JENS FREHSE

ON THE SMOOTHNESS OF SOLUTIONS OF VARIATIONAL INEQUALITIES WITH OBSTACLES

0. Introduction

The first three sections of this contribution are devoted to the question of the regularity of solutions of scalar variational inequalities with obstacles, that is, to problems of the type:

\[ (0.1) \quad \text{Find } u \in K = \{ v \in H^1_0(\Omega) \mid v \geq \psi \text{ in } \Omega \} \text{ such that } \]
\[ \sum_{i,j} a_{ij}(x,u,\nabla u) + \partial_i u - \partial_j \psi \leq 0 \]

for all \( v \in K \).

Here \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), \( H^1_0(\Omega) \) the usual Sobolev space of functions \( u \) which have a generalized gradient in \( L^2(\Omega) \) and vanish on \( \partial \Omega \) in the generalized sense. The scalar product in \( L^2(\Omega) \) is denoted by \( (\cdot,\cdot) \), i.e., \( (u,v) = \int_\Omega u v \). \( \partial_i \) denotes the identity map. The inequality sign \( \geq \psi \) in the definition of \( K \) is to be understood in the sense of \( H^1_0 \), cf. [23] or [37], or in the sense "almost everywhere" (which may be quite different).

We shall assume natural growth and ellipticity conditions for the functions \( a_{ij} \), cf. §1 and §2. For a sufficiently smooth obstacle \( \psi \), say, for \( \psi \in H^{p}\)(\( \Omega \)) (i.e. \( \psi \) having bounded second derivatives), the question of the regularity of solutions of (0.1) has been essentially solved. From the general regularity theory due to Brézis-Stampacchia [8] one obtains that \( u \in H^{p}\)(\( \Omega \)) for all \( p < \infty \), and the final step yielding \( u \in H^{p}\)(\( \Omega \)) was performed in [15], [16], [21], [9]. It is well known that the further regularity condition \( u \in C^{1}(\Omega) \) is false, in general. Cf. also [2], [24], [26], [37] for many other results on regularity and historical remarks.
The results on the regularity of \( u \) are not so complete in the case of an obstacle \( \psi \) being less regular, say, for \( \psi \in C^0(D) \) or \( \psi \in C^{1/2}(D); \, 0 < \alpha \leq 1 \). Here \( C^{1/2}(D) \) is the space of functions on \( \Omega \) which satisfy an interior Hölder condition with exponent \( \alpha \); and \( C^{\mu, \beta} \) is defined by an analogous condition imposed on the first derivatives of the functions. The study of the regularity of solutions \( u \) of (0.1) have thus achieved some importance because of the theory of quasi-variational inequalities, i.e., variational inequalities where the obstacle depends on the unknown function, cf. [2], [3] for examples. The implicit obstacles in the theory of quasi-variational inequalities have a priori a less degree of regularity and thus it is of importance to have regularity theorems for variational inequalities with obstacles having rather measurability regularity properties. The most surprising results in this direction seems to be the one announced in [20]. It states that the solution of an elliptic variational inequality with a discontinuous monotone obstacle (more generally, a one sided Hölder continuous obstacle) is Hölder continuous.

In this paper (§1–§3) we present regularity results for (0.1) with obstacles in \( C^0 \) or \( C^{1/2} \). These results will be used for the study of quasi-variational inequalities, cf. [20], but are also of interest in themselves. In §1 we prove that the solutions of (0.1) are Hölder continuous with some exponent \( \mu \in [0, 1] \) if \( \psi \in C^0 \) for some \( a \). Different sets of conditions are considered; note that we treat also the case of the lower order term \( a_0(a, u, \nabla u) \) having quadratic growth in \( \nabla u \). In §2 we present results of the type like that \( \psi \in C^0 \) implies \( u \in C^0, \, 0 < \alpha \leq 1 \). In §3 we restrict attention to the case of the Laplacian and obtain a corresponding conclusion for \( C^{1/2}, \, 0 < \alpha \leq 1 \).

We do not discuss the obstacle problem for non-linear systems of variational obstacle, since the question of regularity of solutions is not solved satisfactorily yet, even in the case of equations, i.e., without obstacles. In the case of two dimensions, \( C^{1/2} \)-regularity results for the solution of systems of variational inequalities with a non-diagonal principal part have been first presented in [18]. Furthermore, we do not discuss questions concerning other types of obstacles, e.g., thin obstacles, boundary obstacles, or obstacles for \( \nabla u \).

The last section (§4) is devoted to a discussion of the regularity properties of solutions of the polyharmonic variational inequality. We present a simple proof of the boundedness of the second derivatives of the biharmonic variational inequality, assuming only that the second derivatives of the obstacle are bounded from below. Throughout the paper, we shall use the following notations and conventions.

\[ \int = \text{integration over } \Omega. \]

\( H^{\alpha,p}(\Omega) = \text{Sobolev space.} \)

The elements of \( H^{\alpha,p}(\Omega) \) are equivalence classes of real valued functions which are defined in \( \Omega \) up to a set of \( m \)-capacity zero and have generalized derivations in \( L^p \) up to order \( m \) (or up to a set of \( s \)-dimensional Lebesgue measure zero). Two functions lie in the same equivalence class if they coincide in \( \Omega \) except on a set of \( m \)-capacity zero (or of measure zero, respectively). If we consider the elements of \( H^{\alpha,p}(\Omega) \) as (classes of) functions defined up to capacity zero we may understand the inequality \( u \geq \psi \) as \( u \in H^{\alpha,p}(\Omega) \) in the sense "everywhere in \( \Omega \) except a set of \( m \)-capacity zero". The space \( H^{\alpha,p}(\Omega) \) denotes the closure of the test functions with respect to the \( H^{\alpha,p} \)-norm. If \( u \in H^{\alpha,p}(\Omega) \), we consider \( u \) also as a function in \( H^{\alpha,p}(\mathbb{R}^n) \) which vanishes outside \( \Omega \).

For open subsets \( \Omega_1, \Omega_2 \) we write \( \Omega_1 \subset \subset \Omega_2 \) if the closure \( \overline{\Omega_1} \) of \( \Omega_1 \) is contained in \( \Omega_2 \).

\( B_r(x) = \text{ball in } \mathbb{R}^n \text{ of radius } r \text{ with center } x. \)

In the estimates considered in subsequent sections we shall frequently use the same letters \( K, K_1, K \) etc. for different constant \( c \) (a constant which does not depend on the relevant parameters).

The \( m \)-capacity \( (m \text{-cap} E) \) of a closed set \( E \) is defined as

\[ \inf \left\{ \int |P^{m,p}(\varphi)| \, d\lambda : \varphi \in C^0(\Omega), \, \varphi \geq 1 \text{ on } E \right\} \]

and, for an arbitrary set \( E \),

\[ m \text{-cap} E = \sup \left\{ m \text{-cap} K : K \subset E, \, K \text{ closed} \right\}. \]

Here \( Q \) is a ball containing \( E \); \( m \text{-cap} E \) depends also on \( Q \) (which is not relevant).

1. On the continuity of solutions of scalar variational inequalities with obstacles

In this section we consider scalar variational inequalities of the type

(1.1) \( \text{Find } u \in K = \{ v \in H^1(\Omega) : v \geq \psi \text{ in } H^1 \} \text{ such that } \sum_{\rho \in \mathcal{R}} a_\rho(u, v, \nabla u) \leq 0 \)

for all \( v \in K \) such that \( u - v \in L^p(\Omega) \).

Here we assume the following conditions concerning the data \( a_\rho, \psi \):

(1.2) \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \).

(1.3) \( \psi \in C(\overline{\Omega}). \)

The functions \( a_\rho : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) have the following properties:

(1.4) \( a_\rho(a, u, \nabla u) \) is measurable with respect to \( a \in \Omega \) and continuous with respect to \( (u, \nabla u) \in \mathbb{R} \times \mathbb{R}^n \).

\( \mathcal{R} \) denotes the set of rational numbers.
There is a constant $K = K_0$ such that for all $x \in \Omega$, $y \in \mathbb{R}^n$, $u \in \mathbb{R}$, $|u| \leq C$,

$$|a_i(x, y, y)| \leq K|y| + K,$$

and

$$|a_i(x, y, y)| \leq K|y|^2 + K.$$

There are constants $c = c(C) > 0$ and $K = K_0$ such that for all $x \in \Omega$, $y \in \mathbb{R}^n$, $u \in \mathbb{R}$, $|u| \leq C$,

$$\sum_{i=1}^n a_i(x, u, y) y_i \geq c|y|^2 - K.$$

It is well known that the conditions (1.4)–(1.6) guarantee the local Hölder continuity of bounded weak solutions of the equation

$$-\sum_{i=1}^n \delta a_i(x, u, \phi u) + a_i(x, u, \phi u) = 0,$$

cf. the book of Ladyženskaya–Ural’ceva [23]. As regards the Hölder continuity of $u$ up to the boundary of $\Omega$ the following condition is sufficient in the case of equations (1.7) $\partial \Omega$ is Lipschitz continuous.

We shall also be concerned with a weaker Wiener-type condition:

There are positive numbers $c_0$ and $B_0$ such that for all $a_i \in \partial \Omega$, $0 < B < B_0$,

$$\int \{|y|^2 dy| \in C_{B_2(B_0)}(\partial \Omega) \}, \quad \varphi \geq 1 \quad \text{on} \quad \Omega \quad \text{Boundedness of} \quad a_i \quad \text{in} \quad \mathcal{L}^1_{\mu} \quad \text{is contained in} \quad H^1(\Omega).$$

The boundary condition is satisfied since $u \in H^1(\Omega)$ and $k > 0$. Furthermore, $e = e(k, \delta) > 0$ the function $\zeta = \delta \max(|\zeta - k, 0)|/|1 + \delta \max(|\zeta - k, 0)|$ is monotone increasing in $\zeta$. Hence we obtain, in view of $\varphi > y$,

$$\psi = \psi - e \max(y - k, 0),$$

and, therefore, $\psi \in K$.

Thus we may insert the above function $e$ into the variational inequality, obtaining

$$\sum_{i=1}^n a_i(x, u, \phi u) + a_i(x, u, \phi u) \geq 0.$$

We estimate the second summand by using the one-sided condition (1.11):

$$a_i(x, u(x), \phi u(x)) \geq -c_1|\phi u(x)|^2 - K|u(x)|^2 + K.$$
We multiply the last inequality by the factor

$$a(x) = w_{kl}(x)/u(x), \quad u(x) \neq 0.$$ 

Since $k \geq k_0$, we have

$$0 < a(x) \leq 1$$

(even for negative $u(x)$). This yields

$$a_k(x) \geq \frac{u_k(x)}{u(x)} \geq \frac{u_k(x)}{u(x)} - c_1 \left| K \right| \left| u_k(x) - K X_k \right|$$

where $X_k$ is the characteristic function of the set

$$A_k = \{ x \in \Omega \mid u(x) > k \}.$$ 

We apply this estimate to inequality (1.12). Then the term $a_k u_{kl}$ disappears and we may pass to the limit $\delta \to 0$. We obtain

$$\sum_{k \geq 1} \frac{a_k(t, u, V u)}{u_k(u)} - c_1 \int \left| V u \right| u_k dx = K \int u_k |u| dx - K \left| A_k \right|$$

where $u_k = \max(u - k, 0)$, $\left| A_k \right|$ is the Lebesgue measure of $A_k$.

The first term in (1.13) is estimated via the coerciveness condition (1.10) and we obtain

$$\sum_{k \geq 1} \left( a_k - c_1 \right) \int \left| V u_k \right| dx \leq K \left| A_k \right| + K \int u_k |u| dx + K \int u_k |\phi| dx.$$ 

(Recall that $c_1 > 0$.

We rewrite and estimate the term $\int u_k |\phi| dx$ in the following way:

$$\int u_k |\phi| dx = \int u_k |\phi| + 2K \int u_k |\phi| - k^2 |A_k| \leq \int u_k |\phi| + k^2 |A_k| + \frac{1}{2} \int u_k |\phi| dx$$

and hence

$$\int u_k |\phi| dx \leq 2 \int u_k |\phi| + 2K \left| A_k \right|.$$ 

The term $\int u_k |\phi| dx$ is treated in a similar fashion. Thus we conclude that

$$\int \left| V u_k \right| dx \leq K \left| A_k \right| + K \int u_k |\phi| dx.$$ 

According to Sobolev's inequality,

$$\left( \int u_k |\phi| d\nu \right)^{\frac{2}{2+n}} \leq K \int \left| V u_k \right| dx$$

where $\gamma = n/(n-2)$ for $n \geq 2$ and $\gamma > 1$, say $\gamma = 2$, for $n = 2$. Consequently

$$\left( \int u_k |\phi| d\nu \right)^{\frac{2}{2+n}} \leq K \left| A_k \right| + K \int u_k |\phi| dx.$$ 

By Hölder's inequality,

$$\int u_k |\phi| dx \leq \left| A_k \right|^{\frac{2}{2+n}} \left( \int u_k |\phi| d\nu \right)^{\frac{n}{2+n}}.$$ 

For $k \geq k_0 = c_0 |\nu_k|$ we have

$$K \left| A_k \right|^{1-\frac{1}{2}} \leq \frac{1}{2}$$

and we conclude that

$$\int u_k |\phi| d\nu \leq K \left| A_k \right|^{1-\frac{1}{2}}$$

Using again Hölder's inequality we obtain

$$\int u_k |\phi| dx \leq \left| A_k \right|^{1-\frac{1}{2-n}} \left( \int u_k |\phi| d\nu \right)^{\frac{n}{2+n}}$$

and in view of (1.14)

$$\int u_k |\phi| dx \leq K \left| A_k \right|^{1-\frac{1}{2-n}}, \quad \sigma = 1/2 - 1/(2\gamma) > 0, \ k \geq k_0.$$ 

Now, a lemma from [23] (Lemma 5.1, Section 2, p. 71), states that (1.15) implies the boundedness of $u$ from above by a constant depending on $K$, $k_0$, $\sigma$, and $|\nu_k|$.

Since $u \geq \psi$ on $\Omega$, we infer that $u$ is also bounded from below on $\Omega$. This completes the proof of the theorem.

Proof of Remark (iii) to Theorem 1.1. In the above proof we used only the fact that there exists a constant $k_0$ such that $k \geq k_0$ and we saw that the solution $u$ of (1.1) is bounded from above. No further regularity of $\psi$ is needed. The boundedness of $u$ from below follows, since every solution $u$ of (1.1) is also a supersolution, i.e.,

$$\sum_{k \geq 1} \left( a_k - c_1 \right) \left( \int u_k |\phi| d\nu \right)^{\frac{2}{2+n}} \leq K \left| A_k \right| + K \int u_k |\phi| dx.$$ 

We now turn to the discussion of the continuity of bounded solutions $u$ of (1.1). Theorem 1.2 below asserts the interior Hölder continuity of $u$, provided that

$$\psi \in H^{1,p}(\Omega) \text{ for some } p > n,$$

and the Hölder continuity of $u$ up to the (regular) boundary of $\Omega$ if (1.16) holds if and only if

$$\psi \in C^{1}_0(\Omega).$$

Condition (1.17) may be replaced by

$$\psi \in C^{1}_0(\Omega) \text{ for some } p > n,$$

and the Hölder continuity of $u$ up to the (regular) boundary of $\Omega$ if (1.16) holds if

$$\psi \in C^{1}_0(\Omega).$$

The set $K$ contains a function $u_0 \in C^1(\Omega) \cap H^{1,p}(\Omega)$

with suitable constants $\alpha \in [0, 1], \ p > n$.

We consider this result (as well as Theorem 1.1) as a corollary to the general theory of quasilinear elliptic equations [23, 29], since the techniques of proof can be adapted easily. The question whether $u$ remains
continuous when (1.16) is replaced by the condition \( \psi \in \mathcal{C}(\Omega) \), is more difficult, but it has an affirmative answer (cf. Theorem 1.3 where an additional regularity condition is assumed).

**Theorem 1.2.** Under the assumptions (1.2)–(1.6) and (1.14) every solution \( u \in L^p(\Omega) \) of (1.3) is locally Hölder continuous in \( \Omega \) with an exponent \( a \in [0, 1] \). If, in addition, the assumptions (1.7) and (1.17) or (1.18) hold, the solution \( u \) is Hölder continuous up to the boundary of \( \Omega \).

**Remark.** The proof yields an a priori estimate for the Hölder norm in terms of the data and \(|b|_{\infty, 1}\).

**Proof.** Let \( B_1 \subset \Omega \) be a ball of radius \( \varepsilon \) and let

\[
M = \varepsilon \max_{\Omega} \{ |u - \psi|(|x|) \} \quad \text{and} \quad m = \varepsilon \min_{\Omega} \{ |u - \psi|(|x|) \} \quad \forall x \in B_1.
\]

Since \( u \in \mathcal{K} \), we have \( m > 0 \), and for \( l \in [m, M] \) the minimum and the maximum of the two numbers \( |u - \psi|(|x|) \), \( l \) satisfy

\[
(l - m) \leqslant |u - \psi|(|x|) \leqslant (l + m) \quad \forall x \in B_1.
\]

Let \( \tau \) be a Lipschitz continuous function such that

\[
\text{supp} \tau \subset B_2, \quad 0 \leqslant \tau \leqslant 1, \quad \tau = 1 \text{ on } B_{r_{\text{conv}}}, \quad \text{and} \quad |\nabla \tau| \leqslant (\varepsilon a)^{-1},
\]

where \( B_{r_{\text{conv}}} \) is concentric to \( B_1 \), \( 0 < \sigma < 1 \).

For \( \varepsilon \in \mathbb{R} \) define

\[
\begin{align*}
\| \varepsilon \|_+ &= \max \{ |\varepsilon - l|, 0 \}, & \| \varepsilon \|_- &= \min \{ |\varepsilon - l|, 0 \}.
\end{align*}
\]

By (1.19)

\[
\varepsilon_1 := \varepsilon - \tau \varepsilon \varepsilon [u - \psi] \geqslant \psi, & \quad \varepsilon_2 := \varepsilon + \tau \varepsilon [u - \psi] \geqslant \psi.
\]

Thus \( \varepsilon_1, \varepsilon_2 \in \mathcal{K} \) and we may insert the functions \( \varepsilon_1, \varepsilon_2 \) into the variational inequality, obtaining

\[
0 \leqslant \sum_{i=1}^n a_i \left( \varepsilon_i, u, \nabla u \right) + \delta_1 \tau [u - \psi]^2 \|
\]

From this we conclude by a routine analysis, employing the conditions (1.5), (1.6) and \( u \in C^2(\Omega) \),

\[
\int_{\Omega} |u - \psi| \tau |2 \varepsilon_2| \|
\]

where \( \mathcal{A} \) denotes integration over \( \mathcal{A} \).

The last summand in (1.20) of \( (\mathcal{A} \varepsilon_1)^2 \|
\]

We set \( \delta = (2K)^{-1} \) and consider only those numbers \( l \) which satisfy \( (\Omega - \delta) \|
\]

Then (1.20) implies

\[
\int_{\Omega} |u - \psi| \tau |2 \varepsilon_2| \|
\]

Since \( \psi \in \mathcal{L}^p, \|
\]

Using the properties of \( \tau \) we arrive at the inequality

\[
\int_{\Omega} |u - \psi| \tau |2 \varepsilon\|
\]

for \( l \in [M - \delta, M], \|
\]

Here \( \mathcal{A} \) denotes the maximum (respectively, the minimum) of the set \( \mathcal{A} \).

Inequality (1.21) is just the statement that the function \( u - \psi \) is contained in the class

\[
\mathcal{B}(\Omega, (\mathcal{A} \varepsilon_1)^2, K, \delta, 1/p).
\]

as defined in the book of Ladyženskaya–Ural’ceva [22], Section 2.6, p. 81. Therefore, by Theorem 6.1, Section 2.6, p. 90, from [22], the function \( u - \psi \), and hence \( u \), is Hölder continuous on interior domains.

The proof of the boundary regularity presents further technical difficulties and is carried out under an additional natural assumption (1.22), (1.23) below, concerning the principal part of the differential operator. Condition (1.22) allows us to treat the case of an obstacle \( \Omega \) which is merely continuous. This is done in Theorem 1.3 below.

In subsequent considerations we need additional differentiability and ellipticity conditions:

\[
\mathcal{A} \mathcal{E}(\Omega, (\mathcal{A} \varepsilon_1)^2, K, \delta, 1/p).
\]

for all \( x \in \Omega, \mu \in \mathbb{R}, \eta \in \mathbb{R}^p, \xi \in (\mathbb{E}_1, \ldots, \mathbb{E}_n) \in \mathbb{R}^n, \mathcal{A} \mathcal{E}(\Omega, (\mathcal{A} \varepsilon_1)^2, K, \delta, 1/p).
\]
THEOREM 1.3. Under the assumptions (1.2)-(1.6), (1.22), (1.23) every solution \( u \in L^p(\Omega) \) of (1.1) is continuous in \( \Omega \). If, in addition, the assumptions (1.8) and (1.18) are fulfilled, then \( u \in C(\overline{\Omega}) \).

Note that we assume no more regularity of the obstacle \( \psi \) than the mere continuity.

For the proof of Theorem 1.3 we need

Lemmata 1.1. Under the assumptions (1.2)-(1.6), (1.22), (1.23) every solution \( u \in L^p(\Omega) \) satisfies the inequalities

\[
\begin{align*}
\int |F(u)(\cdot)|^n|\omega-x|^{n-\delta}dx & \leq K, \quad n \geq 3, \\
\int |F(u)(\cdot)|^n|\ln|\omega-x||dx & \leq K, \quad n = 2,
\end{align*}
\]

uniformly for all \( x \in \Omega, \omega \in \Omega \) with \( K = K(\Omega) \) denoting a constant depending only on the data. If (1.8) and (1.18) are fulfilled, then (1.24) is satisfied for \( x \in \Omega \).

Remark. The assumption \( \psi \in C(\overline{\Omega}) \) in Lemma 1.1 can be replaced by the assumption that \( \psi \) is bounded from above.

Proof of Lemma 1.1. Let \( a_i(x) = \int a_i(x, u(x), F(u)(x))dx \) for \( x \in \Omega \) and \( a_i(x) = \delta_{ij} \) for \( x \in H^\infty - \Omega, \delta_{ij} = 1 \) for \( i = k, \delta_{ij} = 0 \) for \( i \neq k; i, k = 1, \ldots, n \). Let \( Q \supset \Omega \) be a fixed ball. We define the regularized Green function \( G_\epsilon = G_\epsilon(\epsilon, x, \Omega) = 0, \quad x \in \Omega, \) by the conditions: \( G_\epsilon \in H^1(0) \) and

\[
\sum_{i,k=1}^n (a_{ik} \delta_{ij} G_{ij}(x) - \frac{1}{2} |B_\epsilon|^{-1} \int \phi dx, \quad x \in C^\infty_0(Q).
\]

Here the parentheses \( (\cdot, \cdot) \) denote the scalar product in \( L^2(Q) \); the symbol \( \int \phi dx \) denotes integration over \( B_\epsilon(x) \). The assumptions (1.22), (1.23) guarantee that such a function \( G_\epsilon \) exists and we shall make use of the following properties of \( G_\epsilon \):

\[
G_\epsilon \in L^p(\Omega), \quad G_\epsilon \geq 0,
\]

\[
FG_\epsilon \to FG \text{ weakly in } L^p(\Omega), \quad 1 \leq p < n/(n-1) \quad (\epsilon \to 0),
\]

\[
G_\epsilon \to G \text{ strongly in } L^p(\Omega), \quad 1 \leq r < n/(n-2) \quad (\epsilon \to 0).
\]

Here \( G = G^* \) is the continuous Green function defined by \( G \in H^2(\Omega) \), \( 1 \leq s < n/(n-1) \), and

\[
\sum_{i,k=1}^n (a_{ik} \delta_{ij} G_{ij}(x) - \frac{1}{2} |B_\epsilon|^{-1} \int \phi dx, \quad x \in C^\infty_0(Q),
\]

\[ ((\psi, u) = \int_\Omega F(u)dx). \]

The function \( G \) satisfies the inequalities

\[
\begin{align*}
\epsilon|x-y|^{n-\delta} & \leq G(x) \leq K|x-y|^{n-\delta}, \quad n \geq 3, \\
\epsilon|\ln|x-y|| & \leq G(x) \leq K|\ln|x-y||, \quad n = 2,
\end{align*}
\]

for all \( x \in \Omega \) and \( u \in C(\overline{\Omega}) \), with some constants \( K, \epsilon > 0 \). For the proofs cf. [38].

We first treat the local part of Lemma 1.1.

Let \( \tau \) be a nonnegative Lipschitz continuous function with support in \( Q \), assume \( \tau = 1 \) in a neighborhood of \( x \) and \( \tau \leq 0 \) in a neighborhood of \( y \). Let \( \epsilon > 1 \) be a number; it will be specified later on. We observe that for small \( \epsilon \equiv \epsilon(\|u\|_\infty, \|F\|_\infty, \|G\|) > 0 \) the function \( f \) defined by

\[
f(\xi) = \epsilon^{-1}G(\epsilon^{-1}x)
\]

is monotone increasing in \( \xi \). Hence \( f(u(x)) \geq f(\psi(x)) = \psi(x) \) and

\[
u_i^x = u_i - \epsilon^{-1}G(\epsilon^{-1}x) \in K.
\]

We insert this function \( u_i^x \) into the variational inequality (1.1) and cancel the factor \( \epsilon > 0 \), obtaining

\[
\sum_{\epsilon} \int_{B_\epsilon(x)} (a_{ik} \delta_{ij} u + a_{ik} \tau G_{ij}(\epsilon^{-1}x)) \leq 0.
\]

We now use the identity

\[
a_i(\cdot, \cdot, \psi) = \sum_{\epsilon=1}^n (a_{ik} \delta_{ij} u + a_{ik} \tau G_{ij}(\epsilon^{-1}x))
\]

insert this into (1.29) and estimate the lower order terms via the growth conditions (1.5). This yields

\[
\sum_{\epsilon=1}^n (a_{ik} \delta_{ij} u + a_{ik} \tau G_{ij}(\epsilon^{-1}x)) \leq K_2 \int |F(u)| + \epsilon^{-1}G(\epsilon^{-1}x) \leq K_2 \epsilon^{-1}G(\epsilon^{-1}x) + K_2.
\]

The constants \( K \) and \( K_2 \) are uniform for \( x \in \Omega, \epsilon \to 0 \).

In order to obtain (1.30) we used the fact that \( (1.25), (1.26) \) imply a uniform bound for \( \int \phi dx, \int \phi G dx \), and that \( u \in L^p, a_i(\cdot, \cdot, u, 0) \in L^p \) etc.

From the ellipticity condition (1.23) and Young's inequality (3ab \leq a^2 + b^2) we conclude

\[
\sum_{\epsilon=1}^n (a_{ik} \delta_{ij} u + a_{ik} \tau G_{ij}(\epsilon^{-1}x)) \geq (\epsilon + g)^{-1} \int |F(u)| [u - k]^{n-\delta} G \leq K_2.
\]

\[ (\epsilon \psi, u) = \int_\Omega F(u)dx \]
By the definition of $\mathcal{G}_t$, the second summand on the right hand side of (1.31) is non-negative and can be dropped. This yields

\begin{equation}
(1.32) \quad (q - 1) \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \\
\leq K \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K_q,
\end{equation}

(We passed from $\mathcal{K}_q$ to $\mathcal{K}_q$, etc.). We apply Young's inequality to the first term on the right hand side of (1.32) and obtain

\begin{equation}
(1.33) \quad (q - 1) \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq \frac{K}{r} \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K_q
\end{equation}

uniformly for $q \to 0$, $u \in \Omega$. The constant $K$ does not depend on $q$. We now choose the number $q$ so as to have

\begin{equation}
q = 1, \quad K \int |u - k| \, ds = 1
\end{equation}

and we conclude

\begin{equation}
\int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq K.
\end{equation}

Passing to the limit $q \to 0$ we obtain

\begin{equation}
\int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq K
\end{equation}

uniformly for $x \in \Omega$.

Inequality (1.34) implies that

\begin{equation}
\int \frac{1}{q} |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq K,
\end{equation}

where $\int$ denotes integration over the set

\begin{equation}
\{ x \in \Omega : |u - k| > 1 \}.
\end{equation}

We look once more at (1.33) and now we choose $q = 1$. This yields

\begin{equation}
\int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq \frac{1}{r} \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K
\end{equation}

\begin{equation}
\leq K \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K_q,
\end{equation}

and in view of (1.36)

\begin{equation}
\int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq K \int |Vu|^q |u - k|^{2r-1}G_t^r \, ds + K_q
\end{equation}

\begin{equation}
(g \to 0).
\end{equation}

Taking $l = \frac{1}{r} K^{-1}$ we obtain the uniform bound

\begin{equation}
\int |Vu|^q |u - k|^{2r-1}G_t^r \, ds \leq K
\end{equation}

and the local result (1.24) follows by passing to the limit $q \to 0$ and by (1.27), (1.28).

The estimate (1.24) up to the boundary follows easily if we consider the test function

\begin{equation}
u_s = u - s \mathcal{G}_s[u - u_s], \quad s \geq 1,
\end{equation}

which belongs to $K$ for small $s > 0$. Note that $u - u_s = 0$ outside of $\Omega$. One has to proceed as before, replacing the term $[u - k]$ by the term $[u - u_s]$. Since $\nu_s$ is not necessarily a constant function, there occur certain error terms of the type $K \int |Vu| |Vu|^q |u - k|^{2r-1}G_t^r \, ds$ in our calculations and estimates. However, these terms are bounded uniformly because of (1.25) and (1.26) and our assumption $\nu_s \in L^q$ uniformly. The lemma is proved.

**Proof of Theorem 1.3.** We first prove the interior continuity. Let $\tau$ be a Lipschitz continuous function such that

\begin{equation}
supp \tau \subset B_R(\sigma) \subset \Omega, \quad 0 \leq \tau \leq 1, \quad |Vu| \leq K, \quad \tau = 1 \text{ on } B_R(\sigma).
\end{equation}

Let $G_\tau$ be the function defined in the proof of Lemma 1.1, and let $g \in H^1(\Omega) \cap L^\infty(\Omega), g \geq 0$, be a function, to be defined later on. Since $u \geq \psi$ in $H^1(\Omega)$, or "a.e.", we conclude that for small $\varepsilon = \varepsilon(\tau, g) > 0$

\begin{equation}
\tau a(x) - c(x) g(x) G_\tau(x) (u(x) - u(y) - \psi(x) - \psi(y)) \geq \varepsilon a(x)
\end{equation}

for all $x, y \in \Omega$ except a set of capacity zero (or of measure zero if the inequality $u \geq \psi$ is understood in the sense "almost everywhere").

Let $u_R$ and $\psi_R$ be the average of $u$ and $\psi$ taken over the set $B_R(\sigma)$. From (1.37) we obtain, by averaging over $B_R(\sigma)$ over $B_R(\sigma)$ with respect to $y$,

\begin{equation}
w - c(x) \psi_R G_\tau(w - u_R - \psi + \psi_R) \geq \psi.
\end{equation}

Since $\psi$ is continuous on $\Omega$, we have

\begin{equation}
|\psi - \psi_R| \leq \delta \quad \text{on } B_R(\sigma), \quad R \leq R_0
\end{equation}

where $\delta$ does not depend on $x \in \Omega$. From this and (1.38) we conclude that

\begin{equation}
u_s = u - c(x) G_\tau(u - u_R - \delta) \geq \psi, \quad R \leq R_0
\end{equation}

We now set $\delta = |u - u_R - \delta|^{-1}$, where $\delta \geq 1$ is an exponent which will be specified later on. From (1.39) and the fact that supp $\tau \subset \Omega$ we obtain that

\begin{equation}
u : = u - c(x) G_\tau(u - u_R - \delta) \in K
\end{equation}

where $[\cdot]^2$ is defined as in the proof of Lemma 1.1.

We insert the function $\nu$ into the variational inequality (1.1) and obtain,

\begin{equation}
\sum_{i=1}^n \left[ \partial_{ij}(\gamma, u, Vu) - \partial_{ij}(\tau c(x) G_\tau(u - u_R - \delta)) \right] \leq 0.
\end{equation}
We split the principal part of the differential operator as in the proof ofLemma 1.1. This yields
\[ (1.41) \sum_{i,j=1}^{n} \{ a_{ij} \partial_{ij} u, \partial_{i} (\tau \varphi \partial_{j} u \partial_{j} u - \tilde{u} \tilde{u}) \} \]
\[ \leq \sum_{i,j=1}^{n} \{ a_{ij} \partial_{ij} u, \partial_{i} (\tau \varphi \partial_{j} G_{s} \partial_{j} u - \tilde{u} \tilde{u}) \} \]
\[ \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \sigma \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} . \]

The first summand on the right hand side of (1.41) can be estimated by $K \delta$, $0 < \beta < 1$, since $a_{ij}, u, \varphi \in L^{\infty}$, $\varphi \in L^{p}$, $| \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta$, and $| \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta$, $1 < \delta < n-(n-1)$, uniformly as $\tau \to 0$. The second summand can be estimated via the growth condition (1.5). The left hand side of (1.41) is estimated from below using the ellipticity condition (1.23). Thus we arrive at the inequality
\[ (1.42) \quad (q-1) \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta, \]
\[ \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \sigma \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} . \]

We use the identity
\[ \sum_{i,j=1}^{n} \{ a_{ij} \partial_{ij} u, \partial_{i} (\tau \varphi \partial_{j} G_{s} \partial_{j} u - \tilde{u} \tilde{u}) \} = | B_{\rho} (s) \lor B_{\rho} (s) | \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \]
where $\int_{s}$ denotes integration over the ball $B_{\rho}(s) = B_{\rho}(s)$.

Thus we obtain (employing also Young's inequality)
\[ (q-1) \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta. \]

By the properties of $\tau$ and $G_s$, it is known (23, 38) that $\int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}$ is bounded uniformly.

Since $u \in L^{\infty}, G_{s} \in K \delta + K$ (cf. (38)), the terms $\int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}$ are uniformly bounded for $\tau \to 0$, $s \in \Omega$. Therefore we deduce from (1.43) that
\[ (1.44) \quad (q-1) \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}, \]
\[ \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}. \]

We apply the same trick as in the proof of Lemma 1.1. We first choose $\delta$ large enough to have $(q-\delta) \geq 2 \kappa$, and obtain the bound
\[ \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}. \]

This implies
\[ (1.45) \quad \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}, \]
\[ \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}. \]

Note that we splitted the integral $\int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}$ into two integrals, the one over $A_{u}$, the other one over $\Omega - A_{u}$, and we proceeded estimating as in the proof of Lemma 1.1.

Passing to the limit $\tau \to 0$ we conclude from (1.45) that
\[ (1.46) \quad \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}. \]

By (1.27) we have $| \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}.

(1.47) $\int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}.

where $\chi(M)$ is the characteristic function of a set $M$. (1.47) follows from the inequality
\[ \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} \leq K \delta + K \int | \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p} + K \int | \tau \varphi \partial_{i} G_{s} \partial_{i} u | ^{p}.
The second summand of the last inequality is estimated by the right hand side of (1.47). This estimate follows from the identity
\[(\text{1.48}) \quad \sum_{i=1}^{n} a_{ii} \bar{\partial}_{i}(u - \bar{u})^{2} + \bar{\partial}_{i}(u^{2}) = 0,\]
where \( \bar{\partial}_{i} = 1 \) on \( B_{R}(x) - B_{R}(z) \), \( \bar{\partial}_{i} = 0 \) on \( B_{R}(x) \) and \( R^{2} - B_{R}(z) \), and \( |\bar{\partial}_{i}| \leq K \delta^{-1} \).

Employing ellipticity and Young's inequality we obtain (1.47).

Thus we arrive at the inequality
\[(\text{1.49}) \quad \int_{\Omega} |\nabla u|^{2} + |u|^{2} + |u - \bar{u}|^{2} \\leq K \delta + K \delta^{2-n} \int_{\Omega} |u - \bar{u}|^{2} + K \int_{\Omega} |\nabla u|^{2} \\leq K \delta + K \delta^{2-n} \int_{\Omega} |u - \bar{u}|^{2} + K \int_{\Omega} |\nabla u|^{2} \\leq K \delta + K \int_{\Omega} |\nabla u|^{2},\]
for Lebesgue points \( x \in \Omega \), \( \Omega \subseteq \Omega \), \( R \subseteq R_{0} \). Here \( \int_{\Omega} \) denotes integration over \( B_{R}(x) - B_{R}(z) \).

By Poincaré's inequality
\[\int_{\Omega} |u - \bar{u}|^{2} \\leq K \delta \int_{\Omega} |\nabla u|^{2}\]
and we can simplify (1.49) to
\[(\text{1.50}) \quad \int_{\Omega} |\nabla u|^{2} + |u - \bar{u}|^{2} \\leq K \delta + K \int_{\Omega} |\nabla u|^{2},\]
where \( \int_{\Omega} \) denotes integration over \( B_{R}(x) \).

Using the "hole-filling trick" [39] we conclude from (1.50) that
\[\int_{\Omega} |\nabla u|^{2} \\leq \delta + \theta \int_{\Omega} |\nabla u|^{2}, \quad \theta = K(1+1) < 1.\]

On replacing \( R \) by \( 2R \) this becomes
\[(\text{1.51}) \quad \int_{\Omega} |\nabla u|^{2} \\leq \delta + \theta \int_{B_{R}(x)} |\nabla u|^{2}.\]
From (1.51) we obtain by iteration
\[\int_{\Omega} |\nabla u|^{2} \\leq K \delta + (R/2)|\nabla|^{2} \int_{B_{2R}(x)} |\nabla u|^{2}, \quad R \leq R_{0} \subseteq R_{0},\]
with a certain constant \( a = a(\theta) \in [0, 1] \). Choosing \( R \) small enough, we hence conclude
\[(\text{1.52}) \quad \int_{\Omega} |\nabla u|^{2} \\leq K \delta, \quad R \leq R_{0}, \quad B_{R_{0}}(x) \subseteq \Omega, \text{ uniformly for } x \in \Omega.\]

Applying this to (1.50) we obtain
\[(\text{1.53}) \quad |u - \bar{u}|^{2} \\leq K \delta, \quad R \leq R_{0}, \quad B_{R_{0}}(x) \subseteq \Omega,\]
for Lebesgue points \( x \in \Omega \) and
\[(\text{1.54}) \quad |u(x) - \bar{u}|^{2} \\leq K \delta + |u - \bar{u}|^{2}, \quad R \leq R_{0}, \quad B_{R_{0}}(x) \subseteq \Omega.\]

where \( \bar{u}(x) \) denotes the mean value of \( u \) over the set \( B_{R}(x) - B_{R}(y) \).

If \( |x - y| < R \), we can estimate
\[|u - \bar{u}|^{2} \\leq K \delta + |u - \bar{u}|^{2} \int_{B_{R}} |\nabla u|^{2} \\leq K \delta \int_{B_{R}} |\nabla u|^{2} \\leq K \delta \]
and we conclude from (1.54), for a given \( \delta < 0 \), that
\[|u(x) - \bar{u}(x)| \\leq K \delta, \quad R \leq R_{0}, \quad B_{R_{0}}(x) \subseteq \Omega.\]

for Lebesgue points \( y, x \in \Omega \). This proves the statement about the interior continuity of \( u \). The continuity of \( u \) up to the boundary follows from Lemma 1.2 below. We are stating it in a separate lemma, because we need this fact also in further sections.

Lemma 1.2. Under the assumptions (1.2), (1.4)-(1.6), (1.8) and (1.13) every solution \( u \in L^{\infty}(\Omega) \) is continuous at the points of \( \partial \Omega \).

Note that we do not assume any regularity of the obstacle \( \psi \) besides the existence of the function \( u_{0} \) in condition (1.13). If \( \psi \leq -a < 0 \) in a neighbourhood of \( \partial \Omega \) and \( \psi \) is bounded from above, then such a function \( u_{0} \) exists and Lemma 1.2 implies that the solution \( u \) of (1.1) does not touch the obstacle \( \psi \) in a neighbourhood of \( \partial \Omega \). This neighbourhood can be chosen uniformly for all obstacles \( \psi \) with a common upper bound and a common neighbourhood \( U(\partial \Omega) \) where \( \psi \leq -a_{0} \).

Proof of Lemma 1.2. Let \( G_{1}, \tau, \text{ and } n \) be defined as in the foregoing proof. Then, for small \( \varepsilon > 0 \), the function
\[\varepsilon = u - \alpha_{\psi}|u - u_{0}|^{2} \in K, \quad \varepsilon \geq 1,\]
is an admissible variation. We insert this function \( \varepsilon \) into the variational inequality and we obtain, just as in the proof of the interior continuity and the boundedness of \( \int_{\Omega} |\nabla u|^{2} \), that
\[(\text{1.55}) \quad \int_{\Omega} |\nabla u|^{2} + |u - \bar{u}(x)|^{2} \\leq K \delta + K \int_{\Omega} |\nabla u|^{2} \]
\[\leq K R^{2} + K \int_{\Omega} |\nabla u|^{2} \\leq K \delta.\]

The symbol \( \int_{\Omega} \) denotes integration over \( B_{R_{0}}(x) - B_{R_{0}}(z) \).

The difference in the proofs consists in the fact that \( u_{0} \) is not necessarily constant and the error terms of the type \( \int_{\Omega} |\nabla u_{0}|^{2} \), \( \int_{\Omega} |\nabla u_{0}|^{2} \), etc. occur. However, these terms can be estimated by \( K \delta R^{2} \), since \( \int_{\Omega} |\nabla u_{0}|^{2} \), for some \( p > n \) and \( \psi \in L^{p} \) for all \( r < n/(n-1) \). Since \( u - u_{0} = 0 \) on \( \partial \Omega \) and \( \partial \Omega \) satisfies the Wiener condition (1.13), we may apply Poincaré's
inequality and conclude that
\[ \int_R |u - u_0|^{2^*} dx \leq K \int |V u - V u_0|^{2^*} dx \leq K \int |V u|^{2^*} dx + K |R| \]
for those \( \varepsilon \in \Omega \) for which \( B_{R|u|}(\varepsilon) \cap \Omega \neq \emptyset \). Thus (1.55) is simplified to
\[ (1.56) \quad \int_R |V u|^{2^*} G dx + \int \{|u| - u_0\}^{2^*} \leq K |R|^{2^*} + K \int |V u|^{2^*} G dx. \]
(Recall that \( c|e - e_0|^n \leq G(\varepsilon) \leq K|e - e_0|^n \), \( n \geq 3 \).)

From (1.56) we obtain via the hole-filling technique that
\[ \int_{B_{\varepsilon}} |V u|^{2^*} G dx \leq K_0 R^n, \quad R < R_0, \]
for some \( \varepsilon \in [0, 1] \), provided that \( B_{R_0}(\varepsilon) \cap \Omega \neq \emptyset \).

Using (1.56) again, we conclude that
\[ |u(\varepsilon) - u_0(\varepsilon)| \leq K_0 R^{\alpha}, \quad R < R_0/8, \]
for some \( \varepsilon \in [0, 1] \). Since \( u_0 \in C(\bar{\Omega}) \) and \( u_0 = 0 \) on \( \partial \Omega \), the lemma follows.

**Remarks:**

(i) The proofs of Theorem 1.3 and Lemma 1.2 give also an a priori bound for the modulus of continuity of the solution \( u \). In particular, if \( \{G(\varepsilon)\} \) is a family of equicontinuous obstacles, then the corresponding solutions to (1.1) are equicontinuous in the interior of \( \Omega \). This holds up to the boundary if the functions \( u_0 \) are uniformly bounded in \( C^\alpha \cap H^{2,\alpha} \), \( p > n \).

(ii) There exist also other results on the continuity of the solutions to variational inequalities (1.1) if the obstacle is merely continuous but, to the author's best knowledge, not in the case where \( a(\varepsilon, u, V u) \) has quadratic growth in \( V u \).

If we require that the obstacle \( \psi \) be H"older continuous,
\[ \psi \in C^{\alpha}(\bar{\Omega}) \]
for some \( \alpha \in (0, 1) \), then the proof of Theorem 1.3 yields also the H"older continuity of \( u \) with some exponent \( \alpha' \in (0, 1] \). Furthermore, inequality (1.51), with \( \delta = K_0 R^{\alpha} \), yields
\[ (1.58) \quad \int_{B_{\varepsilon}} |V u|^{2^*} G dx \leq K_0 \]
where \( G(\varepsilon) = G(\varepsilon) = |\varepsilon - e_0|^{2^*} \) for \( n > 3 \), and \( G(\varepsilon) = |\varepsilon - e_0|^{2^*} \) for \( n = 2 \).

The constant \( K \) in (1.57) is uniform for \( 0 < R < R_0 \), \( \varepsilon \in \Omega \). The results extend up to the boundary of \( \Omega \) if (1.8) and (1.19) hold. Summing up, we arrive at the following theorem.

**Theorem 1.4.** Under the assumptions (1.2)-(1.6), (1.22), (1.23) and (1.57), every solution \( u \in L^p(\Omega) \) of (1.1) is H"older continuous in \( \Omega \) and satisfies (1.38). If, in addition, the assumptions (1.8) and (1.18) are fulfilled, then \( u \) is H"older continuous up to the boundary and (1.38) holds uniformly for \( \varepsilon \in \Omega \).

2. \( C^\alpha \)-regularity results if the obstacle is in \( C^\alpha \), \( 0 < \alpha \leq 1 \).

Again we consider the variational inequality
\[ (2.1) \quad \psi \in \mathcal{K} := \{\psi \in H^1(\Omega) \mid \psi \geq \psi_0 \text{ in } \Omega\} \]
such that
\[ \sum_{i=1}^n \langle a_i(\cdot, \cdot, V u), \delta e_i - \delta e_0 \rangle + \langle a(\cdot, \cdot, \psi), u - \psi \rangle \leq 0 \quad \text{for all } \psi \in \mathcal{K}. \]

Regarding the obstacle \( \psi \) we shall assume
\[ \psi \in C^\alpha(\bar{\Omega}) \quad \text{for a certain } \alpha \in (0, 1]. \]

The purpose of this section is to establish that (2.2) implies, under suitable conditions, the H"older continuity of a solution \( u \) of (2.1) with the same H"older exponent \( \alpha \). Results of this type have already been obtained by Biroli [3], [6]. The present results are more general than those in [5], [6], as far as nonlinearity is concerned. The technique of proof is different and yields a certain interesting additional result on the differentiability of the solution \( u \) of (1.1). For applications the results are important, in particular, in the case of \( \alpha = 1 \). For earlier results cf. also [35].

The assumptions on the functions \( a_i \) (differentiability, growth, ellipticity) are the following:

(2.3) The functions \( a_i(\varepsilon, u, p) \) are differentiable in \( (\varepsilon, u, p) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), \( i = 0, 1, \ldots, n \).

(2.4) The derivatives \( a_\alpha(u, u, p) \), \( a_\alpha(u, u, p) \) and \( a_{\delta_\alpha}(u, u, p) \) are measurable in \( u \) and continuous in \( (u, p) \), \( i = 0, 1, \ldots, n \).

(2.5) There exists a constant \( K_0 \) such that
\[ |a_\alpha(\varepsilon, u, p)| + |a_{\delta_\alpha}(\varepsilon, u, p)| + |a_{\delta_\delta}(\varepsilon, u, p)| \leq K_0 |p| + K_0, \]
\[ a_{\delta_\delta}(\varepsilon, u, p) + |a_{\delta_\delta}(\varepsilon, u, p)| \leq K_0 |p|^2 + K_0, \]
\[ |a_{\delta_\delta}(\varepsilon, u, p)| \leq K_0. \]

for all \( \varepsilon \in \Omega \), \( |u| \leq C_0 \), \( p \in \mathbb{R}^n \), \( i = 1, \ldots, n \).

(2.6) There exists a constant \( \psi = c(\bar{\Omega}) > 0 \) such that
\[ \sum_{i=1}^n a_\alpha(\varepsilon, u, p) \cdot \delta e_i \geq c|p|^2 \]
for all \( \varepsilon \in \mathbb{R}^n \), \( \varepsilon \in \bar{\Omega} \), \( |u| \leq C_0 \), \( p \in \mathbb{R}^n \), where \( a_\alpha = (\partial/\partial y_\alpha)a_1 \).

We first state a theorem about the interior regularity of bounded solutions \( u \) of (2.1). If \( \psi < 0 < 0 \) on \( \partial \Omega \) and \( \partial \Omega \in C^\alpha \), then the results of 1 and Theorem 2.1 below yield global \( C^\alpha \)-regularity. After giving the proof of Theorem 2.1 we study the boundary regularity if \( \psi < 0 \) on \( \partial \Omega \).
Theorem 2.1. Under the assumptions (2.2)–(2.6) every solution
\( u \in L^1(\Omega) \) of (2.1) is Hölder continuous in \( \Omega \) with the exponent \( \alpha \) from (2.2). Furthermore, the interior \( C^\alpha \)-norms of \( u \) are uniformly bounded if the constants
\( C = |u|_{L_1}, k_C, c(C) \) in (2.5) and (2.6) are uniformly bounded.

From the proof we shall see the following additional differentiability property:

Corollary to Theorem 2.1. Let \( y \) be Lipschitz continuous on \( \Omega \) and let
\[
M_\Omega^+ = \sup_{e \in \mathbb{R}^n, |e| = 1} \left[ \tau u \right]_{\Omega},
M_\Omega^- = \inf_{e \in \mathbb{R}^n, |e| = 1} \left[ \tau u \right]_{\Omega},
\]
where \( D_{\alpha} y(x) = h^{-1} \left( y(x+he) - y(x) \right). \)
Then
\[
\left[ \tau u - M_\Omega^+ \right]_{\Omega} \in B_1(\Omega)
\]
where \( B_1 = \max(\xi, 0) \) and \( \left[ \tau \right]_{\Omega} = \min(\xi, 0). \)

Remark. The differentiability assumptions on the \( a_i \) are not optimal. For example, one could also include lower order terms which are merely in \( L^\infty \).

We shall use the abbreviations:
\[
D_{\alpha}^\pm w(x) = h^{-1} \left( w(x+he) - w(x) \right), \quad D_{\alpha}^\pm w(x) = h^{-1} \left( w(x) - w(x-he) \right)
\]
for \( e \in \mathbb{R}^n, |e| = 1, h \geq 0 \).

For the proof of Theorem 2.1 we need the following simple

Lemma 2.1. Let \( u, \psi, \tau \) be real functions such that
\[
u \geq \psi \text{ a.e. in } \Omega
\]
and \( 0 \leq \tau \leq 1 \) a.e. in \( \Omega \). Let \( a \in [0, 1] \) be given and let
\[
M_\Omega^+ = \sup_{e \in \mathbb{R}^n, |e| = 1} \left[ \tau u \right]_{\Omega} > 0, \quad M_\Omega^- = \inf_{e \in \mathbb{R}^n, |e| = 1} \left[ \tau u \right]_{\Omega} < \infty,
\]
for all \( e \in \mathbb{R}^n, |e| = 1 \). Then the functions \( u^\pm \) defined by
\[
\begin{align*}
u^+ &= u + \frac{1}{h} D_{\alpha}^\pm \left[ \tau D_{\alpha}^+ u - h \cdot \nabla M_\Omega^+ \right], \\
u^- &= u + \frac{1}{h} D_{\alpha}^\pm \left[ \tau D_{\alpha}^- u - h \cdot \nabla M_\Omega^- \right]
\end{align*}
\]
satisfy the inequality
\[
u^+ \geq \psi
\]
for almost all \( x \in \Omega \) such that \( x + he, x \in \Omega \).

If the inequalities \( \nu \geq \psi \) and \( 0 \leq \tau \leq 1 \) hold in \( \Omega \) except a set of capacity zero then (2.7) holds for all \( x \) as above except a set of capacity zero (which may depend on \( h \)).

Proof of Lemma 2.1. We have for \( \xi, x+he \in \Omega \)
\[
u^+ = u + \frac{1}{h} D_{\alpha}^+ \left[ \tau D_{\alpha}^+ u - h \cdot \nabla M_\Omega^+ \right] - \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^+.
\]
Since the terms \( [\xi - u] \cdot \nabla M_\Omega^+ \) and \( -[\tau u - \xi] \cdot \nabla M_\Omega^- \) are monotone increasing in \( \xi \) and since \( u \geq \psi \) a.e. in \( \Omega \), we conclude that for almost all \( x, x+he \in \Omega \),
\[
u^+ \geq u + \frac{1}{h} D_{\alpha}^\pm \left[ \tau (x+he) - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^+ - \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^-.
\]
where \( \psi(\xi) = \xi + \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^+ - \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^- \).

Now, the function \( \psi \) is monotone increasing, as it can be easily seen by calculating its one-sided derivatives. Hence \( \psi(u(x)) \geq \psi(\nu(x)) \)
\[
u^+ \geq \psi(\nu(x)) = \psi(x) + d(x),
\]
where
\[
2d(x) = \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^+ - \frac{1}{h} \left( x+he \right) \left[ \tau u - \nu \right]_{\Omega} - h \cdot \nabla M_\Omega^-.
\]
for almost all \( x \in \Omega \) with \( x+he \in \Omega \).

From the definition of \( M_\Omega^+ \) we see that
\[
\psi(x+he) - \psi(x) \leq h \cdot M_\Omega^+,
\]
for almost all \( (x, x+he) \in \Omega \).

This yields \( d(x) \geq 0 \) and hence
\[
u^+ \geq \psi(x),
\]
which was to be shown. A similar argument works for \( \psi^- \) and for the “capacity-formulation”. The lemma is proved.

We prove Theorem 2.1 first for the case of the Laplacian, i.e., for
\[
-\Delta u + f(x) \in H_0^1(\Omega),
\]
where the key-idea of the proof can be easily seen.

Proof of Theorem 2.1 in the case of (2.8). Let \( G, \sigma \in L^\infty(\Omega) \) be the solution of the equation
\[
-\Delta G = \delta(x), \quad \delta(x) = |B_1|^{-1} \chi_{\{B_1(x)\}}.
\]
\[ \chi(B_r(\sigma)) \] is the characteristic function of the ball \( B_r(\sigma) \subset \Omega \) with radius \( r \) and centre \( \sigma \in \Omega \). By Lemma 2.1
\[ u_k^e \coloneqq u + e h \mathcal{D}_{a_k}^E \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right), \]
where \( e = e(\varepsilon, \phi) > 0, \varepsilon \in C_0^0(\Omega), \phi \equiv 1 \) in a neighbourhood of \( \sigma \). Inserting \( u_k^e \) into the variational inequality we obtain (writing down only the case \( a_k^e \))
\[ - \{ f, \mathcal{D}_{a_k}^E \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \} \leq - \{ f, \mathcal{D}_{a_k}^E \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \}. \]
Here we have cancelled the factor \( e h^2 \).

By partial summation we obtain
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \leq \int \mathcal{D}_{a_k} f, \mathcal{G}_k^E \left( \mathcal{D}_{a_k} u - M_k^a \right) \]
for \( 0 < h < h_0((\partial \Omega, \zeta)) \). We rewrite the left hand side of (2.9) and estimate the right hand side. This gives
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \leq \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
where \( \rho(\omega) = \rho(\omega) \) if \( \mathcal{D}_{a_k} u(\omega) \geq M_k^a \) and \( \rho(\omega) = 0 \) otherwise. We now use the identity
\[ \{ \mathcal{D}_{a_k}^E u - M_k^a, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \} = \{ \mathcal{D}_{a_k}^E u - M_k^a, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \} + \frac{1}{2} A + \frac{1}{2} B, \]
where
\[ A = \{ \mathcal{D}_{a_k}^E u - M_k^a, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \} + \frac{1}{2} A + \frac{1}{2} B, \]
and
\[ B = - \{ \mathcal{D}_{a_k}^E u - M_k^a, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \} + \frac{1}{2} A + \frac{1}{2} B \]
From the definition of \( G_k^E \)
\[ \int_j \int \mathcal{D}_{a_k}^E u, \mathcal{G}_k^E \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
where \( \mathcal{G}_k^E \) denotes integration over \( \mathcal{B}_n(\sigma) \). From (2.10) to (2.13)
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
we arrive at the inequality
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
from which we infer that
\[ \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \leq K \]
uniformly for \( h \to 0, e = 1. \) Similarly we obtain
\[ \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \leq K \]

The theorem (in case of (2.8)) follows.
The corollary follows by the following consideration. Taking into account that \( a = 1 \) and that Theorem 2.1 implies \( \mathcal{G}_a \subseteq \mathcal{G}_a \), we obtain from (2.10)
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
Here \( G \) is the Green function of \(-A\), i.e.,
\[ \left( \mathcal{G}_a, \mathcal{G}_a \right) = \varphi(\sigma), \quad \varphi \in C_0^0(\Omega), G \in H^1_\text{curl}(\Omega). \]
Taking into account that \( \mathcal{D}_{a_k}^E u \subseteq L^2(\Omega), Q_a = Q \), \( \mathcal{G}_a \subseteq \mathcal{G}_a \), \( \mathcal{G}_a \subseteq L^\infty \), we obtain for points \( \sigma_a \in Q_a \)
\[ \frac{1}{2} \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
\[ \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty + \frac{1}{2} \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
From Lemma 1.3 we conclude that
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
uniformly for \( 0 < h < h_0 \), where \( f \) denotes integration over \( \mathcal{S} \). We proceed by estimating
\[ \| \mathcal{D}_{a_k}^E u - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]
and
\[ \| \mathcal{D}_{a_k}^E u - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]
We used the fact that \( \int \mathcal{G}_a \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]
according to Theorem 1.4, for some \( \sigma \in \mathcal{S}, \xi = 1 \). Choosing a point \( \sigma_0 \) such that
\[ \xi(\sigma_0) \mathcal{D}_{a_k} u(\sigma_0) - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]
we arrive at the inequality
\[ \int \mathcal{D}_{a_k}^E u, \mathcal{D}_{a_k} \left( (\mathcal{G}_k^E)^{1/2} (\mathcal{D}_{a_k} u - M_k^a) \right) \]
from which we infer that
\[ \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]
uniformly for \( h \to 0, e = 1. \) Similarly we obtain
\[ \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \leq \mathcal{K}(\xi) \| \mathcal{D}_{a_k} u - M_k^a \| L_\infty \]

The theorem (in case of (2.8)) follows.
uniformly for \( h \to 0 \), \(|e| = 1 \). The symbol \( \int_{\Omega} \) denotes integration over \( \Omega \), \( c \in \Omega \). Thus

\[
\sup \left\{ \left| \int_{\Omega} \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \phi \right| \, |\phi| \leq 1 \right\} \leq K, \quad i = 1, \ldots , n,
\]

and

\[
\sup \left\{ \left| \left( \int_{\Omega} \phi - M_{\lambda}^{+} \right) \phi \right| \, |\phi| \leq 1 \right\} \leq K, \quad i = 1, \ldots , n.
\]

The corollary follows.

**Proof of Theorem 2.1.** In the general case. We use a similar variation \( u_{\xi}^{e} \) as before, namely

\[ u_{\xi}^{e} := u + \varepsilon \xi \frac{\partial}{\partial u} \left( c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right) \in K, \]

\[ \xi \in H^{1,\infty} (\Omega), \sup \left\{ |\xi| \leq \zeta \right\} \leq 1; \]

however, this time \( G_{\xi}^{e} = G_{\xi} \left( \cdot , \beta \right) \) is the Green function of the operator

\[ L = - \sum \beta \partial_{i} (a_{\xi} \partial_{i} u), \]

where

\[ a_{\xi} (u, \psi, \phi) = \int_{\Omega} \psi (u + \varepsilon \xi \phi) \left( \int \rho (u + \varepsilon \xi \phi) \right) dt; \]

\[ a_{\xi} (u, \psi, \phi) = (\partial_{i} \partial_{j} c_{\xi} (u, \psi, \phi)). \]

We have \( \hat{G}_{\xi} \in H_{l}^{s} (\Omega) \cap L^{\infty} (\Omega) \)

and

\[ \sum_{i=1}^{n} \partial_{i} a_{\xi} = (B_{1})^{-1} \int_{\Omega} \beta \left( \int \rho (u + \varepsilon \xi \phi) \right) dt; \quad \varphi \in C_{c}^{0} (\Omega). \]

We insert the above function \( u_{\xi}^{e} \) into the variational inequality (2.1) (we do not treat the case \( a_{\xi} \), which is analogous) and we obtain

\[ \sum_{i=1}^{n} \left( D_{\alpha}^{0} a_{\xi} (u, \psi, \phi), \partial_{i} (c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right) \right) \leq 0, \]

\( \xi = \text{identity} \). This yields

\[ \sum_{i=1}^{n} \left( a_{\xi} \partial_{i} D_{\alpha}^{0} u, \partial_{i} (c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right) \right) + A_{\xi} + B_{\xi} \leq 0, \]

where

\[ A_{\xi} = \sum_{i=1}^{n} \left( a_{\xi} \partial_{i} D_{\alpha}^{0} u, \partial_{i} (c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right) \right), \]

\[ B_{\xi} = h^{-a} \sum_{i=1}^{n} a_{\xi} (\varepsilon \xi, \phi) \theta \left( \int \rho (u + \varepsilon \xi \phi) \right) dt; \]

Here \( a_{\xi} (x, \xi, \phi) \) is defined as in (2.17),

\[ a_{\xi} (x, \xi, \phi) = (\partial_{i} \partial_{j} c_{\xi} (x, \xi, \phi)). \]

We rewrite the first summand in (2.18) and estimate it from below via ellipticity:

\[ \sum_{i=1}^{n} \left( a_{\xi} \partial_{i} D_{\alpha}^{0} u, \partial_{i} (c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right) \right) \]

\[ \geq - c \int \left| D_{\alpha}^{0} u \right|^{2} G_{\xi} \left| c^{0} \right| ds + \frac{1}{2} \sum_{i=1}^{n} \left( a_{\xi} \partial_{i} (c^{0} G_{\xi}^{e} \left[ \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right] \right), \partial_{i} G_{\xi} \right) + F_{\xi} + F_{\xi}, \]

where

\[ F_{\xi} = \sum_{i=1}^{n} \left( a_{\xi} \partial_{i} D_{\alpha}^{0} u, \partial_{i} (c^{0} G_{\xi}^{e} \left( D_{\alpha}^{0} u - M_{\lambda}^{+} \right) \right) \right), \]

\[ F_{\xi} = h^{-a} \sum_{i=1}^{n} a_{\xi} (\varepsilon \xi, \phi) \theta \left( \int \rho (u + \varepsilon \xi \phi) \right) dt; \]

Here \( \left| D_{\alpha}^{0} u \right|^{2} \) is defined as in the proof before.

We take into account that \( G_{\xi} \in L^{\infty} (\Omega_{\xi} - U(x)) \), where \( \Omega_{\xi} \in \Omega \) and \( U(x) \) is a neighbourhood of \( x \), and that the \( L^{\infty} \)-bound of \( G_{\xi} \) taken over \( \Omega_{\xi} - U(x) \) remains bounded as \( e \to 0 \). Thus we may estimate the term \( F_{\xi} \) by

\[ F_{\xi} \geq - \frac{c}{2} \int \left| D_{\alpha}^{0} u \right|^{2} G_{\xi} \left| c^{0} \right| ds - K_{\xi} \int \left| D_{\alpha}^{0} u \right|^{2} \left| V \right| \left| \partial_{i} \right| ds \]

\[ \geq - \frac{c}{2} \int \left| D_{\alpha}^{0} u \right|^{2} G_{\xi} \left| c^{0} \right| ds - K_{\xi}, \]

where \( K_{\xi} \) remains bounded for \( e \to 0, 0 < h \leq h_{0} \). Recall that \( V \approx 0 \) in \( U(x) \). Since \( \left| G_{\xi} \partial_{i} \right|_{L^{\infty}} \leq K_{\xi} \), we may also estimate the term \( F_{\xi} \) by

\[ F_{\xi} \geq - K_{\xi} \left| D_{\alpha}^{0} u - M_{\lambda}^{+} \right|_{L^{\infty}} \left| \int \left| D_{\alpha}^{0} u \right|^{2} \left| V \right| \right| ds \]

\[ \geq - K_{\xi} \left| D_{\alpha}^{0} u - M_{\lambda}^{+} \right|_{L^{\infty}} \left| \int \left| D_{\alpha}^{0} u \right|^{2} \right| ds \].
Thus we arrive at the inequality

\[
(2.19) \quad \sum_{\lambda \in \Lambda} \left| a_{n_{\lambda}} \overline{b}_{n_{\lambda}} \langle \mathcal{F}_{n_{\lambda}} (D_{\alpha} u - M_{+}^{+} \zeta) \rangle \right|
\geq \frac{c}{2} \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + |B_{\alpha}|^{-1} \int \mathcal{C}^{2} (D_{\alpha} u - M_{+}^{+} \zeta) \, ds - K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} - K_{\alpha}
\]

Here \( f_{\alpha} \) denotes integration over \( B_{\alpha}(\varepsilon) \).

We now estimate the terms \( A_{\lambda}, \ldots, E_{\lambda} \), uniformly for \( \varepsilon \to 0, h \to 0, \sigma \in \Omega, \) and also the \( C^{2} \)-regularity results of \( \S \), which establish that

\[
(2.20) \quad \int |F_{\lambda}|^{2} |\sigma| - \sigma^{2} \, d\sigma \leq K_{\alpha}
\]

(at least locally) with \( \mu > 0 \) and \( K_{\alpha} \) denoting some constants.

\textit{Estimation of \( A_{\lambda} \).} By (2.5) and Hölder's inequality, we obtain

\[
|A_{\lambda}| \leq c_{1} \left( \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + K_{\alpha} \right) \| D_{\alpha} u \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma + K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma + K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma.
\]

Here \( f_{\alpha} \) denotes integration over \( B_{\alpha}(\varepsilon) \).

By (2.20) the quantities \( f_{\alpha} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma \) and \( f_{\alpha} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma \) are small if the support of \( \zeta \) is contained in a ball with radius sufficiently small. (Estimate \( f_{\alpha} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma \) by \( |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma \).

Thus we obtain for every \( \varepsilon_{0}, \varepsilon_{1} > 0 \)

\[
(2.21) \quad |A_{\lambda}| \leq c_{1} \left( \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + K_{\alpha} \right) \| D_{\alpha} u \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma + K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma.
\]

provided that the support of \( \zeta \) is chosen sufficiently small (in accordance with \( \varepsilon_{0}, \varepsilon_{1} \)). Note that we estimate \( \int |F_{\lambda} \sigma| \sigma^{2} \, d\sigma \) by \( \mathcal{C}^{2} = 0 \) in \( U \). (2.5) The term \( f_{\alpha} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma \) can be estimated by \( K_{\alpha} \| F_{\lambda} \|_{\infty} \).

\textit{Estimation of \( B_{\lambda} \).} This term behaves slightly better than the term \( A_{\lambda} \) and can be estimated similarly. This yields

\[
(2.22) \quad |B_{\lambda}| \leq c_{1} \left( \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma.
\]

As we conclude, we can estimate \( \zeta \in C^{2}(\Omega) \) with support sufficiently small, so

\[
(2.23) \quad |C_{\lambda}| + |D_{\lambda}| + |E_{\lambda}| \leq c_{1} \left( \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma + 
+ K_{\alpha} \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \int \mathcal{C}^{2} (|F_{\lambda}| + 1) |G_{\alpha} |d\sigma.
\]

From (2.18), (2.19), (2.21), (2.22) and (2.23) we see that

\[
(2.24) \quad \int |D_{\alpha} u|^{2} |G_{\alpha}^{2} |d\sigma + |B_{\alpha}|^{-1} \int \mathcal{C}^{2} (|D_{\alpha} u - M_{+}^{+} \zeta| \sigma) \, d\sigma \leq K_{\alpha}.
\]

We choose \( \varepsilon_{0} \) and \( \varepsilon_{1} \) small enough (say, \( \varepsilon_{0} = \varepsilon_{1} = 1/8 \) and pass to the limit \( \varepsilon \to 0 \). This yields

\[
(2.25) \quad \| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \leq K_{\alpha}.
\]

and we obtain from (2.24)

\[
\| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \leq 2 K_{\alpha}.
\]

(Recall that \( \zeta = 1 \) in \( U \)). In a similar fashion we get

\[
\| D_{\alpha} u - M_{+}^{+} \zeta \|_{\infty} \leq 2 K_{\alpha}.
\]

This completes the proof of Theorem 2.1.

The corollary follows from (2.24), which implies also an estimate for \( \| D_{\alpha} u \|_{2} |G_{\alpha} |d\sigma \).

\textbf{Remark.} (i) The case of two Hölder continuous obstacles \( y_{1}, y_{2} \), \( y_{1} \leq y \leq y_{2} \), can be reduced by a partition of unity to the case of one obstacle if \( y_{1} < y \) and \( y_{2} \) is already known to be continuous. For the latter question, cf. \( \S \).

(ii) If the obstacle is only Hölder continuous with exponent \( \alpha \) in the direction of some unit vector \( e \) and if \( u \in C^{\alpha} \) for some \( \mu \in [0, 1] \), then \( u \) is Hölder continuous with exponent \( \alpha \) in the direction of \( e \). This holds also for thin obstacles and “boundary obstacles” \( u \gg v \) on \( \partial \Omega \).
In the latter case, if \( \psi : \partial \Omega \to \mathbb{R} \) is Lipschitz, the method of proof shows that the derivatives tangential to \( \partial \Omega \) are bounded. (Here we assume that \( \partial \Omega \) is smooth, say, \( \partial \Omega \in C^1 \).)

(iii) If \( \psi \equiv 0 \) on \( \partial \Omega \), \( \psi \in H^{1,\infty}(\Omega) \), \( \partial \Omega \in C^1 \), and \( \mathbf{K} = \{ v \in H^1_0(\Omega) \mid v \geqslant \psi \} \) is nonempty, then the method of the proof of Theorem 2.1 shows that the derivatives of \( u \) tangential to \( \partial \Omega \) are bounded. To obtain the regularity of the normal derivatives, one has to establish certain differential inequalities for the solution \( u \) as it has been done in [20], [31].

(iv) If we require the obstacle \( \psi \) to be merely continuous, then one can still prove that \( u \) is continuous, provided that the lower order terms which are quadratic in \( P_u \) (cf. the proof of Theorem 2.1) do not occur. (Clearly, Theorem 1.3 is much stronger.)

For the simple case of \( (2.8) \) we obtain

**Theorem 2.2.** Let \( u \in \mathbf{K} = \{ v \in H^1_0(\Omega) \mid v \geqslant \psi \} \) be a solution of

\[
(V_u, V_u - P_u) \leqslant \mathbf{f}, \quad v \in \mathbf{K},
\]

where \( f \in H^{1,\infty}(\Omega) \) and \( \psi \in C(\Omega) \). Then \( u \in C(\Omega) \).

For the proof, let \( \Omega_h \subset \subset \Omega \) and let

\[
\omega(h) = \omega(h, \Omega_h) = \sup \{ |(v(x+se) - v(x))| \mid x, x+se \in \Omega_h, 0 < s < h, |e| = 1, e \in \mathbb{R}^n \}.
\]

Define

\[
D_h \omega(x) = \omega(h) - \omega(h, \Omega_h - x(h)).
\]

Then the method of the proof of Theorem 2.1 can be easily adapted to the new situation and it yields an estimate for the modulus of continuity of \( u \); namely we get

\[
|D_h \omega(x)| \leqslant K, \quad x, x+he \in \Omega_h, |e| = 1, e \in \mathbb{R}^n.
\]

**3. \( C^{1,\infty} \)-regularity results for obstacles in \( C^{1,\infty} \), \( 0 < a \ll 1 \)**

In this section we consider the simplest elliptic variational inequality over a bounded domain \( \Omega \subset \mathbb{R}^n \):

Find \( u \in \mathbf{K} = \{ v \in H^1_0(\Omega) \mid \psi \geqslant v \), a.e. in \( \Omega \} \) such that

\[
(V_u, V_u - P_u) \leqslant (f, u - v)
\]

for all \( v \in \mathbf{K} \).

Here and in the following \( (w, e) = \int_\Omega w e dx \).

Inequality \( (3.1) \) implies that \( u \) minimizes

\[
\frac{1}{2}(V_u, V_u) - (f, u)
\]

on \( \mathbf{K} \) and vice versa.

We shall impose the following conditions on the data:

(3.2) \( \partial \Omega \) satisfies the Wiener condition \( (1.8) \),

(3.3) \( f \in L^{1,\infty}(\Omega) \) for some \( a \in ]0, 1[ \),

which implies \( 1 - a f \in C^{1,\infty}(\Omega) \),

(3.4) \( \psi \leqslant -\delta \) in a neighbourhood of \( \partial \Omega \),

(3.5') \( \psi \in C^{1,\infty}(\Omega) \).

Condition \( (3.5') \) may be replaced by

(3.5) \( \psi = \inf \{ \delta \omega(x) \mid h > 0, |e| = 1; x, x+he \in \Omega \} > -\infty \).

Here \( D_h \omega(x) = h^{-1} \omega(x+he) - \omega(x) \).

If we leave out \( (3.5) \), we have to assume some regularity of \( \psi \), say

(3.6) \( \psi \in C(\Omega) \),

or, which is weaker,

(3.6') \( \psi \in \mathcal{L}(\Omega) \) and \( \mathbf{K} \neq \emptyset \).

Under these assumptions we prove the following

**Theorem 3.1.** Let \( u \) be the solution of the variational inequality \( (3.1) \), whose data satisfy \( (3.2)-(3.6) \). Then \( u \in C^{1,\infty}(\Omega) \).

We remind the reader that in our notation \( u \in C^{1,\infty}(\Omega) \) does not imply \( u \in C^{1,\infty}(\partial \Omega) \). If \( \partial \Omega \) is smooth enough, then \( u \in C^{1,\infty}(\partial \Omega) \), since the solution \( u \) of the variational inequality does not touch the obstacle in a neighbourhood of \( \partial \Omega \).

**Proof.** We first prove the theorem under the assumption \( (3.6') \) instead of \( (3.6) \). From the results of §1 it follows that \( u \in C(\Omega) \) and \( u > \psi \) in \( U \cap \Omega \), where \( U \) is an open neighbourhood of \( \partial \Omega \). Hence the set \( I = \{ \xi \in \partial \Omega \mid u(x) = u(\xi) \) is closed and contained in an open subset \( \Omega_h \subset \subset \partial \Omega \) with smooth boundary \( \partial \Omega_h \subset \subset \partial \Omega \). The assumptions on the data imply \( u \in C^{1,\infty}(U \cap \Omega) \) for the restriction of \( u \) to \( U \cap \Omega \). The proof consists in showing that the quantities

\[
\delta_h \omega(x) = h^{-1} \{ u(x+he) - 2u(x) + u(x-he) \}
\]

are uniformly bounded for \( x \in \Omega_h \), \( e \in \mathbb{R}^n \), \( |e| = 1, 0 < h < h_0 := \inf \{ |y_x| \mid y_x \in \partial \Omega, |y_x| \geqslant 1 \} \).

From this the theorem follows via a result of the theory of Sobolev–Bèsov-spaces. The proof of the boundedness of \( \delta_h \omega \) is similar to the author’s proof [15] of the boundedness of the second derivatives of \( u \) in the case where \( \psi \in C^{1,\infty} \) and \( f \in C^1 \); now, however, additional technical difficulties arise.
From (3.1) and the closedness of $I$ we obtain that in the sense of distributions
\begin{equation}
\Delta u = f \quad \text{on } \Omega - I
\end{equation}
and
\begin{equation}
\Delta u \leq -f \quad \text{on } \Omega.
\end{equation}

To see this, choose $v = u + \varepsilon \varphi$, $\varphi \in C_0^0(\Omega - I)$, $\varepsilon$ small, or $v = u + \varphi$, $\varphi \in C_0^0(\Omega)$, $\varphi \geq 0$.

From (3.7) and (3.8) we conclude that
\begin{equation}
\Delta \delta^\alpha_{u}(u - s) \leq -\delta^\alpha_{u}f \quad \text{on } \Omega - I, \quad 0 < h < h_1, \varphi \in R^3, |\varphi| = 1.
\end{equation}

We solve the equation
\[-\Delta s = f \quad \text{in } \Omega\]
and observe that (3.3) implies $z \in C^{1+\alpha}(\Omega)$. From (3.9) we obtain
\[
\delta^\alpha_{u}(u - s) \leq 0 \quad \text{on } \Omega - I
\]
and from the maximum principle (which is proved by truncation methods here, cf. [36])
\[
\delta^\alpha_{u}(u - s) \geq \mu_0 \quad \text{on } \Omega_0, \quad 0 < h < h_1, \varphi \in R^3, |\varphi| = 1,
\]
where
\[
\mu_0 = \inf \{ \delta^\alpha_{u}(u - s)(\varphi) : \varphi \in \partial \Omega_0 \cup I, \varphi \in R^3, |\varphi| = 1, 0 < h < h_1 \}, \quad h_1 = \frac{1}{2} \inf \{ \| y - y_0 \| : y_0 \in \partial \Omega_0 \cup I, y_0 \in \partial \Omega_0 \}.
\]

Since $u \geq \psi$ on $\Omega$, we have
\[
\delta^\alpha_{u}u \geq \delta^\alpha_{u}\psi \geq m \quad \text{on } \Omega, \quad 0 < h < h_1, \varphi \in R^3, |\varphi| = 1.
\]

Since $-\Delta f = f$ in $\Omega$ and $\Delta u = f$ in $\Omega - I$, $I \subset \Omega_1$, we conclude that
\[
\zeta_{\max} := \sup \{ \delta^\alpha_{u}(u)(\varphi) : \varphi \in \Omega_0 \cup I, \varphi \in R^3, |\varphi| = 1, 0 < h < h_1 \} \leq \zeta \leq \sup \{ \delta^\alpha_{u}(u - s)(\varphi) : \varphi \in \Omega_0 \cup I, \varphi \in R^3, |\varphi| = 1, 0 < h < h_1 \} \leq \zeta_{\max}.
\]

Recall that $h_1 = \frac{1}{2} \inf \{ \| y - y_0 \| : y_0 \in \partial \Omega_0 \cup I, y_0 \in \partial \Omega_0 \} > 0$. Thus
\[
\mu_\alpha \geq \mu := \min \{ \zeta_{\max}, \zeta_{\min} \} > 0.
\]

where $m$ is defined in (3.3). From (3.10) and Lemma 3.1 applied to $v = u - s$ we obtain
\[
\delta^\alpha_{u}(u - s) \leq -C(n, a) \mu_\alpha \quad \text{in } \Omega,
\]

where $C(n, a) > 0$ and $\delta^\alpha_{u}(u)(\varphi) = h^{-n}(\varphi(x + h_\alpha a) - 2 \varphi(\varphi) + \varphi(x + h_\alpha a))$.

Hence
\[
\delta^\alpha_{u}(u - s) \leq -C(n, a) \mu_\alpha - \sum_{j=1}^{n} \delta^\alpha_{u}(u - s)(j) \quad (j \neq i, j = 1, \ldots, n),
\]

and in view of (3.10) and (3.12)
\[
\mu \leq \delta^\alpha_{u}(u - s) \leq -[n - 1 + \frac{1}{2} - C(n, a)] \mu \quad \text{in } \Omega_0, \quad i = 1, \ldots, n.
\]

Since $\alpha \geq 0$, we see that $\delta^\alpha_{u}$ is bounded uniformly on $\Omega_0$ as $h \to 0$.

The theorem then follows from a result of the theory of Besov-spaces, cf. [10], p. 229, Theorem 4.1.4. In the case of (3.6') being replaced by (3.6) we argue via an approximation argument: Let $v \in L^\alpha, v_0 \in R$. Since $v \leq 0$ in $\bar{U}(\partial \Omega)$, there is a test function $\varepsilon \in C_0^0(\Omega)$ such that $v := \varepsilon v_0 \in R$. We extend $v$ outside $\Omega$ by $-v$, and denote by $J^\alpha v$ the usual modification operator which convolutes a function with nonnegative mean functions converging to the Dirac measure $\delta (x \to 0)$, cf. [1]. Then $J^\alpha v < 0$ in a neighbourhood $U_0 \subset \partial \Omega$ uniformly for $0 < h < h'$. Furthermore $J^\alpha v \in H^{n+1}(\Omega)$ and $J^\alpha v \geq 0$.

We consider the variational inequality (3.1) with the obstacle $J^\alpha v$ instead of $v$. Since $J^\alpha v \geq J^\alpha v \geq J^\alpha v \in H^{n+1}(\Omega)$, the corresponding admissible set is non-empty and the variational inequality has a continuous solution $u_0 \geq J^\alpha v$. Since $J^\alpha v < -4[2]$ in a neighbourhood $U_0 \subset \partial \Omega$, $0 < h < h'$, the functions $u_0$ are uniformly Hölder continuous in a neighbourhood $U_1 \subset \partial \Omega$. This was proved in § 1, Lemma 1.2. Hence there exist open sets $U_0 \subset U_1 \subset U$ such that the sets of coincidence $I = \{ x \in I : u_0(x) = J^\alpha v(x) \}$ are contained in $U_0$, and that $|u_0 - J^\alpha v| > 0$ on $\partial \Omega - U_1$.

Since, further,
\[
\delta^\alpha_{u}(J^\alpha v)(x) \geq m, \quad x \in U_0, e \in R^3, |\varphi| = 1, 0 < h < h_0, 0 < h < h',
\]
we may apply the method of proof used in the case of $v \in C(\Omega)$ in order to obtain a uniform bound for $\delta^\alpha_{u}u_0$ on $\Omega_0$.

By well known perturbation theorems on variational inequalities (cf. e.g. [1], [28], [30]) it follows that the sequence $(u_k)_{k=1}^\infty$ tends to the unique solution $u_0$ of (3.1) in the weak (or strong) topology of $H^{n+1}(\Omega)$. This can be proven easily using theorems on weak compactness in $H^{n+1}$. Hence also $\delta^\alpha_{u_0}$ is bounded uniformly on $\Omega_0$, $0 < h < h_0$. Applying Theorem 4.1.4 of [10] once more, we infer that $u \in C^{n+\alpha}(\Omega)$. Since $|u - v| \geq \varepsilon > 0$, a.e. on $\partial \Omega - U_1$, we obtain $u \in C^{n+\alpha}(\Omega - U_1)$, and thus the theorem. It remains to prove

**Lemma 3.1.** Let $v \in C(\Omega) \cap H^{n+1}(\Omega)$ and suppose that $\Delta v \leq 0$ in $\Omega$ in the sense of distributions. Assume further that for an open subset $\Omega_1 \subset \Omega$ we have
\[
\mu_\alpha := \inf \{ \delta^\alpha_{u}(u)(\varphi) : \varphi \in \Omega_0, e \in R^3, |\varphi| = 1, 0 < h < h_0 \} > -\infty,
\]

where $h_0 = \frac{1}{2} \inf \{ \| y - y_0 \| : y_0 \in \partial \Omega_0 \cup I, y_0 \in \partial \Omega_0 \}$.
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Then

\[ \sum_{i=1}^{n} \delta_i \leq - C(m, a) \mu_0 \quad \text{on} \quad \Omega_0, \quad 0 < h < h_0, \]

with \( C(m, a) \) being a constant.

Proof. Let \( Q = \Omega \) be the cube with centre \( x_0 \in \Omega_0 \) and with edges of length \( 2h \) parallel to the coordinate axes. Let \( G_a \) be the discrete Green function defined by

\[ -\Delta G_a = \delta_a, \quad G_a \in H^1_0(Q), \]

where \( \delta_a = |E_a|^{-1} \) on \( E_a(x_0) \), and \( \delta_a = 0 \) otherwise, \( 0 < q < h \). The function \( G_a \) has the following properties:

\begin{align*}
(3.13) & \quad G_a \geq 0 \quad \text{on} \quad Q, \\
(3.14) & \quad G_a \in C(B), \\
(3.15) & \quad \nabla G_a \in L^2(\Omega), \\
(3.16) & \quad \nabla^2 G_a \geq 0 \quad \text{a.e.} \quad \text{on} \quad \partial Q, \\
(3.17) & \quad \int_{\partial \Omega} \nabla G_a \cdot ds = 1,
\end{align*}

where \( \nu(x) \) is the inner normal at \( x \in \partial \Omega \).

From the hypothesis that \( \Delta \leq 0 \) in the sense of distributions we obtain

\[ \langle Pf, \nabla G_a \rangle \leq 0 \]

and using the definition of \( G_a \) we conclude that

\[ \int_{\Omega} \nu \nabla G_a \cdot ds - \nu \psi_0(x_0) \leq 0, \quad \psi_0(x_0) = (\delta_a, \psi). \]

We split the integral over \( \partial \Omega \) into \( 2n \) integrals over the \((n-1)\)-dimensional faces of \( \Omega \) and denote by \( I_{+i}, I_{-i} \) the integration over the face \( \Omega_{+i} \) through \( x_0 + h e_i \), resp. \( x_0 - h e_i \) (\( e_i \) is the \( i \)-th unit vector). We want to examine the term

\[ D_i = \int_{\Omega_{+i}} \nu \nabla G_a \cdot dy = \frac{1}{n} \nu \psi_0(x_0) + \int_{\Omega_{-i}} \nu \nabla G_a \cdot dy. \]

Note that (3.18) implies

\[ \sum_{i=1}^{n} D_i \leq 0. \]

On account of the symmetry properties of \( G_a \) we have \( \int_{\Omega} \nabla G_a \cdot dy = 1/2n \) and hence

\[ D_i = \frac{1}{2n} [\psi_0(x_0 + h e_i) - 2 \psi_0(x_0) + \psi_0(x_0 - h e_i)] + \int_{\Omega_{+i}} \nu \psi_0(x_0 + h e_i) \nabla G_a \cdot dy + \int_{\Omega_{-i}} \nu \psi_0(x_0 - h e_i) \nabla G_a \cdot dy. \]

We may split each \( Q_{+i} \) and \( Q_{-i} \) into two congruent rectangles. We thus get \((n-1)\)-dimensional rectangular parallelepipeds \( R_i \) such that

\[ Q_{\pm i} = (x_0 \pm h e_i + R_i) \cup (x_0 \pm h e_i - R_i). \]

On account of symmetry

\[ \int_{\partial \Omega} \nu \nabla G_a \cdot ds = \frac{1}{4n}, \quad i = 1, \ldots, n, \]

where \( \int_{\partial \Omega} \) denotes integration over \( x_0 \pm h e_i + R_i \). Thus we may write

\[ \int_{\Omega} \nu [\psi_0(x_0 \pm h e_i) + \nu_0(x_0 \pm h e_i) + \psi_0(x_0 \pm h e_i - y)] \nabla G_a \cdot dy = \int_{\Omega} [\psi_0(x_0 \pm h e_i + y) - 2 \psi_0(x_0 \pm h e_i) + \psi_0(x_0 \pm h e_i - y)] \nabla G_a \cdot dy \]

\[ \geq \int_{\Omega} h^{1+1} \psi_0(x_0 \pm h e_i) \nabla G_a \cdot dy = h^{1+1} \psi_0(x_0 \pm h e_i). \]

Here we used the definition of \( \mu_a \) and the notation \( \psi_0 = \psi_0^0, \psi = |y|^{-1} \psi. \)

From the last inequality and (3.20) we obtain

\[ D_i \leq \frac{1}{2n} [\psi_0(x_0 + h e_i) - 2 \psi_0(x_0) + \psi_0(x_0 - h e_i)] + \psi_0(x_0, a) h^1+1 \]

and in view of (3.19)

\[ \frac{1}{2n} \sum_{i=1}^{n} [\psi_0(x_0 + h e_i) - 2 \psi_0(x_0) + \psi_0(x_0 - h e_i)] \leq - \psi_0(x_0, a) h^{1+1}. \]

Passing to the limit \( \varepsilon \to 0 \) we obtain

\[ \frac{1}{2n} \sum_{i=1}^{n} h^{1+1} \delta_a \psi_0(x_0) \leq - 2 \psi_0(x_0, a) h^{1+1}. \]

The lemma follows.

Remarks. (i) D. Kinderlehrer and L. Caffarelli have announced a result similar to Theorem 3.1, which they obtained independently.

(ii) The generalization of the proof of Theorem 3.1 to operators with variable coefficients or non-linear operators causes greater difficulties than in the case \( a = 1 \), cf. [16], [31], since the symmetry properties of the Green function which we have used do not hold any more in the general case. One has to be more careful while splitting the integral over \( \partial \Omega \).

(iii) Lemma 3.1 and the inequalities (3.10) and (3.12) yield an a priori bound for the \( C^{1+1}(\partial \Omega_0) \)-norm of \( u \).
4. Higher order variational inequalities with obstacles

We restrict the discussion to the polyharmonic variational inequality, since this case seems to be sufficiently characteristic to indicate what regularity properties can be expected in the general case, and not many results are known anyhow so far. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $V = H^{m-1}_0(\Omega)$ the closure of the test functions $C^\infty_0(\Omega)$ in the norm $\|\cdot\|_{m,2}$ of the Sobolev space $H^{m-1}(\Omega)$, where

$$\|w\|_{m,2} = \sum_{j=1}^n \|P_j w\|_2, \quad \|w\|_2 = \left( \int_\Omega |w|^2 \, dx \right)^{1/2}.$$ 

We denote the natural pairing of elements $v \in V$ and $l \in V^*$ (dual of $V$) by $\langle l, v \rangle$.

Let $g \in H^{m-1}(\Omega)$ be a function which represents the boundary condition and let $\psi : \Omega \to \mathbb{R}$ be a function.

We define

$$K = \{ v + g + \psi \in V \mid v \geq \psi \text{ in } \Omega \}.$$ 

The inequality sign in the definition of $K$ can be understood in the sense of $H^{m-1}(\Omega)$, cf. the definition in [24, p. 135 or in the sense "almost everywhere in $\Omega"]. If $\psi$ is not smooth, the set $K$ and the solution of the variational inequality (4.1) can of course depend quite strongly on the choice between these two possibilities.

Finally, let $l \in V^*$ and $m = N$, $m \geq 2$, be given. We consider the variational inequality:

(4.1) \text{Find } u \in K \text{ such that}

$$\langle (-1)^m \Delta^m u, v - u \rangle \leq \langle l, v - u \rangle$$

for all $v \in K$.

It is well known that a solution of (4.1) exists if $K \neq \emptyset$. Setting $v = u + \psi$, $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$, we have $v \in K$ and we deduce from (4.1) that

$$\langle (-1)^m \Delta^m u, \varphi \rangle \geq 0.$$

Hence $(-1)^m \Delta^m u - l$ is a measure, cf. [34]. If $l$ is a measure, i.e., if $l$ satisfies some weak regularity assumption, we have

$$\sup \{ \langle \varphi, \varphi \rangle \mid \|\varphi\|_{m,2} = 1, \varphi \in C^\infty_0(\Omega) \} < \infty,$$

and hence

$$\sup \{ \langle (-1)^m \Delta^m u, \varphi \rangle \mid \|\varphi\|_{m,2} = 1, \varphi \in C^\infty_0(\Omega) \} < \infty.$$

By Sobolev’s inequality also

$$\sup \{ \sum_{j=1}^n \langle (-1)^m \partial_j \Delta^m u, \partial_j \varphi \rangle \mid \int |P_j \varphi|^2 \, dx = 1, \varphi \in C^\infty_0(\Omega) \} < \infty.$$

If $m > 0$, and by Garding’s inequality in $H^\delta$, cf. [35], we obtain that $\mathcal{P} \Delta^{m-1} u \in L^{\delta-1}_2(\Omega)$ for $0 < \delta < 1$.

This holds without any restriction on the obstacle $\psi$. We state this result (which we consider to be well known) as

**Theorem 4.1.** Let $u \in K$ be the solution of (4.1). Then

$$\mathcal{P} \Delta^{m-1} u \in L^{\delta-1}_2(\Omega)$$

for all $\delta \in [0, 1[$.

**Remarks.** (i) An a priori bound of the $L^{\delta-1}_2$-norms of $\mathcal{P} \Delta^{m-1} u$ in terms of the $H^{m-1}$-norm of $u$ can be easily given.

(ii) Consider the penalty approximation to problem (4.1):

$$(-1)^m \Delta^m u + \varepsilon^{-1}(u - \psi)_- = l$$

where $\varepsilon > 0$, and $(u - \psi)_- = \min(0, u - \psi)$. If, say, $\psi \in L^1, l \in L^1$, then $u_\varepsilon$ exists and one can prove the existence of a uniform bound for the $L^1_{\text{loc}}$-norm of $u_\varepsilon$ as $\varepsilon \to 0$, cf. [19].

(iii) The statement in Theorem 4.1 gives rise to the question whether $\mathcal{P} \Delta^{m-1} u \in L^{\delta}_2(\Omega)$ for regular obstacles. The answer is affirmative for $m = 1, 2$, cf. Theorem 4.2.

Another simple method of obtaining a stronger assertion concerning the differentiability of the solutions to elliptic variational inequalities with obstacles consists in the classical finite difference procedure; cf. [1], [39] for PDE’s, [36] for variational inequalities. Here we impose the following condition on the obstacle, which allows a one-sided irregularity.

(4.2) \text{There exist functions } g_i \in H^{m-1}_0(\Omega), i = 0, \ldots, n, \text{ such that for all } \Omega_j \subset \subset \Omega

$$A_h \psi + \frac{1}{h} \sum_{i=1}^n D_i g_i \quad \text{in } \Omega, \quad 0 < h < h_0,$$

where $h_0 = \frac{1}{2} \inf \{ |y_i - y_j| \mid y_i, y_j \in \partial \Omega \}$ and

$$D_i \Delta \varphi(x) = \pm \varepsilon^{m-1} \varphi(\pm h_0 - \varepsilon \varphi(x)) \quad (\varepsilon = i-th \text{ unit vector})$$

$$A_h = \sum_{i=1}^n D_i^2 (D_i^2)^{1/2}.$$ 

Condition (4.2) is satisfied if $\varphi \in H^{m,\infty}(\Omega)$.

As regards the right hand side of (4.1), we assume:

(4.3) \text{There are functions } f_i \in L^1(\Omega), \text{ where } f_i \in L^1(\Omega).$

Under these additional assumption we obtain

**Theorem 4.2.** Under the assumptions (4.2) and (4.3) the solution $u \in K$ to the variational inequality (4.1) satisfies $u \in H^{m,\infty}(\Omega)$. 

Remarks. For obstacles \( \psi \in H^{m+1,1} \) this theorem can be found in [26].

In the case where the set \( K \) is defined by a two-sided restriction \( \psi_1 \leq u \leq \psi_2 \), a technique different from that applied in [26] and the one described here has to be used, cf. [14]. The proof of the differentiability of \( u \) on the boundary gives additional difficulties, which were attacked in [33] for the boundary obstacle case.

If one wants to prove the analogue of Theorem 4.2 for elliptic operators with variable coefficients, one has to replace the inequality (4.2) for \( \Delta_y \psi \) by a corresponding one for \( D_y D_y \psi \).

Proof of Theorem 4.2. Since \( u \geq \psi \) in \( \Omega \) (in the sense of \( H^{m,1} \) or in the sense “almost everywhere”), we conclude that

\[
u + \frac{1}{2n} \beta^2 \tau \Delta \psi \geq \psi \quad \text{in } \Omega_0, \quad \Omega_0 \subset \subset \Omega, \quad 0 < h < h_3, \quad \tau \in C_0^\infty(\Omega), \quad 0 \leq \tau \leq 1.
\]

Hence

\[
u + \frac{1}{2n} \beta^2 \tau \Delta \psi \geq \psi + \frac{1}{2n} \beta^2 \tau \left( \psi_1 + \sum_{i=1}^n D_i g_i \right)
\]

and

\[
u_0 := \nu + \frac{1}{2n} \beta^2 \tau \left( \Delta \psi - \psi_1 - \sum_{i=1}^n D_i g_i \right) \geq \psi,
\]

and thus \( \nu_0 \in K \).

We insert \( v = \nu_0 \) into the variational inequality (4.1), cancel the factor \( (1/2n)^{1/2} \) and obtain

\[
\langle (1-1/n) \Delta \nu - l, \tau \Delta \nu - \psi \rangle - \tau \sum_{i=1}^n D_i g_i \leq 0.
\]

By routine estimates, this yields a uniform bound for the \( L^\infty \) norms \( ||D_\nu \Delta \nu||_{L^\infty} \) as \( h \to 0 \) and thus the theorem.

In the case of thebiharmonic variational inequality \( (m = 2) \) a more detailed analysis of the regularity properties of the solution of the variational inequality can be given. It can be easily seen that the solution of thebiharmonic variational inequality cannot have continuous third order derivatives — even in the case of dimension one \( (n = 1) \).

In [17] we proved that the second order derivatives are bounded. A simplification of the proof which extends it to the case of irregular obstacles is given below. The experts suspect that the second order derivatives of the solution of thebiharmonic variational inequality are continuous if \( \psi \) is smooth. This was recently proved by A. Friedman and L. Caffarelli in the case of two dimensions. They established a logarithmic estimate for the modulus of continuity. For the proof of the boundedness of the second order derivatives of the solution we need only the following one-sided condition on the obstacle \( \psi \).

(4.4) In every subdomain \( \Omega_0 \subset \subset \Omega \) we have

\[
\Delta_y \psi \geq -c_0, \quad 0 < h < h_4
\]

where \( h_4 = \frac{1}{2} \inf \{ |y_1 - y_2| \mid y_1, y_2 \in \partial \Omega_0, y_1, y_2 \in \partial \Omega \} \) and \( c_0 = c_0(\Omega_0) \) is some constant.

Regarding the right hand side of (4.1) we assume:

(4.5) \( \langle \nu_0 - \sum_{i=1}^n f_i \delta_i \psi, \psi \rangle \leq C_\nu \in L^\infty(\Omega) \),

where \( f_i \in L^{m+1}(\Omega) \) with some \( \delta \geq 0 \).

Theorem 4.3. Under the assumptions (4.4) and (4.5) the solution \( u \) of thebiharmonic variational inequality (4.1), \( m = 2 \), has second order derivatives in \( L^\infty(\Omega) \). Furthermore, the third order derivatives of \( u \) satisfy

(4.6) \( \int_{\Omega_0} |\nabla^2 u|^2 d\sigma \leq K \langle \Omega_0 \rangle, \quad \zeta \in \mathbb{R}^n, \quad \Omega_0 \subset \subset \Omega, \)

where \( G \in C^\infty(\Omega) \) for \( n \geq 2 \), \( G \in C^\infty(\Omega_0) \) for \( n = 2 \).

Proof. By Theorem 4.2 we know already that \( u \in H^m_{\text{loc}}(\Omega) \). Let \( a_0 \) denote the usual modification operation which convolutes a function \( f \) with a non-negative mean function \( a_0 \in C^\infty \) with support in \( B(0) \). Let \( \mathcal{G}_f := a_0 \ast f \). Then \( \mathcal{G}_f \geq 0, \mathcal{G}_f \in C^\infty \), and \( (D_y \mathcal{G}_f, D_y f) = a_0 \ast f(-\zeta) \) for \( f \in H^{m,1} \), \( \supp f \subset B(\xi) \) \( (\xi \leq \text{exp}^{-1/2} 2 \) for \( n = 2 \). Finally, let \( \xi \in C_0^\infty(\Omega), \) \( \tau = 1 \) in a neighbourhood of \( \xi \). Then we get for \( 0 < h < h_0 = h_0(\tau) \)

\[
\nu + c_0 h \mathcal{G}_f \Delta \psi \geq \psi
\]

if \( c_0 = c_0(h, \xi, \text{etc.}) > 0 \) is chosen to be small enough. Hence and from (4.5) we obtain

\[
u_0 = \nu + c_0 \mathcal{G}_f (\Delta \psi + c_0) \geq \psi
\]

and so \( u_0 \in K \).

We insert this function \( u_0 \) into the variational inequality (4.1), which yields, after cancelling \( c_0 > 0 \),

(4.7) \( -\langle \Delta_y u - l, \mathcal{G}_f (\Delta \psi + c_0) \rangle \geq 0 \).

We may solve the equation

\[
\Delta_y u = l = \sum_{i=1}^n g_i \delta_i,
\]
in a ball containing $D$ in its interior. The functions $f_i$ are equal to zero outside $D$. Since $f_i \in L^{n+1}$, we have
\[ u_\varepsilon \in C^{n+1} \cap H^{n+1}, \quad \alpha = \alpha(\varepsilon), \]
and setting $\varepsilon = u + u_\varepsilon$ we obtain from (4.7) that
\[ -\langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq 0. \]
Since $u$, $u_\varepsilon$ and $\varepsilon$ lie in $H^{n+1}(\Omega)$, we may write
\[ \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq 0 \]
and passing to the limit $\varepsilon \to 0$ we conclude that
\[ \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq 0. \]
We rewrite this in the form
\[ \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq B_\varepsilon + C_\varepsilon, \]
where
\[ A_\varepsilon = \int \langle d x, \tau^2 [d x + c_\varepsilon - \mu_\varepsilon] G' \rangle, \]
\[ B_\varepsilon = 2 \int \langle d x, \tau^2 [G'([d x + c_\varepsilon - \mu_\varepsilon])\rangle \rangle \| V \tau \| \| V G' \| \| d x, \]
\[ C_\varepsilon = \frac{1}{2} \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq 0. \]
Since $V_\varepsilon = 0$ in a neighbourhood of the singularity $\xi$, the term $D_\varepsilon$ remains bounded as $\varepsilon \to 0$. Finally, since $A_\varepsilon \in C^1$, $\mu_\varepsilon = A_\varepsilon(\varepsilon)$, the term $|\mu_\varepsilon - A_\varepsilon(\varepsilon)|$ behaves like $k_0 |x - \xi|^{-n}$. The factor $[G'(\varepsilon)]^{-1}$ behaves like $[|x - \xi|^{-n}]$. Hence the term $E_\varepsilon$ remains bounded as $\varepsilon \to 0$ and we obtain the uniform bound
\[ \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq K \]
as $\varepsilon \to 0$, $\xi \in \Omega$. We pass to the limit $\varepsilon \to 0$ and conclude that
\[ \int \langle d x, \tau^2 G'([d x + c_\varepsilon - \mu_\varepsilon]) \rangle \leq K, \quad \xi \in \Omega. \]
The last part of the proof consists in showing that $A_\varepsilon \in L^p(\Omega)$ implies $A_\varepsilon \in L^p(\Omega)$. This is done with the help of an idea from the author's paper [17].
which can be checked by simple computations. Thus, for \( r \in C_0^\infty(\Omega) \),
\[ r \geq 0, \quad r = 1 \text{ in } U(x_0), \]
\[ \Phi_2 := \tau e_0 \ast [(\delta_2 - \frac{1}{2} A) g + e] \in C_0^\infty, \]
and \( \Phi_2 \geq 0 \).

Therefore, \( u + \Phi_2 \in K \), and we obtain

\[ -\langle \Delta A u - I, \Phi_2 \rangle < 0; \]

using the function \( u_0 \) and \( z \) defined above we get

\[ -\langle \Delta A z, \Phi_2 \rangle > 0 \]

and

\[ \left| \nabla A z, \nabla [\tau e_0 \ast [(\delta_2 - \frac{1}{2} A) g + e]] \right| < 0. \]

By partial integrations, we may move the operator \( \delta_2 - \frac{1}{2} A \) to the left factor and the operation \( \nabla A \) to the right factor. This yields

\[ [(\delta_2 - \frac{1}{2} A) z, \tau e_0 \ast A_0 g] \geq -E_2, \]

where \( E_2 \) contains several error terms which arise while performing the partial integrations, according to Leibniz rule. However, these error terms remain bounded as \( \varepsilon \to 0 \), cf. [17]. Thus we obtain

\[ [e_0 \ast [(\delta_2 - \frac{1}{2} A) z](x_0)] \geq -E_2 \]

uniformly as \( \varepsilon \to 0 \). Hence

\[ \tau \langle \delta_2 - \frac{1}{2} A z \rangle \geq -E_2 \quad \text{a.e. in } \Omega, \]

i.e., the functions \( \delta_2 z - \frac{1}{2} A z \) are locally uniformly bounded from below.

Since \( A = \sum a_i^2 z + A_0 z \in L^\infty(\Omega) \), we conclude that \( \delta_2 z \in L^\infty(\Omega) \)
and, finally, \( z \in L^\infty(\Omega) \).

The boundedness of the mixed derivatives \( \delta_i \delta_j z \) is shown by an orthogonal transformation of the problem, which yields the boundedness of \( (\delta_i \pm \delta_i)^2 z \). This is possible, since \( A' \) is invariant under orthogonal transformations and the lower order term \( \sum \langle f_i, \delta_i (u - v) \rangle \) is transformed into a similar one.

Remark: Theorem 4.3 can be extended to operators whose principal part is the product of two second order operators with smooth coefficients; cf. [12].

For applications to engineering problems cf. [12], where also further references can be found.

References

ON EXISTENCE AND NONEXISTENCE RESULTS FOR NONLINEAR SCHröDINGER EQUATIONS

HERBERT GAJEWSKI

Akademia der Wissenschaften der DDR, Institut für Mathematik, Berlin, DDR.

Introduction

In this paper we shall speak about existence and nonexistence results for initial value problems for equations of the form

\[ iu_t + Au + f(|u|^2)u = 0, \quad u_0 = \partial u/\partial t, \]

where \( A \) is the \( d \)-dimensional Laplacian and \( f \) is a continuous real function. In the special case \( f(q) = q, q = |\cdot| \) const., (1) is the dimensionless standard form of the nonlinear Schrödinger equation which has sometimes been called Ginsburg–Landau equation or recently also Zakharov–Shabat equation. The latter notation is due to the fact that Zakharov and Shabat [18] were the first to see that Cauchy’s problem for the spatially one-dimensional Schrödinger equation can be solved globally by means of the inverse scattering method. This famous method was discovered by Gardner, Greene, Kruskal and Miura [4] and firstly applied to Cauchy’s problem for the Korteweg–de Vries equation. Unfortunately the approach of Zakharov–Shabat does not seem to generalize neither to higher space dimensions nor to other functions \( f \) than \( f(q) = q \). Since we are interested in more general cases we do not go into details of the inverse scattering method here.

In the last decade, existence and nonexistence results for initial value problems for (1) have been published by many authors. In this paper we take into account existence results of Shabat [13], Strauss [15], Balcon, Cazenave & Figueira [11] and Ginibre & Velo [5] as well as nonexistence results of Tshalov [16], Shabat [18], Zakharov, Sobolev & Synch [19], Kudrashov [8] and GLasse [6].