)v\big)(x) = \big(4\pi i(1-i\varepsilon)t\big)^{-n/2} \int \exp\Big(i\,|x-y|^2/\big(4(1-i\varepsilon)t\big)\Big)\,v(y)\,dy$$

from which we see that

(3.12) $$\|U_\varepsilon(t)v\|_\infty \leqslant (2\pi t)^{-n/2}\,\|v\|_1, \qquad \forall v \in L^1.$$

On account of the Riesz–Thorin theorem (cf. [12]) (3.11) and (3.12) imply

(3.13) $$\|U_\varepsilon(t)v\|_q \leqslant ct^{n(1/2-1/p)}\|v\|_p, \qquad \forall v \in L^p,\ 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty,\quad 1/p+1/q = 1.$$

Now we rewrite (3.5) in the equivalent form

$$u(t) = U_\varepsilon(t)\varphi_\varepsilon + i\int_0^t U_\varepsilon(t-s)\,g\big(u(s)\big)\,ds.$$

Using (1.2), (3.2), (3.8), (3.9) and (3.13) with $q = 2/(1-a) \in \big(n, 2n/(n-2)\big)$, assuming without loss of generality that $p_2/a < 1$, $2 \leqslant p_3/a$, we get with $D = \partial/\partial x_j$, $j = 1, \ldots, n$,

$$\begin{aligned}
\|Du(t)\|_q &\leqslant \|U_\varepsilon(t)D\varphi_\varepsilon\|_q + \int_0^t \big\|U(t-s)\,Dg\big(u(s)\big)\big\|_q\,ds\\
&\leqslant c\Big(\|U_\varepsilon(t)D\varphi_\varepsilon\|_{1,2} + \int_0^t (t-s)^{n(1/2-1/p)}\,\|g'(u)\,Du\|_p\,ds\Big)\\
&\leqslant c\Big(\|D\varphi_\varepsilon\|_{1,2} + \int_0^t (t-s)^{-an/2}\big\|\big(|u|^{p_2}+|u|^{p_3}\big)\,|Du|\big\|_p\,ds\Big)\\
&\leqslant c\Big(\|\varphi_\varepsilon\|_{2,2} + \int_0^t (t-s)^{-an/2}\big(\|u\|^{p_2}_{2p_2/a}\|Du\| + \|u\|^{p_3}_{p_3/a}\,\|Du\|_q\big)\,ds\Big)
\end{aligned}$$

$$\begin{aligned}
&\leqslant c\Big(1 + \int_0^t (t-s)^{-an/2}\big(\|u\|^{p_2}_W\,\|(1+|x|)^{2p_2/(p_2-a)}\|_1^{(a-p_2)/2} + \|Du\|_q\big)\,ds\Big)\\
&\leqslant c\Big(1 + \int_0^t (t-s)^{-an/2}\,\|Du\|_q\,ds\Big).
\end{aligned}$$

Hence it follows that $\|Du(t)\|_q \leqslant y(t)$, where $y(t)$ is the solution of the integral equation

$$y(t) = c\Big(1 + \int_0^t (t-s)^{-na/2}\,y(s)\,ds\Big).$$

(Note that $an/2 < 1$ by assumption.) Since $q > n$, (1.1) implies (3.10).

(iv) Finally we need the estimate

(3.11) $$\|u_\varepsilon\|_{C(0,T;H^2)} + \|u_{\varepsilon t}\|_{C(0,T;H)} \leqslant C(T).$$

Using (3.8), (3.10) and

(3.12) $$\|u_{\varepsilon t}\| \leqslant (1+\varepsilon)\,\|\Delta u_\varepsilon\| + \|g(u_\varepsilon)\| \leqslant c(1+\|\Delta u_\varepsilon\|)$$

we conclude from (3.5) that

$$\begin{aligned}
0 &= 2\,\mathrm{Re}\int_0^t \big(iu_t + (1-i\varepsilon)\,\Delta u + g(u),\ (1-i\varepsilon)\,\Delta u_t\big)\,ds\\
&= \int_0^t \Big(\varepsilon\,\|\nabla u_t\|^2 - 2\,\mathrm{Re}\big((g(u))_t,\ (1-i\varepsilon)\,\Delta u\big)\Big)\,ds +\\
&\quad + (1+\varepsilon^2)\big(\|\Delta u(t)\|^2 - \|\Delta\varphi_\varepsilon\|^2\big) + 2\,\mathrm{Re}\big[(1+i\varepsilon)\big(g(u),\ \Delta u\big)\big]_0^t\\
&\geqslant -c\Big(1 + \int_0^t \|g'(u)\,u_t\|\,\|\Delta u\|\,ds\Big) + \tfrac{1}{2}\|u(t)\|^2\\
&= -c\Big(1 + \int_0^t \|\Delta u\|^2\,ds\Big) + \tfrac{1}{2}\|\Delta u(t)\|^2.
\end{aligned}$$

Thus Gronwall's lemma and (3.12) imply (3.11).

We are now going to take the limit $\varepsilon \to 0$. Noting that the embedding from $H^1 \cap W$ into H is compact, we conclude from (3.11) by means of a well-known compactness lemma (cf. [9], I, Theorem 5.1) that the set (u_ε) is compact in $L^2(0, T; H)$. Consequently, there exist a sequence $(u_j) = (u_{\varepsilon_j})$, $\varepsilon_j \to 0$, and a function u such that

$$\begin{gathered}
u \in C_w(0, T; H^2 \cap W) \cap C(0, T; H^1) \cap C^1_w(0, T; H),\\
u_j \to u \ \text{(strongly) in } L^2(0, T; H),\\
u_j \rightharpoonup u \ \text{(weakly) in } L^2(0, T; H^2 \cap W),\\
u_{jt} \rightharpoonup u_t \ \text{in } L^2(0, T; H).
\end{gathered}$$

By standard arguments (cf. [2], [9], [14]) one shows that u is the (unique) solution of (1.3) and that in fact $u \in X^2(T)$.

Theorem 2 is proved.

COROLLARY 2. *Let $\varphi \in H^2$. Suppose the functions f and g fulfill the hypotheses of Theorem 2 with $p_2 \geqslant a$. Then for each $T < \infty$ the problem (1.3) has a unique solution $u \in C(0, T; H^2) \cap C^1(0, T; H)$.*

Proof. Let (φ_j) be a sequence with $\varphi_j \in H^2 \cap W$, $\varphi_j \to \varphi$ in H^2, and let u_j be the solution of (1.3) corresponding to the initial value φ_j. It is easy to check that u_j satisfies a priori estimates like (3.8), (3.10) and (3.11). Now for $u_{jk} = u_j - u_k$ we find by (3.10)

$$\|u_{jk}(t)\|^2 \leqslant \|\varphi_{jk}\|^2 + c\int_0^t \|u_{jk}\|^2 ds$$

and by Gronwall's lemma $u_{jk} \to 0$ as $j, k \to \infty$. Thus the sequence (u_j) is compact in $C(0, T; H)$ and we can proceed as in the proof of Theorem 2.

Remark 3.1. Evidently the functions $f(s^2) = qs^p$ with $\max\big(0, (n-2)/(n+2)\big) < p < 4/n$ $\big(\max\big(0, (n-2)/n\big) < p < 4/n\big)$ satisfy the hypotheses of Theorem 2 (Corollary 2). (Clearly, the assertions of Theorem 2 and Corollary 2 hold also for the linear case $p = 0$.)

Remark 3.2. As we shall show in the next section, global solutions to (1.3) do not exist in general for $p \geqslant 4/n$. Nevertheless, for initial values φ with sufficiently small L^2-norm solutions may exist globally. The existence of such "small" solutions for $f(s^2) = qs^p$ Baillon and *al.* [1] proved for $n = p = 2$ and $n = 3$, $p = 4/3$ and Strauss [15] for sufficiently large p.

4. Nonexistence

In this section we prove a blow up result for $X^2(T)$-solutions to Cauchy's problem (1.3). Apparently the first nonexistence result for the nonlinear Schrödinger equation is due to Talanov [16] (cf. also Shabat [13]), who found an explicite example of a solution to (1.3) blowing up in finite time for $n = 2$ and $f(s^2) = s^2$. A nonexistence result for the spherically symmetrical case when $n = 3$ and $f(s^2) = s^2$ has been given by Zakharov, Sobolev & Synach [19]. More recently Kudrashov [8] and Glassey [6] have independently proved blow up results for sufficiently smooth solutions to (1.3). The main tool of all the mentioned papers are identities like those we have stated in Lemma 1.

THEOREM 3. *Suppose that the function f satisfies*

$$(2+n)F(s) \leqslant nsf(s) \quad \forall s \geqslant 0. \tag{4.1}$$

Suppose that $u \in X^2(T)$, $T > 0$, is a solution to (1.3) and $I_2(\varphi) = b < 0$. Then necessarily $T \leqslant t_0$, where t_0 is the positive root of the equation

$$4bt^2 + \dot{e}(0)t + e(0) = 0 \tag{4.2}$$

and $e(0) = \||x|\varphi\|^2$, $\dot{e}(0) = 4\operatorname{Im}(\nabla\varphi, x\varphi)$. If in addition the function $g(z) = f(|z|^2)z$ satisfies the hypotheses of Theorem 1, then there exists a $T_0 \in (0, t_0]$ such that

$$\|u(t)\|_{l,2} \to \infty \quad as \quad t \to T_0^-.$$

Moreover, if $n \leqslant 3$ and $l = [n/2]+1$, we have

$$\|u(t)\|_\infty \to \infty \quad as \quad t \to T_0^-.$$

Proof. We shall show that the hypothesis $T > t_0$ leads to a contradiction. From (4.1) and Lemma 1 it follows that

$$e(t) = e(0) + \int_0^t \Big(\dot{e}(0) + \int_0^s \ddot{e}(\tau)d\tau\Big) ds \leqslant e(0) + \dot{e}(0)t + 4bt^2.$$

Consequently, there exists a $t_1 \leqslant t_0$ such that $e(t_1) = 0$ and thus $u(t_1) = 0$. But this contradicts the fact that $\|u(t_1)\|^2 = \|\varphi\|^2 > 0$ as a consequence of $I_2(\varphi) \neq 0$.

The remaining statements follow immediately from Corollary 1.

Remark 4.1. Evidently the function $f(s^2) = qs^p$, $q = \bar{q} > 0$, satisfies (4.1) if $p \geqslant 4/n$.

The following proposition covers our results concerning the case $f(s^2) = qs^p$, $n \leqslant 3$.

PROPOSITION 1. *Let $n \leqslant 3$ and $p > \min\big(0, (n-2)/(n+2)\big)$. Suppose $\varphi \in H^2 \cap W$ and $I_2(\varphi) = b < 0$. Then the Cauchy problem*

$$iu_t + \Delta u + q|u|^p u = 0, \quad u(0) = \varphi, \ q > 0,$$

for each $T > 0$ has a unique solution $u \in X^2(T)$ if and only if $p < 4/n$. If $p \geqslant 4/n$ then there exists a unique local solution. This solution blows up in finite time $T_0 \leqslant t_0$ such that

$$\|u(t)\|_\infty \to \infty \quad as \quad t \to T_0^-.$$

Here t_0 is the positive solution of equation (4.2).

Remark 4.2. Evidently, for arbitrary $\varphi \in H^2$, $\varphi \neq 0$, the function $\varphi_\lambda = \lambda\varphi$ for sufficiently large $|\lambda|$ satisfies

$$I_2(\varphi_\lambda) = \int \Big(|\nabla\varphi_\lambda|^2 - \frac{2q}{p+2}|\varphi_\lambda|^{p+2}\Big)dx < 0, \quad \text{if} \quad p, q > 0.$$

5. One-dimensional initial-boundary value problems

In this section we carry over some results of the preceding sections to initial-boundary value problems. Unfortunately we can handle only one-dimensional problems (apart from the n-dimensional spherically symmetrical case). The proof at least of global existence and non-existence theorems for higher dimensional initial-boundary value problems seems to require new ideas. Especially, we do not see how to prove suitable equivalents for the identity (1.7) and the estimate (3.13) when boundary conditions are posed and $n \geqslant 1$.

We consider initial-boundary value problems of the form

$$iu_t + u_{xx} + f(|u|^2)u = 0, \quad u(0) = \varphi, \tag{5.1}$$

$$u_x(t, 0) = ku(t, 0), \quad u(t, 1) = 0, \quad 0 \leqslant k = \bar{k} \leqslant \infty, \tag{5.2}$$

and set now

$$H = L^2(0, 1), \quad (v, w) = \int_0^1 v\bar{w}\,dx, \quad H^2 = H^2(0, 1),$$

$$V = \{v \in H^2 |\ v_x(0) = kv(0),\ v(1) = 0\};$$

$$Y(T) = C(0, T; V) \cap C^1(0, T; H).$$

LEMMA 1'. *Let $u \in Y(T)$, $T > 0$, be a solution of* (5.1), (5.2) *and*

$$I_1(u(t)) = \int_0^1 |u(t)|^2 dx, \quad I_2(u(t)) = \int_0^1 \big(|u_x(t)|^2 - F(|u(t)|^2)\big)dx + k|u(t, 0)|^2,$$

$$e(t) = \int_0^1 x^2 |u(t)|^2 dx.$$

Then for $t \leqslant T$ the following identities hold:

$$I_1(u(t)) = I_1(\varphi), \quad I_2(u(t)) = I_2(\varphi), \quad \dot{e}(t) = 4\,\mathrm{Im}\,(xu_x(t), u(t)),$$

$$\ddot{e}(t) = 4\Big[2I_2(u(t)) + \int_0^1 \big(3F(|u(t)|^2) - f(|u(t)|^2)|u(t)|^2\big)dx - \\ - |u_x(t, 1)|^2 - k|u(t, 0)|^2\Big].$$

The proof of this lemma is analogous but easier than the proof of Lemma 1.

THEOREM 1'. *Let $\varphi \in V$. Suppose the function g is continuously differentiable such that*

$$\|g(v)\|_{1,2} \leqslant \varrho(\|v\|_{1,2}) \qquad \forall v \in H^1$$

with a function ϱ as in Theorem 1. *Then the problem* (5.1), (5.2) *has a unique solution $u \in Y(T_0)$, where $[0, T_0)$ is the existence interval of the solution to the ordinary differential equation*

$$\dot{y}(t) = \varrho(y(t)), \quad y(0) = (1 + \|\varphi\|_{1,2}^2 + k|\varphi(0)|^2)^{1/2}.$$

Sketch of the proof. As in the proof of Theorem 2 we find that the regularized problem

$$iu_t + (1 - i\varepsilon)u_{xx} + g_r(u) = 0, \quad u(0) = \varphi_\varepsilon \in C^\infty(0, 1) \cap V, \quad \varphi_\varepsilon \to \varphi \text{ in } V$$

has a unique solution $u_{\varepsilon r} \in Y(T_0)$ satisfying

$$\big(\|u_{\varepsilon r}(t)\|_{1,2}^2 + k|u_{\varepsilon r}(t, 0)|^2\big)_t \leqslant 2\,\mathrm{Im}\,\big(g_r(u_{\varepsilon r}), u_{\varepsilon rxx}\big)(t) \\ \leqslant 2\big(\|(g_r(u_{\varepsilon r}))_x\|\,\|u_{\varepsilon rx}\|\big)(t) \\ \leqslant 2\varrho\big(\|u_{\varepsilon r}(t)\|_{1,2}\big)\|u_{\varepsilon r}(t)\|_{1,2}.$$

Since $\|\varphi_\varepsilon\|_{1,2}^2 + k|\varphi_\varepsilon(0)|^2 \leqslant (y(0))^2$ for sufficiently small ε, we get the a priori estimate

$$\|u_{\varepsilon r}(t)\|_{1,2}^2 + k|u_{\varepsilon r}(t, 0)|^2 \leqslant (y(t))^2, \quad t < T_0,$$

which implies

$$\|u_{\varepsilon r}(t)\|_\infty \leqslant y(t).$$

The remainder of the proof is essentially the same as that of Theorem 2.

From Theorem 1' we deduce immediately

COROLLARY 1'. *Under the hypotheses of Theorem* 1' *the problem* (5.1), (5.2) *either has a unique solution $u \in Y(T)$ for each $T < \infty$ or there is a finite time T_0 such that $\|u(t)\|_\infty \to \infty$ as $t \to T_0^-$.*

THEOREM 2'. *Let $\varphi \in V$. Suppose that*

$$f(s^2) \leqslant c(1 + s^p), \quad s \geqslant 0, \ 0 \leqslant p < 4,$$

and that g is continuously differentiable. Then the problem (5.1) *has a unique solution $u \in Y(T)$ for each $T \in [0, \infty)$.*

The proof proceeds substantially as that of Theorem 2 for $n = 1$.

Using Lemma 1' and Corollary 1' one can prove the following non-existence result in much the same way as Theorem 3.

THEOREM 3'. *Suppose the function f satisfies*

$$3F(s) \leqslant sf(s), \quad s \geqslant 0.$$

Suppose $u \in Y(T)$, $T > 0$, is a solution to (5.1), (5.2) *and $I_2(\varphi) = b < 0$. Then $T \leqslant t_0 < \infty$, where t_0 is the positive root of the equation $4bt^2 + \dot{e}(0)t + e(0) = 0$ and $e(0) = (x^2, |\varphi|^2)$, $\dot{e}(0) = 4\,\mathrm{Im}\,(x\varphi_x, \varphi)$.*

If, in addition, the function g satisfies the hypotheses of Theorem 1' *then there exists a $T_0 \leqslant t_0$ such that $\|u(t)\|_\infty \to \infty$ as $t \to T_0^-$.*

Remark 5.1. Theorem 3' yields also a nonexistence result for the initial value problem (5.1) under the boundary conditions

$$u_x(t, 0) = ku(t, 0), \quad u_x(t, 1) = -ku(t, 1). \tag{5.3}$$

Indeed, let φ be an initial value satisfying these conditions with $\varphi(x) = -\varphi(1-x)$, $0 \leqslant x \leqslant 1$. For a solution u of (5.1), (5.3) evidently $v(t,x) = -u(t, 1-x)$ is also solution. Suppose u is unique. Then we have $u = v$ and, especially, $u(t, 1/2) = v(t, 1/2) = -u(t, 1/2)$, that is $u(t, 1/2) = 0$. Thus the problem (5.1), (5.3) for special initial values is reduced to a problem like (5.1), (5.2).

Similarly as (5.1), (5.2), we can handle the following n-dimensional spherically symmetrical initial-boundary value problem:

$$iu_t + x^{1-n}(x^{n-1}u_x)_x + f(|u|^2)u = 0, \quad u(0) = \varphi, \tag{5.5}$$

$$u_x(t, 0) = 0, \quad u(t, 1) = 0. \tag{5.6}$$

THEOREM 3″. *Suppose the function f satisfies*

$$(n+2)F(s) \leqslant nsf(s), \quad s \geqslant 0.$$

Suppose $u \in C(0, T; H^2) \cap C(0, T; H)$, $T > 0$, is a solution to (5.5), (5.6) *and*

$$I_2(\varphi) = \int_0^1 \big(|\varphi_x|^2 - F(|\varphi|^2)\big) x^{n-1} dx = b < 0.$$

Then the first statement of Theorem 3′ holds with $e(0) = (x^{n+1}, |\varphi|^2)$ and $\dot{e}(0) = 4\,\mathrm{Im}(x^n \varphi_x, \varphi)$.

Sketch of the proof. The theorem follows essentially from the identities

$$I_1\big(u(t)\big) = \big(x^{n-1}, |u(t)|^2\big) = I_1(\varphi), \quad I_2\big(u(t)\big) = I_2(\varphi),$$

$$\ddot{e}(t) = 4\Big[2I_2\big(u(t)\big) + \Big(\int_0^1 \big((2+n)F(|u|^2) - nf(|u|^2)|u|^2\big) x^{n-1} dx\Big)(t) - |u_x(t, 1)|^2\Big].$$

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Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978