We are interested in the question whether T is a mapping *onto* $\boldsymbol{R}^n$ if (18) holds.

A simple consequence of our theorem is

THEOREM A. *Let* $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ *be a polynomial mapping such that for all* x, $\nabla T(x)$ *is positively definite. Then* T *is a mapping onto* $\boldsymbol{R}^n$.

Proof. By Theorem 1 the range of T is linear and hence closed. By (18), it is open and, therefore, all $\boldsymbol{R}^n$.

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ON THE SMOOTHNESS OF SOLUTIONS OF VARIATIONAL INEQUALITIES WITH OBSTACLES

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0. Introduction

The first three sections of this contribution are devoted to the question of the regularity of solutions of scalar variational inequalities with obstacles, that is, to problems of the type:

(0.1) *Find* $u \in \boldsymbol{K} = \{v \in H_0^1(\Omega) \mid v \geqslant \psi \text{ in } \Omega\}$ *such that*

$$\sum_{i=0}^{n} \big(a_i(x, u, \nabla u), \partial_i u - \partial_i v\big) \leqslant 0$$

for all $v \in \boldsymbol{K}$.

Here Ω is a bounded open subset of $\boldsymbol{R}^n$, $H_0^1(\Omega)$ the usual Sobolev space of functions u which have a generalized gradient in $L^2(\Omega)$ and vanish on $\partial\Omega$ in the generalized sense. The scalar product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, i.e. $(w, z) = \int wz\,dx$; ∂_0 denotes the identity map. The inequality sign $v \geqslant \psi$ in the definition of $\boldsymbol{K}$ is to be understood in the sense of H^1, cf. [25] or [37], or in the sense "almost everywhere" (which may be quite different).

We shall assume natural growth and ellipticity conditions for the functions a_i, cf. § 1 and § 2. For a sufficiently smooth obstacle ψ, say, for $\psi \in H^{2\infty}(\Omega)$ (i.e. ψ having bounded second derivatives), the question of the regularity of solutions of (0.1) has been essentially solved. From the general regularity theory due to Brézis–Stampacchia [8] one obtains that $u \in H^{2,p}(\Omega)$ for all $p < \infty$, and the final step yielding $u \in H^{2,\infty}(\Omega)$ was performed in [15], [16], [21], [9]. It is well known that the further regularity condition $u \in C^2(\Omega)$ is false, in general. Cf. also [2], [24], [26], [27], [37] for many other results on regularity and historical remarks.

The results on the regularity of u are not so complete in the case of an obstacle ψ being less regular, say, for $\psi \in C^{\alpha}(\Omega)$ or $\psi \in C^{1+\beta}(\Omega)$; $0 < \alpha \leqslant 1$, $0 < \beta < 1$. Here $C^{\alpha}(\Omega)$ is the space of functions on Ω which satisfy an interior Hölder condition with exponent α; and $C^{1+\beta}$ is defined by an analogous condition imposed on the first derivatives of the functions. The study of the regularity of solutions u of (0.1) have recently achieved some importance because of the theory of quasi-variational inequalities, i.e. variational inequalities where the obstacle depends on the unknown function, cf. [2], [3] for examples. The implicit obstacles in the theory of quasi-variational inequalities have a priori a less degree of regularity and thus it is of importance to have regularity theorems for variational inequalities with obstacles having rather meagre regularity properties. The most surprising result in this direction seems to be the one announced in [20]. It states that the solution of an elliptic variational inequality with a *discontinuous* monotone obstacle (more generally, a one sided Hölder continuous obstacle) is Hölder continuous.

In this paper (§ 1–§ 3) we present regularity results for (0.1) with obstacles in C^{α} or $C^{1+\alpha}$. These results will be used for the study of quasi-variational inequalities, cf. [20], but are also of interest in themselves. In § 1 we prove that the solutions of (0.1) are Hölder continuous with some exponent $\mu \in]0,1[$ if $\psi \in C^{\alpha}$ for some α. Different sets of conditions are considered; note that we treat also the case of the lower order term $a_0(x, u, \nabla u)$ having quadratic growth in ∇u. In § 2 we present results of the type like that $\psi \in C^{\alpha}$ implies $u \in C^{\alpha}$, $0 < \alpha \leqslant 1$. In § 3 we restrict attention to the case of the Laplacian and obtain a corresponding conclusion for $C^{1+\alpha}$, $0 < \alpha \leqslant 1$.

We do not discuss the obstacle problem for non-linear *systems* of variational obstacle, since the question of regularity of solutions is not solved satisfactorily yet, even in the case of equations, i.e without obstacles. In the case of two dimensions, $C^{1+\alpha}$-regularity results for the solution of *systems* of variational inequalities with a *non-diagonal* principal part have been first presented in [18]. Furthermore, we do not discuss questions concerning other types of obstacles, e.g. thin obstacles, boundary obstacles, or obstacles for ∇u.

The last section (§ 4) is devoted to a discussion of the regularity properties of solutions of the polyharmonic variational inequality. We present a simple proof of the boundedness of the second derivatives of the biharmonic variational inequality, assuming only that the second derivatives of the obstacle are bounded from below. Throughout the paper, we shall use the following notations and conventions:

$\int$ = integration over Ω.

$H^{m,p}(\Omega)$ = Sobolev space.

The elements of $H^{m,p}(\Omega)$ are equivalence classes of real valued functions

which are defined in Ω up to a set of m-p-capacity zero and have generalized derivations in L^p up to order m (or up to a set of n-dimensional Lebesgue measure zero). Two functions lie in the same equivalence class if they coincide in Ω except on a set of m-p-capacity zero (or of measure zero, respectively). If we consider the elements of $H^{m,p}(\Omega)$ as (classes of) functions defined up to capacity zero we may understand the inequality $u \geqslant \psi$ ($u \in H^{m,p}(\Omega)$) in the sense "everywhere in Ω except a set of m-p-capacity zero". The space $H_0^{m,p}(\Omega)$ denotes the closure of the test functions with respect to the $H^{m,p}$-norm. If $u \in H_0^{m,p}(\Omega)$, we consider u also as a function in $H^{m,p}(\mathbf{R}^n)$ which vanishes outside Ω.

For open subsets Ω_1, Ω_2 we write $\Omega_1 \subset\subset \Omega_2$ if the closure $\bar{\Omega}_1$ of Ω_1 is contained in Ω_2.

$B_R(z)$ = ball in $\mathbf{R}^n$ of radius r with center z.

In the estimates considered in subsequent sections we shall frequently use the same letters K, K_0, $\bar{K}$ etc. for different constants (a constant = a number which does not depend on the relevant parameters).

The m-p-capacity (m-p-cap$E$) of a closed set E is defined as

$$\inf\Big\{\int |\nabla^m \varphi|^p dx \,\Big|\, \varphi \in C_0^{\infty}(Q),\ \varphi \geqslant 1 \text{ on } E\Big\}$$

and, for an arbitrary set E,

$$m\text{-}p\text{-cap}E = \sup\{m\text{-}p\text{-cap}K \mid K \subset E,\ K \text{ closed}\}.$$

Here Q is a ball containing E; m-p-cap$E$ depends also on Q (which is not relevant).

1. On the continuity of solutions of scalar variational inequalities with obstacles

In this section we consider scalar variational inequalities of the type

(1.1) *Find* $u \in \mathbf{K} = \{v \in H_0^1(\Omega) \mid v \geqslant \psi \text{ in } H^1\}$ *such that*

$$\sum_{i=0}^{n} \big(a_i(x, u, \nabla u), \partial_i u - \partial_i v\big) \leqslant 0$$

for all $v \in \mathbf{K}$ *such that* $u - v \in L^{\infty}(\Omega)$.

Here we assume the following conditions concerning the data Ω, a_i, ψ:

(1.2) Ω *is a bounded open subset of* $\mathbf{R}^n$.

(1.3) $\psi \in C(\bar{\Omega})$.

The functions a_i: $\Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ have the following properties:

(1.4) $a_i(x, \mu, \eta)$ *is measurable with respect to* $x \in \Omega$ *and continuous with respect to* $(\mu, \eta) \in \mathbf{R} \times \mathbf{R}^n$.

(1.5) *There is a constant* $K = K_C$ *such that for all* $x \in \Omega$, $\eta \in \mathbf{R}^n$, $u \in \mathbf{R}$, $|u| \leqslant C$,

$$|a_i(x, u, \eta)| \leqslant K|\eta| + K, \qquad i = 1, \ldots, n$$

and

$$|a_0(x, u, \eta)| \leqslant K|\eta|^2 + K.$$

(1.6) *There are constants* $c = c(C) > 0$ *and* $K = K_C$ *such that for all* $x \in \Omega$, $\eta \in \mathbf{R}^n$, $u \in \mathbf{R}$, $|u| \leqslant C$

$$\sum_{i=1}^{n} a_i(x, u, \eta)\eta_i \geqslant c|\eta|^2 - K.$$

It is well known that the conditions (1.4)–(1.6) guarantee the local Hölder continuity of *bounded* weak solutions of the *equation*

$$-\sum_{i=1}^{n} \partial_i a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = 0,$$

cf. the book of Ladyženskaya–Ural'ceva [23]. As regards the Hölder continuity of u up to the boundary of Ω the following condition is sufficient in the case of equations:

(1.7) $\partial\Omega$ *is Lipschitz continuous.*

We shall also be concerned with a weaker *Wiener-type condition*:

(1.8) *There are positive numbers* c_0 *and* R_0 *such that for all* $x_0 \in \partial\Omega$, $0 < R < R_0$,

$$\inf\Big\{\int |\nabla\varphi|^2 dx| \;\; \varphi \in C_0^\infty\big(B_{2R}(x_0)\big),\ \varphi \geqslant 1 \text{ on } \complement\Omega - B_R(x_0)\Big\} \geqslant c_0 R^{n-2}.$$

(Integration runs over the support of φ; $\complement\Omega = \mathbf{R}^n - \Omega$.)

At is can be seen by simple counterexamples, the solution u of (1.1) need not be bounded in L^∞, even in the case of an equation. This is due to the fact that we allow a quadratic growth of $a_0(x, u, \nabla u)$ in ∇u.

Thus we first present a theorem which asserts the boundedness of the solutions of (1.1) under the additional assumptions (1.9)–(1.11) given below. Then we continue the discussion about the continuity of solutions of (1.1).

The additional conditions are as follows:

(1.9) *There is a constant* K *such that for all* $x \in \Omega$, $\eta \in \mathbf{R}^n$, $\mu \in \mathbf{R}$,

$$|a_i(x, \mu, \eta)| \leqslant K|\eta| + K|\mu| + K, \qquad i = 1, \ldots, n,$$

and

$$|a_0(x, \mu, \eta)| \leqslant K|\eta|^2 + K|\mu|^2 + K.$$

(1.10) *There are constants* $c_0 > 0$ *and* K *such that for all* $x \in \Omega$, $\eta \in \mathbf{R}^n$, $\mu \in \mathbf{R}$,

$$\sum_{i=1}^{n} a_i(x, \mu, \eta)\eta_i \geqslant c_0|\eta|^2 - K|\mu|^2 - K.$$

(1.11) *There are constants* $c_1 < c_0$ (cf. (1.10)) *and* K *such that for all* $x \in \Omega$, $\eta \in \mathbf{R}^n$, $\mu \in \mathbf{R}$,

$$a_0(x, \mu, \eta)\mu \geqslant -c_1|\eta|^2 - K|\mu|^2 - K.$$

Condition (1.11) is called a "one-sided condition".

THEOREM 1.1. *Under the assumptions* (1.2), (1.3), (1.9)–(1.11), *every solution* u *of* (1.1) *is essentially bounded on* Ω.

Remarks. (i) There are many generalizations and variants of (1.9)–(1.11) which still yield the boundedness of u, for example those involving Sobolev's inequality.

(ii) The proof of Theorem 1.1 implies also an a priori-estimate for u in terms of the data.

(iii) It suffices to assume that ψ is bounded from above.

Proof of Theorem 1.1. Let k and δ be numbers such that $\delta > 0$

$$k \geqslant \psi \text{ on } \Omega, \qquad k \geqslant 0.$$

Write

$$u_{k\delta} = \max\big(u(x) - k, 0\big)/\big(1 + \delta\max\big(u(x) - k, 0\big)\big).$$

Then the function v defined by

$$v = u - \varepsilon u_{k\delta}$$

is contained in $H_0^1(\Omega)$. The zero boundary condition is satisfied since $u \in H_0^1(\Omega)$ and $k \geqslant 0$. Furthermore, $v - u \in L^\infty(\Omega)$, and for $\varepsilon = \varepsilon(k, \delta) > 0$ the function $\zeta - \varepsilon\max(\zeta - k, 0)/\big(1 + \delta\max(\zeta - k, 0)\big)$ is monotone increasing in ζ. Hence we obtain, in view of $u \geqslant \psi$,

$$v \geqslant \psi - \varepsilon\max(\psi - k, 0)/\big(1 + \delta\max(\psi - k, 0)\big),$$

and, therefore, $v \in \mathbf{K}$.

Thus we may insert the above function v into the variational inequality, obtaining

$$\sum_{i=1}^{n}\big(a_i(\cdot, u, \nabla u), \partial_i u_{k\delta}\big) + \big(a_0(\cdot, u, \nabla u), u_{k\delta}\big) \leqslant 0. \tag{1.12}$$

We estimate the second summand by using the one-sided condition (1.11):

$$a_0\big(x, u(x), \nabla u(x)\big)u(x) \geqslant -c_1|\nabla u(x)|^2 - K|u(x)|^2 - K.$$

We multiply the last inequality by the factor

$$a(x) = u_{k\delta}(x)/u(x), \quad u(x) \neq 0.$$

Since $k \geqslant 0$, we have

$$0 \leqslant a(x) \leqslant 1$$

(even for negative $u(x)$). This yields

$$a_0\big(x, u(x), \nabla u(x)\big)u_{k\delta}(x) \geqslant -c_1|\nabla u(x)|^2\chi_k - Ku_{k\delta}(x)|u(x)| - K\chi_k$$

where χ_k is the characteristic function of the set

$$A_k = \{x \in \Omega |\ u(x) \geqslant k\}.$$

We apply this estimate to inequality (1.12). Then the term $a_0u_{k\delta}$ disappears and we may pass to the limit $\delta \to 0$. We obtain

$$(1.13) \qquad \sum_{i=1}^{n}\big(a_i(\cdot, u, \nabla u), \partial_i u_k\big) - c_1\int|\nabla u|^2\chi_k dx \leqslant K\int u_k|u|\,dx - K|A_k|$$

where $u_k = \max(u-k, 0)$, $|A_k| =$ Lebesgue measure of A_k.

The first term in (1.13) is estimated via the coerciveness condition (1.10) and we obtain

$$(c_0 - c_1)\int|\nabla u_k|^2dx \leqslant K|A_k| + K\int u_k|u|\,dx + K\int|u|^2\chi_k dx.$$

(Recall that $c_0 - c_1 > 0$.)

We rewrite and estimate the term $\int u^2\chi_k dx$ in the following way:

$$\int u^2\chi_k dx = \int u_k^2dx + 2k\int u\chi_k dx - k^2|A_k| \leqslant \int u_k^2dx + k^2|A_k| + \tfrac{1}{2}\int u^2\chi_k dx$$

and hence

$$\int u^2\chi_k dx \leqslant 2\int u_k^2dx + 2k^2|A_k|.$$

The term $\int u_k|u|dx$ is treated in a similar fashion. Thus we conclude that

$$\int|\nabla u_k|^2dx \leqslant Kk^2|A_k| + K\int u_k^2dx.$$

According to Sobolev's inequality,

$$\Big(\int u_k^{2\gamma}dx\Big)^{1/\gamma} \leqslant K\int|\nabla u_k|^2dx$$

where $\gamma = n/(n-2)$ for $n \geqslant 2$ and $\gamma > 1$, say $\gamma = 2$, for $n = 2$. Consequently

$$\Big(\int u_k^{2\gamma}dx\Big)^{1/\gamma} \leqslant Kk^2|A_k| + K\int u_k^2dx.$$

By Hölder's inequality,

$$\int u_k^2dx \leqslant |A_k|^{1-1/\gamma}\Big(\int u_k^{2\gamma}dx\Big)^{1/\gamma}.$$

For $k \geqslant k_0 = k_0(\|u\|_2)$ we have

$$K|A_k|^{1-1/\gamma} \leqslant \tfrac{1}{2}$$

and we conclude that

$$(1.14) \qquad \Big(\int u_k^{2\gamma}dx\Big)^{1/\gamma} \leqslant Kk^2|A_k|.$$

Using again Hölder's inequality we obtain

$$\int u_k dx \leqslant |A_k|^{1-1/(2\gamma)}\Big(\int u_k^{2\gamma}dx\Big)^{1/2\gamma}$$

and in view of (1.14)

$$(1.15) \qquad \int u_k dx \leqslant Kk|A_k|^{1+\sigma}, \quad \sigma = 1/2 - 1/(2\gamma) > 0, \ k \geqslant k_0.$$

Now, a lemma from [23] (Lemma 5.1, Section 2, p. 71), states that (1.15) implies the boundedness of u from above by a constant depending on K, k_0, σ, and $\|u\|_1$.

Since $u \geqslant \psi$ on Ω, we infer that u is also bounded from below on Ω. This completes the proof of the theorem.

Proof of Remark (iii) *to Theorem* 1.1. In the above proof we used only the fact that there exists a constant k_0 such that $k \geqslant \psi_0$ on Ω and we saw that the solution u of (1.1) is bounded from above. No further regularity of ψ is needed. The boundedness of u from below follows, since every solution u of (1.1) is also a supersolution, i.e.,

$$\sum_{i=1}^{n}(a_i, \partial_i\varphi) + (a_0, \varphi) \geqslant 0 \quad \text{for} \quad \varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0.$$

We now turn to the discussion of the *continuity* of *bounded* solutions u of (1.1). Theorem 1.2 below asserts the interior Hölder continuity of u, provided that

$$(1.16) \qquad \psi \in H^{1,p}(\Omega) \quad \textit{for some } p > n,$$

and the Hölder continuity of ψ up to the (regular) boundary of Ω if (1.16) holds and if

(1.17) $\psi \leqslant -a_0 < 0$ *in a neighbourhood of* $\partial\Omega$ *with a certain constant* a_0.

Condition (1.17) may be replaced by:

(1.18) *The set* $\mathbf{K}$ *contains a function*

$$u_0 \in C^{\tilde{\alpha}}(\bar{\Omega}) \cap H^{1,p}(\Omega)$$

with suitable constants $\tilde{\alpha} \in \,]0, 1[$, $p > n$.

We consider this result (as well as Theorem 1.1) as a corollary to the general theory of quasilinear elliptic equations [23], [29], since the techniques of proof can be adapted easily. The question whether u remains continuous when (1.16) is replaced by the condition $\psi \in C(\bar{\Omega})$, is more difficult, but it has an affirmative answer (cf. Theorem 1.3 where an additional regularity condition is assumed).

THEOREM 1.2. *Under the assumptions* (1.2)–(1.6) *and* (1.16) *every solution* $u \in L^\infty(\Omega)$ *of* (1.1) *is locally Hölder continuous in* Ω *with an exponent* $\alpha \in]0,1[$. *If, in addition, the assumptions* (1.7) *and* (1.17) *or* (1.18) *hold, the solution* u *is Hölder continuous up to the boundary of* Ω.

Remark. The proof yields an a priori-estimate for the Hölder norm in terms of the data and $\|u\|_{1,2}$.

Proof. Let $B_\varrho \subset \Omega$ be a ball of radius ϱ and let

$$M = \operatorname{ess\,max}\{(u-\psi)(x) \mid x \in B_\varrho\}, \qquad m = \operatorname{ess\,min}\{(u-\psi)(x) \mid x \in B_\varrho\}.$$

Since $u \in \boldsymbol{K}$, we have $m \geqslant 0$, and for $l \in [m, M]$ the minimum and the maximum of the two numbers $(u-\psi)(x)$, l satisfy:

$$\text{(1.19)} \qquad \min\{(u-\psi)(x), l\} \geqslant 0, \qquad \max\{(u-\psi)(x), l\} \geqslant 0 \qquad \text{for } x \in B_\varrho.$$

Let τ be a Lipschitz continuous function such that

$$\operatorname{supp}\tau \subset B_\varrho, \quad 0 \leqslant \tau \leqslant 1, \ \tau = 1 \text{ on } B_{\varrho-\sigma\varrho}, \text{ and } |\nabla\tau| \leqslant (\sigma\varrho)^{-1},$$

where $B_{\varrho-\sigma\varrho}$ is concentric to B_ϱ, $0 < \sigma < 1$.

For $\zeta \in \boldsymbol{R}$ define:

$$[\zeta]_l^+ = \max\{\zeta - l, 0\}, \qquad [\zeta]_l^- = \min\{\zeta - l, 0\}.$$

By (1.19)

$$v_1 := u - \tau^2[u-\psi]_l^+ \geqslant \psi, \qquad v_2 := u - \tau^2[u-\psi]_l^- \geqslant \psi.$$

Thus $v_1, v_2 \in \boldsymbol{K}$ and we may insert the functions v_1, v_2 into the variational inequality, obtaining

$$\sum_{i=1}^n \big(a_i(\cdot, u, \nabla u), \ \partial_i(\tau^2[u-\psi]_l^\pm)\big) \leqslant 0.$$

From this we conclude by a routine analysis, employing the conditions (1.5), (1.6) and $u \in L^\infty$, that

$$\text{(1.20)} \qquad \int_{l\varrho} |\nabla u - \nabla\psi|^2\tau^2 dx \leqslant K \int_{l\varrho} |\nabla\psi|^2\tau^2 dx + K \int_{l\varrho} |\nabla\tau|^2 |\xi - \psi - l|^2 dx + \\ + K|A_{l\varrho}| + K \int_{l\varrho} |\nabla u - \nabla\psi|^2 |u - \psi - l| \tau^2 dx,$$

where $A_{l\varrho} = \{x \in B_\varrho \mid u(x) - \psi(x) \geqslant l \ (\text{or} \leqslant l)\}$ and $\int_{l\varrho}$ denotes integration over $A_{l\varrho}$.

The last summand in (1.20) arises from the term

$$a_0(\cdot, u, \nabla u)\tau^2[u-\psi]_l^\pm.$$

We set $\delta = (2K)^{-1}$ and consider only those numbers l which satisfy

$$M - l \leqslant \delta \quad \text{or} \quad m - l \geqslant -\delta.$$

Then (1.20) implies

$$\int_{l\varrho} |\nabla u - \nabla\psi|^2\tau^2 dx \leqslant K \int_{l\varrho} |\nabla\psi|^2\tau^2 dx + K \int_{l\varrho} |\nabla\tau|^2 |u - \psi - l|^2 dx + K|A_{l\varrho}|.$$

Since $\nabla\psi \in L^p$, $p > n$, we may estimate

$$\int_{l\varrho} |\nabla\psi|^2\tau^2 dx \leqslant K|A_{l\varrho}|^{1-2/p}.$$

Using the properties of τ we arrive at the inequality

$$\text{(1.21)} \qquad \int_{l,\varrho-\sigma\varrho} |\nabla u - \nabla\psi|^2 dx \leqslant K|A_{l\varrho}|^{1-2/p} + K(\sigma\varrho)^{-2}|A_{l\varrho}| \max_{l\varrho}(u-\psi-l) \\ \leqslant K\big(1 + \sigma^{-2}\varrho^{-2+2n/p}\max_{l\varrho}(u-\psi-l)\big)|A_{l\varrho}|^{1-2/p}$$

for $l \in [M-\delta, M]$, or $l \in [m, m+\delta]$.

Here $\max_{l\varrho}$ denotes the maximum (respectively, the minimum) over the set $A_{l\varrho}$.

Inequality (1.21) is just the statement that the function $u-\psi$ is contained in the class

$$\mathscr{B}(\Omega, \|u-\psi\|_\infty, K, \delta, 1/p)$$

as defined in the book of Ladyženskaya–Ural'ceva [23], Section 2.6, p. 81. Therefore, by Theorem 6.1, Section 2.6, p. 90, from [23], the function $u-\psi$, and hence u, is Hölder continuous on interior domains.

The proof of the boundary regularity presents further technical difficulties and is carried only under an additional natural assumption (1.22), (1.23) below, concerning the principal part of the differential operator. Condition (1.22) allows us to treat the case of an obstacle Ω which is merely continuous. This is done in Theorem 1.3 below.

In subsequent considerations we need additional differentiability and ellipticity conditions:

(1.22) *The partial derivatives*

$$a_{ik}(x, \mu, \eta) = (\partial/\partial\eta_k) a_i(x, \mu, \eta)$$

exist for $x \in \Omega$, $\mu \in \boldsymbol{R}$, $\eta \in \boldsymbol{R}^n$, $i, k = 1, \ldots, n$, *and are continuous in* $(\mu, \eta) \in \boldsymbol{R} \times \boldsymbol{R}^n$, *and measurable in* $x \in \Omega$.

$$\text{(1.23)} \qquad \sum_{i,k=1}^n a_{ik}(x, \mu, \eta)\xi_i\xi_k \geqslant C_0|\xi|^2$$

for all $x \in \Omega$, $\mu \in \boldsymbol{R}$, $\eta \in \boldsymbol{R}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n$, *with a certain constant* $c_0 > 0$.

THEOREM 1.3. *Under the assumptions* (1.2)–(1.6), (1.22), (1.23) *every solution* $u \in L^\infty(\Omega)$ *of* (1.1) *is continuous in* Ω. *If, in addition, the assumptions* (1.8) *and* (1.18) *are fulfilled, then* $u \in C(\bar{\Omega})$.

Note that we assume no more regularity of the obstacle ψ than the mere continuity.

For the proof of Theorem 1.3 we need

LEMMA 1.1. *Under the assumptions* (1.2)–(1.6), (1.22), (1.23) *every solution* $u \in L^\infty(\Omega)$ *satisfies the inequalities*

$$(1.24)\qquad \begin{aligned} &\int |\nabla u(x)|^2 |x-z|^{2-n}dx \leqslant K, \qquad n \geqslant 3,\\ &\int |\nabla u(x)|^2 \big|\ln|x-z|\big|\,dx \leqslant K, \qquad n = 2, \end{aligned}$$

uniformly for all $z \in \Omega_0 \subset\subset \Omega$ *with* $K = K(\Omega_0)$ *denoting a constant depending only on the data. If* (1.8) *and* (1.18) *are fulfilled, then* (1.24) *is satisfied for* $z \in \Omega$.

Remark. The assumption $\psi \in C(\bar{\Omega})$ in Lemma 1.1 can be replaced by the assumption that ψ is bounded from above.

Proof of Lemma 1.1. Let $a_{ik}(x) = \int_0^1 a_{ik}\big(x, u(x), t\nabla u(x)\big)dt$ for $x \in \bar{\Omega}$ and $a_{ik}(x) = \delta_{ik}$ for $x \in \mathbf{R}^n - \Omega$, $\delta_{ik} = 1$ for $i = k$, $\delta_{ik} = 0$ for $i \neq k$; $i, k = 1, \ldots, n$. Let $Q \supset\supset \Omega$ be a fixed ball. We define the regularized Green function $G_\varrho = G_\varrho^z$, $z \in \Omega$, by the conditions: $G_\varrho \in H_0^1(Q)$ and

$$\sum_{i,k=1}^n (a_{ik}\partial_k\varphi, \partial_i G_\varrho)_Q = |B_\varrho|^{-1}\int_\varrho \varphi\,dx, \qquad \varphi \in C_0^\infty(Q).$$

Here the parentheses $(\cdot,\cdot)_Q$ denote the scalar product in $L^2(Q)$; the symbol $\int_\varrho$ denotes integration over $B_\varrho(z)$. The assumptions (1.22), (1.23) guarantee that such a function G_ϱ exists and we shall make use of the following properties of $G_\varrho = G_\varrho^z$:

$$G_\varrho \in L^\infty(Q), \qquad G_\varrho \geqslant 0,$$

$$(1.25)\qquad \nabla G_\varrho \to \nabla G \text{ weakly in } L^s(Q), \quad 1 \leqslant s < n/(n-1) \quad (\varrho \to 0),$$

$$(1.26)\qquad G_\varrho \to G \text{ strongly in } L^r(Q), \quad 1 \leqslant r < n/(n-2) \quad (\varrho \to 0).$$

Here $G = G^z$ is the continuous Green function defined by $G \in H_0^{1,s}(Q)$, $1 \leqslant s < n/(n-1)$, and

$$\sum_{i,k=1}^n (a_{ik}\partial_k\varphi, \partial_i G)_Q = \varphi(z), \qquad \varphi \in C_0^\infty(Q),$$

($(v, w)_Q = \int_Q vw\,dx$). The function G satisfies the inequalities

$$(1.27)\qquad c|x-z|^{2-n} \leqslant G^z(x) \leqslant K|x-z|^{2-n}, \qquad n \geqslant 3,$$

$$(1.28)\qquad c\big|\ln|x-z|\big| \leqslant G^z(x) \leqslant K\big|\ln|x-z|\big|, \qquad n = 2,$$

for all x in a neighbourhood of z, with some constants $K, c > 0$. For the proofs cf. [38].

We first treat the local part of Lemma 1.1.

Let τ be a nonnegative Lipschitz continuous function with support in Ω, assume $\tau = 1$ in a neighbourhood of z an let k be an upper bound of ψ. Let $q \geqslant 1$ be a number; it will be specified later on. We observe that for small $\varepsilon = \varepsilon(\|u\|_\infty, k, \varrho, q, \text{etc.}) > 0$ the function f defined by

$$f(\zeta) = \zeta - \varepsilon\tau^2 G_\varrho[\zeta - k]^q, \qquad [\xi]^q = |\xi|^{q-1}\xi,$$

is monotone increasing in ζ. Hence $f\big(u(x)\big) \geqslant f\big(\psi(x)\big) \geqslant \psi(x)$ and

$$u_\varepsilon := u - \varepsilon\tau^2 G_\varrho[u-k]^q \in \mathbf{K}.$$

We insert this function u_ε into the variational inequality (1.1) and cancel the factor $\varepsilon > 0$, obtaining

$$(1.29)\qquad \sum_{i=1}^n \big(a_i(\cdot, u, \nabla u), \partial_i(\tau^2 G_\varrho[u-k]^q)\big) + \big(a_0(\cdot, u, \nabla u), \tau^2 G_\varrho[u-k]^q\big) \leqslant 0.$$

We now use the identity

$$a_i(\cdot, u, \nabla u) = \sum_{k=1}^n a_{ik}\partial_k u + a_i(\cdot, u, 0),$$

nsert this into (1.29) and estimate the lower order terms via the growth conditions (1.5). This yields

$$(1.30)\qquad \sum_{i,k=1}^n \big(a_{ik}\partial_k u, \partial_i(\tau^2 G_\varrho[u-k]^q)\big) \leqslant K_q\int |\nabla u|\,|u-k|^{q-1}G_\varrho\tau^2 dx + K\int |\nabla u|^2|u-k|^q G_\varrho\tau^2 dx + K_q.$$

The constants K and K_q are uniform for $z \in \Omega$, $\varrho \to 0$.

In order to obtain (1.30) we used the fact that (1.25), (1.26) imply a uniform bound for $\int G_\varrho dx$, $\int |\nabla G_\varrho| dx$, and that $u \in L^\infty$, $a_i(\cdot, u, 0) \in L^\infty$ etc.

From the ellipticity condition (1.23) and Young's inequality ($2ab \leqslant \varepsilon a^2 + \varepsilon^{-1}b^2$) we conclude

$$(1.31)\qquad \sum_{i,k=1}^n \big(a_{ik}\partial_k u, \partial_i(\tau^2 G_\varrho[u-k]^q)\big) \geqslant (q-\tfrac{1}{4})c_0\int |\nabla u|^2|u-k|^{q-1}\tau^2 G_\varrho dx + (q+1)^{-1}\sum_{i,k=1}^n \big(a_{ik}\partial_k(|u-k|^{q+1}\tau^2), \partial_i G_\varrho\big) - K_q.$$

By the definition of G_ϱ the second summand on the right hand side of (1.31) is non-negative and can be dropped. This yields

$$(1.32)\qquad (q-\tfrac{1}{4})\int |\nabla u|^2|u-k|^{q-1}G_\varrho\tau^2dx$$
$$\leqslant K_q\int|\nabla u|\,|u-k|^{q-1}G_\varrho\tau^2dx+K\int|\nabla u|^2|u-k|^qG_\varrho\tau^2dx+K_q.$$

(We passed from $c_0^{-1}K_q$ to K_q etc.). We apply Young's inequality to the first term on the right hand side of (1.32) and obtain

$$(1.33)\qquad (q-\tfrac{1}{2})\int|\nabla u|^2|u-k|^{q-1}G_\varrho\tau^2dx\leqslant K\int|\nabla u|^2|u-k|^qG_\varrho\tau^2dx+\overline{K}_q$$

uniformly for $\varrho\to 0$, $z\in\Omega$. The constant K does not depend on q. We now choose the number q so as to have

$$q-\tfrac{1}{2}\geqslant K\|u-k\|_\infty+1$$

and we conclude

$$(1.34)\qquad \int|\nabla u|^2|u-k|^{q-1}G_\varrho\tau^2dx\leqslant\tilde{K}.$$

Passing to the limit $\varrho\to 0$ we obtain

$$(1.35)\qquad \int|\nabla u|^2|u-k|^{q-1}G\tau^2dx\leqslant\tilde{K}$$

uniformly for $z\in\Omega$.

Inequality (1.34) implies that

$$(1.36)\qquad \int_l|\nabla u|^2G_\varrho\tau^2dx\leqslant\overline{K}_l,\qquad l\to 0\qquad(\varrho\to 0)$$

where $\int_l$ denotes integration over the set

$$\{x\in\Omega|\ |u-k|(x)>l\}.$$

We look once more at (1.33) and now we choose $q=1$. This yields

$$\int|\nabla u|^2G_\varrho\tau^2dx\leqslant K\int|\nabla u|^2|u-k|G_\varrho\tau^2dx+K$$
$$\leqslant Kl\int|\nabla u|^2G_\varrho\tau^2dx+\tilde{K}\int_l|\nabla u|^2G_\varrho\tau^2dx+K$$

and in view of (1.36)

$$\int|\nabla u|^2G_\varrho\tau^2dx\leqslant Kl\int|\nabla u|^2G_\varrho\tau^2dx+K_0\qquad(\varrho\to 0).$$

Taking $l=\tfrac{1}{2}K^{-1}$ we obtain the uniform bound

$$\int|\nabla u|^2G_\varrho\tau^2dx\leqslant K$$

and the local result (1.24) follows by passing to the limit $\varrho\to 0$ and by (1.27), (1.28).

The estimate (1.24) up to the boundary follows easily if we consider the test function

$$u_\varepsilon=u-\varepsilon G_\varrho[u-u_0]^q,\qquad q\geqslant 1,$$

which belongs to $\boldsymbol{K}$ for small $\varepsilon>0$. Note that $u-u_0=0$ outside of Ω. One has to proceed as before, replacing the term $[u-k]$ by the term $[u-u_0]$. Since u_0 is not necessarily a constant function, there occur certain error terms of the type $K\int|\nabla u_0|\,|\nabla G_\varrho|dx$, $K\int|\nabla u_0|^2G_\varrho dx$ in our calculations and estimates. However, these terms are bounded uniformly because of (1.25) and (1.26) and our assumption $\nabla u_0\in L^p$, $p>n$. The lemma is proved.

Proof of Theorem 1.3. We first prove the interior continuity. Let τ be a Lipschitz continuous function such that

$$\operatorname{supp}\tau\subset B_{2R}(z)\subset\Omega,\qquad 0\leqslant\tau\leqslant 1,\ |\nabla\tau|\leqslant R^{-1},\ \tau=1\ \text{on}\ B_R(z).$$

Let G_ϱ be the function defined in the proof of Lemma 1.1, and let $g\in H_0^1(\Omega)\cap\cap L^\infty(\Omega)$, $g\geqslant 0$, be a function, to be defined later on. Since $u\geqslant\psi$ (in $H^1(\Omega)$, or "a.e."), we conclude that for small $\varepsilon=\varepsilon(\varrho,g)>0$

$$(1.37)\qquad u(x)-\varepsilon\tau^2(x)g(x)G_\varrho(x)\big(u(x)-u(y)-\psi(x)+\psi(y)\big)\geqslant\psi(x)$$

for all $x,y\in\Omega$ except a set of capacity zero (or of measure zero if the inequality $u\geqslant\psi$ is understood in the sense "almost everywhere").

Let $\tilde{u}_R$ and $\tilde{\psi}_R$ be the average of u and ψ taken over the set $B_{2R}(z)-B_R(z)$. From (1.37) we obtain, by averaging over $B_{2R}(z)-B_R(z)$ with respect to y,

$$(1.38)\qquad u-\varepsilon\tau^2gG_\varrho(u-\tilde{u}_R-\psi+\tilde{\psi}_R)\geqslant\psi.$$

Since ψ is continuous on $\overline{\Omega}$, we have

$$|\psi-\tilde{\psi}_R|\leqslant\delta\quad\text{on}\quad B_{2R}(z),\ R\leqslant R_\delta,$$

where δ does not depend on $z\in\Omega$. From this and (1.38) we conclude that

$$(1.39)\qquad u-\varepsilon\tau^2gG_\varrho(u-\tilde{u}_R-\delta)\geqslant\psi,\qquad R\leqslant R_\delta.$$

We now set $g=|u-\tilde{u}_R-\delta|^{q-1}$, where $q\geqslant 1$ is an exponent which will be specified later on. From (1.39) and the fact that $\operatorname{supp}\tau\subset\Omega$ we obtain that

$$v:=u-\varepsilon\tau^2G_\varrho[u-\tilde{u}_R-\delta]^q\in\boldsymbol{K}$$

where $[\cdot]^q$ is defined as in the proof of Lemma 1.1.

We insert the function v into the variational inequality (1.1) and obtain, after cancelling $\varepsilon>0$,

$$(1.40)\qquad \sum_{i=0}^n\big(a_i(\cdot,u,\nabla u),\partial_i(\tau^2G_\varrho[u-\tilde{u}_R-\delta]^q)\big)\leqslant 0.$$

We split the principal part of the differential operator as in the proof of Lemma 1.1. This yields

$$(1.41)\qquad \sum_{i,k=1}^{n}\big(a_{ik}\partial_k u,\ \partial_i(\tau^2 G_\varrho[u-\tilde u_R-\delta]^q)\big)$$

$$\leqslant \sum_{i=1}^{n}\big(a_i(\cdot,u,0),\ \partial_i(\tau^2 G_\varrho[u-\tilde u_R-\delta]^q)\big)+\big(a_0(\cdot,u,\nabla u),\ \tau^2 G_\varrho[u-\tilde u_R-\delta]^q\big).$$

The first summand on the right hand side of (1.41) can be estimated by KR^β, $0<\beta<1$, since $a_i(\cdot,u,0)\in L^\infty$, $u\in L^\infty$, $|\nabla\tau|\leqslant R^{-1}$, and $\|G_\varrho\|_s\leqslant K_s$, $1\leqslant s<n/(n-1)$, uniformly as $\varrho\to 0$. The second summand can be estimated via the growth condition (1.5). The left hand side of (1.41) is estimated from below using the ellipticity condition (1.23). Thus we arrive at the inequality

$$(1.42)\qquad qc_0\int|\nabla u|^2|u-\tilde u_R-\delta|^{q-1}\tau^2G_\varrho dx+$$

$$+(q+1)^{-1}\sum_{i,k=1}^{n}\big(a_{ik}\partial_k|u-\tilde u_R-\delta|^{q+1},\ \partial_i(\tau^2G_\varrho)\big)$$

$$\leqslant KR^\beta+K\int|\nabla u|^2|u-\tilde u_R-\delta|^q\tau^2G_\varrho dx.$$

We use the identity

$$\sum_{i,k=1}^{n}\big(a_{ik}\partial_k(|u-\tilde u_R-\delta|^{q+1}\tau^2),\ \partial_iG_\varrho\big)=|B_\varrho|^{-1}\int_\varrho|u-\tilde u_R-\delta|^{q+1}dx$$

where $\int_\varrho$ denotes integration over the ball $B_\varrho(z)\subset B_R(z)$. With this, we estimate the second summand in (1.42) from below by

$$(q+1)^{-1}|B_\varrho|^{-1}\int_\varrho|u-\tilde u_R-\delta|^{q+1}dx-K\int|\nabla u|\,|u-\tilde u_R-\delta|^q|\nabla\tau|\tau G_\varrho dx-$$

$$-K\int|u-\tilde u_R-\delta|^{q+1}|\nabla\tau|\,|\nabla G_\varrho|dx.$$

Thus we obtain (employing also Young's inequality)

$$(1.43)$$

$$(q-\tfrac14)c_0\int|\nabla u|^2|u-\tilde u_R-\delta|^{q-1}\tau^2G_\varrho dx+(q+1)^{-1}|B_\varrho|^{-1}\int_\varrho|u-\tilde u_R-\delta|^{q+1}dx$$

$$\leqslant KR^\beta+K\int|\nabla u|^2|u-\tilde u_R-\delta|^q\tau^2G_\varrho dx+$$

$$+K\int|u-\tilde u_R-\delta|^{q+1}(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx,$$

$$\varrho\to 0,\ z\in\Omega,\ \varrho\leqslant R\leqslant R_\delta,\ B_{2R}(z)\subset\Omega.$$

By the properties of τ and G_ϱ it is known ([22], [38]) that $\int(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx$ is bounded uniformly.

Since $u\in L^\infty$, $G_\varrho\leqslant KG+K$ (cf. [38]), the terms $\int|\nabla u|^2G_\varrho$ are uniformly bounded for $\varrho\to 0$, $z\in\Omega$. Therefore we deduce from (1.43) that

$$(1.44)\qquad (q-\tfrac14)c_0\int|\nabla u|^2|u-\tilde u_R|^{q-1}\tau^2G_\varrho dx+(q+1)^{-1}|B_\varrho|^{-1}\int_\varrho|u-\tilde u_R|^{q+1}dx$$

$$\leqslant K_q\delta+K\int|\nabla u|^2|u-\tilde u_R|^q\tau^2G_\varrho dx+K\int|u-\tilde u_R|^{q+1}(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx,$$

$$R\leqslant R_\delta.$$

We apply the same trick as in the proof of Lemma 1.1. We first choose q large enough to have $(q-\frac14c_0)\geqslant 2K$, and obtain the bound

$$\int|\nabla u|^2|u-\tilde u_R|^{q-1}\tau^2G_\varrho dx\leqslant K\delta+K\int|u-\tilde u_R|^2(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx.$$

This implies

$$\int_l|\nabla u|^2\tau^2G_\varrho dx\leqslant K\delta+K\int|u-\tilde u_R|^2(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx$$

where $\int_l$ denotes integration over the set $A_l=\{x\in B_{2R}(z)\mid\ |u-\tilde u_R|(x)\geqslant l\}$. We then use (1.44) again, with $q=1$ and $l=\frac14c_0/K$. This results in the inequality

$$(1.45)\qquad \int|\nabla u|^2\tau^2G_\varrho dx+|B_\varrho|^{-1}\int|u-\tilde u_R|^2dx$$

$$\leqslant K\delta+K\int|u-\tilde u_R|^2(|\nabla\tau|\,|\nabla G_\varrho|+|\nabla\tau|^2G_\varrho)dx.$$

Note that we splitted the integral $K\int|\nabla u|^2|u-\tilde u_R|\tau^2G_\varrho dx$ into two integrals, the one over A_l, the other one over $\Omega-A_l$, and we proceeded estimating as in the proof of Lemma 1.1.

Passing to the limit $\varrho\to 0$ we conclude from (1.45) that

$$(1.46)\qquad \int|\nabla u|^2\tau^2G\,dx+|u(z)-\tilde u_R|^2$$

$$\leqslant K\delta+K\int|u-\tilde u_R|^2(|\nabla\tau|\,|\nabla G|+|\nabla\tau|^2G)dx.$$

By (1.27) we have $|\nabla\tau|^2G\leqslant KR^{-n}$, $\nabla\tau=0$ on $B_R(z)$. (We do not treat the case $n=2$ separately, since it can be reduced to the case $n=3$ via a dummy variable.) Furthermore, by a simple trick one can derive the estimate

$$(1.47)\qquad \int|u-\tilde u_R|^2|\nabla\tau|\,|\nabla G|dx\leqslant KR^{-n}\int|u-\tilde u_R|^2\chi\big(B_{4R}(z)-B_{R/2}(z)\big)dx+$$

$$+K\int|\nabla u|^2G\chi\big(B_{4R}(z)-B_{R/2}(z)\big)dx$$

where $\chi(M)$ is the characteristic function of a set M. (1.47) follows from the inequality

$$\int|u-\tilde u_R|^2|\nabla\tau|\,|\nabla G|dx$$

$$\leqslant\int|u-\tilde u_R|^2\nabla|\tau^2|R^{2-n}dx+\int|u-u_R|^2|\nabla G|^2R^{n-2}\chi(B_{2R}-B_R)dx.$$

The second summand of the last inequality is estimated by the right hand side of (1.47). This estimate follows from the identity

$$\sum_{i,k=1}^{n}\left(a_{ik}\partial_k(|u-\tilde{u}_R|^2G\tilde{\tau}^2),\ \partial_i G\right)=0, \tag{1.48}$$

where $\tilde{\tau}=1$ on $B_{2R}(z)-B_R(z)$, $\tilde{\tau}=0$ on $B_{R/2}(z)$ and $\boldsymbol{R}^n-B_{4R}(z)$, and $|\nabla\tilde{\tau}|\leqslant KR^{-1}$.

Employing ellipticity and Young's inequality we obtain (1.47).

Thus we arrive at the inequality

$$\int|\nabla u|^2G\tau^2dx+|u(z)-\tilde{u}_R|^2 \leqslant K\delta+KR^{2-n}\int_*|u-\tilde{u}_R|^2dx+K\int_*|\nabla u|^2G\,dx \tag{1.49}$$

for Lebesgue points $z\in\Omega$, $B_R(z)\subset\Omega$, $R\leqslant R_\delta$. Here $\int_*$ denotes integration over $B_{4R}(z)-B_{R/2}(z)$.

By Poincaré's inequality

$$\int_*|u-\tilde{u}_R|^2dx\leqslant KR^2\int_*|\nabla u|^2dx$$

and we can simplify (1.49) to

$$\int_R|\nabla u|^2G\,dx+|u(z)-\tilde{u}_R|^2\leqslant K\delta+K\int_*|\nabla u|^2G\,dx \tag{1.50}$$

where $\int_R$ denotes integration over $B_R(z)$.

Using the "hole-filling-trick" [39] we conclude from (1.50) that

$$\int_{R/2}|\nabla u|^2G\,dx\leqslant\delta+\theta\int_{4R}|\nabla u|^2G\,dx,\qquad \theta=K/(K+1)<1.$$

On replacing R by $2R$ this becomes

$$\int_R|\nabla u|^2G\,dx\leqslant\delta+\theta\int_{8R}|\nabla u|^2G\,dx. \tag{1.51}$$

From (1.51) we obtain by iteration

$$\int_R|\nabla u|^2G\,dx\leqslant K_0\delta+(R/R_0)^\alpha\int_{R_0}|\nabla u|^2G\,dx,\qquad R<R_0\leqslant R_\delta$$

with a certain constant $\alpha=\alpha(\theta)\in\,]0,1[$. Choosing R small enough, we hence conclude

$$\int_R|\nabla u|^2G\,dx\leqslant K\delta,\qquad R\leqslant R_\delta,\qquad B_{R_\delta}(z)\subset\Omega,\ \text{uniformly for } z\in\Omega. \tag{1.52}$$

Applying this to (1.50) we obtain

$$|u(z)-\tilde{u}_R|^2\leqslant\bar{K}\delta,\qquad R\leqslant R_\delta,\ B_{R_\delta}(z)\subset\Omega \tag{1.53}$$

for Lebesgue points $z\in\Omega$ and

$$|u(y)-u(z)|\leqslant K_0\delta^{1/2}+|\tilde{u}_R(y)-\tilde{u}_R(z)|, \qquad R\leqslant R_\delta,\quad B_{R_\delta}(y)\cup B_{R_\delta}(z)\subset\Omega, \tag{1.54}$$

where $\tilde{u}_R(y)$ denotes the mean value of u over the set $B_{2R}(y)-B_R(y)$.

If $|y-z|\leqslant R$, we can estimate

$$|\tilde{u}_R(y)-\tilde{u}_R(z)|^2\leqslant KR^{2-n}\int_{3R}|\nabla u|^2dx\leqslant K\int_{3R}|\nabla u|^2G\,dx\leqslant K\delta$$

and we conclude from (1.54), for a given $\delta>0$, that

$$|u(y)-u(z)|\leqslant 2K_0\delta^{1/2},\qquad R\leqslant R_\delta,\qquad B_{R_\delta}(y)\cup B_{R_\delta}(z)\subset\Omega$$

for Lebesgue points $y,z\in\Omega$. This proves the statement about the interior continuity of u. The continuity of u up to the boundary follows from Lemma 1.2 below. We are stating it in a separate lemma, because we need this fact also in further sections.

LEMMA 1.2. *Under the assumptions* (1.2), (1.4)–(1.6), (1.8) *and* (1.18) *every solution $u\in L^\infty(\Omega)$ is continuous at the points of $\partial\Omega$.*

Note that we do not assume any regularity of the obstacle ψ besides the existence of the function u_0 in condition (1.18). If $\psi\leqslant-\alpha_0<0$ in a neighbourhood of $\partial\Omega$ and ψ is bounded from above, then such a function u_0 exists and Lemma 1.2 implies that the solution u of (1.1) *does not touch the obstacle ψ in a neighbourhood of $\partial\Omega$*. This neighbourhood can be chosen uniformly for all obstacles ψ with a common upper bound and a common neighbourhood $U(\partial\Omega)$ where $\psi\leqslant-\alpha_0$.

Proof of Lemma 1.2. Let G_ϱ, τ, and $[\zeta]_l^q$ be defined as in the foregoing proof. Then, for small $\varepsilon>0$, the function

$$v=u-\varepsilon\tau^2G_\varrho[u-u_0]^q\in\boldsymbol{K},\qquad q\geqslant1,$$

is an admissible variation. We insert this function v into the variational inequality and we obtain, just as in the proof of the interior continuity and the boundedness of $\int|\nabla u|^2G\,dx$, that

$$\int_R|\nabla u|^2G\,dx+|u(z)-u_0(z)|^2 \leqslant KR^\beta+K\int_*|\nabla u|^2G\,dx+KR^{-n}\int_*|u-u_0|^2dx. \tag{1.55}$$

The symbol $\int_*$ denotes integration over $B_{8R}(z)-B_R(z)$.

The difference in the proofs consists in the fact that u_0 is not necessarily constant and the error terms of the type $\int|\nabla u_0|\,|\nabla G|\tau^2dx$, $\int|\nabla u_0|^2G\tau^2dx$ etc. occur. However, these terms can be estimated by KR^β, since $\nabla u_0\in L^p$ for some $p>n$ and $\nabla G\in L^r$ for all $r<n/(n-1)$. Since $u-u_0=0$ on $\partial\Omega$ and $\partial\Omega$ satisfies the Wiener condition (1.18), we may apply Poincaré's

inequality and conclude that

$$\int_* |u-u_0|^2 dx \leqslant KR^2 \int_* |\nabla u - \nabla u_0|^2 dx \leqslant KR^2 \int_* |\nabla u|^2 dx + KR^\beta$$

for those $z \in \Omega$ for which $B_{2R}(z) \cap \complement\Omega \neq 0$. Thus (1.55) is simplified to

(1.56) $$\int_R |\nabla u|^2 G\,dx + |u(z)-u_0(z)|^2 \leqslant KR^\beta + K \int_* |\nabla u|^2 G\,dx.$$

(Recall that $c|x-z|^{2-n} \leqslant G(x) \leqslant K|x-z|^{2-n}$, $n \geqslant 3$.)

From (1.56) we obtain via the hole-filling technique that

$$\int_R |\nabla u|^2 G\,dx \leqslant K_0 R^\alpha, \qquad R \leqslant R_0$$

for some $\alpha \in]0,1[$, provided that $B_{2R_0}(z) \cap \complement\Omega \neq \emptyset$.

Using (1.56) again, we conclude that

$$|u(z)-u_0(z)| \leqslant KR^\gamma, \qquad R \leqslant R_0/8,$$

for some $\gamma \in]0,1[$. Since $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, the lemma follows.

Remarks. (i) The proofs of Theorem 1.3 and Lemma 1.2 give also an a priori bound for the modulus of continuity of the solution u. In particular, if $\{\psi\}$ is a family of equicontinuous obstacles, then the corresponding solutions to (1.1) are equicontinuous in the interior of Ω. This holds up to the boundary if the functions u_0 are uniformly bounded in $C^\alpha \cap H^{1,p}$, $p > n$.

(ii) There exist also other results on the continuity of the solutions to variational inequalities (1.1) if the obstacle is merely continuous but, to the author's best knowledge, not in the case where $a(x, u, \nabla u)$ has quadratic growth in ∇u.

If we require that the obstacle ψ be Hölder continuous,

(1.57) $$\psi \in C^\mu(\Omega)$$

for some $\mu \in]0,1]$, then the proof of Theorem 1.3 yields also the Hölder continuity of u with some exponent $\alpha \in]0,\mu]$. Furthermore, inequality (1.51), with $\delta = KR^\mu$, yields

(1.58) $$\int_{B_R} |\nabla u|^2 G\,dx \leqslant KR^\alpha$$

where $G(x) = G^z(x) = |x-z|^{2-n}$ for $n \geqslant 3$, and $G(x) = \big|\ln|x-z|\big|$ for $n = 2$.

The constant K in (1.57) is uniform for $0 < R < R_0$, $z \in \Omega_0 \subset\subset \Omega$. The results extend up to the boundary of Ω if (1.8) and (1.18) hold. Summing up, we arrive at the following

THEOREM 1.4. *Under the assumptions* (1.2)–(1.6), (1.22), (1.23) *and* (1.57), *every solution $u \in L^\infty(\Omega)$ of* (1.1) *is Hölder continuous in Ω and satisfies* (1.58). *If, in addition, the assumptions* (1.8) *and* (1.18) *are fulfilled, then u is Hölder continuous up to the boundary and* (1.58) *holds uniformly for $z \in \Omega$.*

2. C^α-regularity results if the obstacle is in C^α, $0 < \alpha \leqslant 1$

Again we consider the variational inequality

(2.1) *Find $u \in \boldsymbol{K} := \{v \in H_0^1(\Omega) |\ v \geqslant \psi \text{ in } \Omega\}$ such that*

$$\sum_{i=1}^n \big(a_i(\cdot, u, \nabla u), \partial_i u - \partial_i v\big) + \big(a_0(\cdot, u, \nabla u), u-v\big) \leqslant 0 \quad \textit{for all } v \in \boldsymbol{K}.$$

Regarding the obstacle ψ we shall assume

(2.2) $$\psi \in C^\alpha(\overline{\Omega}) \quad \textit{for a certain } \alpha \in]0,1].$$

The purpose of this section is to establish that (2.2) implies, under suitable conditions, the Hölder continuity of a solution u of (2.1) with the *same* Hölder exponent α. Results of this type have been already obtained by Biroli [5], [6]. The present results are more general than those in [5], [6], as far as nonlinearity is concerned. The technique of proof is different and yields a certain interesting additional result on the differentiability of the solution u of (1). For applications the results are important, in particular, in the case of $\alpha = 1$. For earlier results cf. also [25].

The assumptions on the functions a_i (differentiability, growth, ellipticity) are the following:

(2.3) *The functions $a_i(x, u, p)$ are differentiable in $(x, u, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$, $i = 0, 1, \ldots, n$.*

(2.4) *The derivatives $a_{ix}(x, u, p)$, $a_{iu}(x, u, p)$ and $a_{ip}(x, u, p)$ are measurable in x and continuous in (u, p), $i = 0, 1, \ldots, n$.*

(2.5) *There exists a constant K_C such that*

$$|a_i(x,u,p)| + |a_{ix}(x,u,p)| + |a_{iu}(x,u,p)| \leqslant K_C|p| + K_C,$$
$$|a_0(x,u,p)| + |a_{0x}(x,u,p)| + |a_{0u}(x,u,p)| \leqslant K_C|p|^2 + K_C,$$
$$|a_{ip}(x,u,p)| \leqslant K_C,$$
$$|a_{0p}(x,u,p)| \leqslant K_C|p| + K_C$$

for all $x \in \Omega$, $|u| \leqslant C$, $p \in \boldsymbol{R}^n$, $i = 1, \ldots, n$.

(2.6) *There exists a constant $c = c(C) > 0$ such that*

$$\sum_{i,k=1}^n a_{ik}(x,u,p)\,\xi_i\xi_k \geqslant c|\xi|^2$$

for all $\xi \in \boldsymbol{R}^n$, $x \in \Omega$, $|u| \leqslant C$, $p \in \boldsymbol{R}^n$, where $a_{ik} = (\partial/\partial p_k) a_i$.

We first state a theorem about the interior regularity of bounded solutions u of (2.1). If $\psi < c_0 < 0$ on $\partial\Omega$ and $\partial\Omega \in C^\alpha$, then the results of §1 and Theorem 2.1 below yield global C^α-regularity. After giving the proof of Theorem 2.1 we study the boundary regularity if $\psi \leqslant 0$ on $\partial\Omega$.

THEOREM 2.1. *Under the assumptions* (2.2)–(2.6) *every solution* $u \in L^\infty(\Omega)$ *of* (2.1) *is Hölder continuous in* Ω *with the exponent* a *from* (2.2). *Furthermore, the interior* C^a*-norms of* u *are uniformly bounded if the constants* $C = \|u\|_\infty$, K_C, $c(C)$ *in* (2.5) *and* (2.6) *are uniformly bounded.*

From the proof we shall see the following additional differentiability property:

COROLLARY TO THEOREM 2.1. *Let* ψ *be Lipschitz continuous on* Ω *and let*

$$M_e^+ = \sup\{D_{eh}\psi(x) |\ x, x+he \in \Omega,\ h>0\}, \qquad e \in \boldsymbol{R}^n,\ |e| = 1,$$

$$M_e^- = \inf\{D_{eh}\psi(x) |\ x, x+he \in \Omega,\ h>0\}, \qquad e \in \boldsymbol{R}^n,\ |e| = 1,$$

where $D_{eh}\psi(x) = h^{-1}\big(\psi(x+he)-\psi(x)\big)$. *Then*

$$[(e\nabla)u - M_e^\pm]_\pm \in H^1_{\mathrm{loc}}(\Omega)$$

where $[\xi]_+ = \max(\xi, 0)$, $[\xi]_- = \min(\xi, 0)$.

Remark. The differentiability assumptions on the a_i are not optimal. For example, one could also include lower order terms which are merely in L^∞.

We shall use the abbreviations:

$$D^a_{eh}w(x) = h^{-a}\big(w(x+he)-w(x)\big), \qquad D^a_{-eh}w(x) = h^{-a}\big(w(x)-w(x-he)\big)$$

for $e \in \boldsymbol{R}^n$, $|e| = 1$, $h > 0$.

For the proof of Theorem 2.1 we need the following simple

LEMMA 2.1. *Let* u, ψ, τ *be real functions such that*

$$u \geqslant \psi \text{ a.e. in } \Omega$$

and $0 \leqslant \tau \leqslant 1$ *a.e. in* Ω. *Let* $a \in]0, 1]$ *be given and let*

$$M_e^+ = \sup\{D^a_{eh}\psi(x) |\ h>0;\ x, x+he \in \Omega\} < \infty,$$

$$M_e^- = \inf\{D^a_{eh}\psi(x) |\ h>0;\ x, x+he \in \Omega\} > \infty$$

for all $e \in \boldsymbol{R}^n$, $|e| = 1$. *Then the functions* $u_h^\pm$ *defined by*

$$u_h^+ = u + \tfrac{1}{2}h^a D^a_{-eh}\{\tau[h^a D^a_{eh}u - h^a M_e^+]_+\},$$

$$u_h^- = u + \tfrac{1}{2}h^a D^a_{-eh}\{\tau[h^a D^a_{eh}u - h^a M_e^-]_-\}$$

satisfy the inequality

$$u_h^\pm(x) \geqslant \psi(x) \tag{2.7}$$

for almost all x *in* Ω *such that* $x+he$, $x \in \Omega$.

If the inequalities $u \geqslant \psi$ *and* $0 \leqslant \tau \leqslant 1$ *hold in* Ω *except a set of capacity zero then* (2.7) *holds for all* x *as above except a set of capacity zero (which may depend on* h).

Proof of Lemma 2.1. We have for $x, x+he \in \Omega$

$$u_h^+(x) = u(x) + \tfrac{1}{2}\tau(x)[u(x+he)-u(x)-h^aM_e^+]_+ - \\ -\tfrac{1}{2}\tau(x-he)[u(x)-u(x-he)-h^aM_e^+]_+.$$

Since the terms $[\zeta - u(x) - h^aM_e^+]_+$ and $-[u(x)-\zeta-h^aM_e^+]_+$ are monotone increasing in ζ, and since $u \geqslant \psi$, a.e. in Ω, we conclude that for almost all $x, x+he \in \Omega$,

$$u_h^+(x) \geqslant u(x) + \tfrac{1}{2}\tau(x)[\psi(x+he)-u(x)-h^aM_e^+]_+ - \\ -\tfrac{1}{2}\tau(x-he)[u(x)-\psi(x-he)-h^aM_e^+]_+ =: \varphi\big(u(x)\big),$$

where

$$\varphi(\xi) = \xi + \tfrac{1}{2}\tau(x)[\psi(x+he)-\xi-h^aM_e^+]_+ - \\ -\tfrac{1}{2}\tau(x-he)[\xi-\psi(x-he)-h^aM_e^+]_+.$$

Now, the function φ is monotone increasing, as it can be easily seen by calculating its one-sided derivatives. Hence $\varphi\big(u(x)\big) \geqslant \varphi\big(\psi(x)\big)$ and

$$u_h^+(x) \geqslant \varphi\big(\psi(x)\big) = \psi(x) + d(x),$$

where

$$2d(x) = \tau(x)[\psi(x+he)-\psi(x)-h^aM_e^+]_+ - \\ -\tau(x-he)[\psi(x)-\psi(x-he)-h^aM_e^+]_+$$

for almost all $x \in \Omega$ with $x+he \in \Omega$.

From the definition of M_e^+ we see that

$$\psi(x+he)-\psi(x) \leqslant h^aM_e^+, \qquad \psi(x)-\psi(x-he) \leqslant h^aM_e^+.$$

This yields $d(x) \geqslant 0$ and hence

$$u_h^+(x) \geqslant \psi(x),$$

which was to be shown. A similar argument works for u_h^- and for the "capacity-formulation". The lemma is proved.

We prove Theorem 2.1 first for the case of the Laplacian, i.e., for

$$-\sum_{i=1}^n \partial_i a_i(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = -\Delta u + f(\cdot), \qquad f \in H^{1,\infty}(\Omega), \tag{2.8}$$

where the key-idea of the proof can be easily seen.

Proof of Theorem 2.1 *in the case of* (2.8). Let $G_\varrho \in L^\infty \cap H_0^{1,2}(\Omega)$ be the solution of the equation

$$-\Delta G_\varrho = \delta_\varrho \quad \text{where} \quad \delta_\varrho = |B_\varrho|^{-1}\chi\big(B_\varrho(x_0)\big);$$

$\chi(B_\varrho(x_0))$ is the characteristic function of the ball $B_\varrho(x_0) \subset \Omega$ with radius ϱ and centre $x_0 \in \Omega$. By Lemma 2.1

$$u_h^\pm := u + \varepsilon h^\alpha D_{-eh}^\alpha(\zeta^2 G_\varrho h^\alpha [D_{eh}^\alpha u - M_e^\pm]_\pm) \in \boldsymbol{K},$$

where $\varepsilon = \varepsilon(\zeta, \varrho) > 0$, $\zeta \in C_0^\infty(\Omega)$, $\zeta = 1$ in a neighbourhood of x_0. Inserting $u_h^\pm$ into the variational inequality we obtain (writing down only the case u_h^+)

$$-\big(\nabla u, \nabla\big(D_{-eh}^\alpha(\zeta^2 G_\varrho[D_{eh}^\alpha u - M_e^+]_+)\big)\big) \leqslant -\big(f, D_{-eh}^\alpha(\zeta^2 G_\varrho[D_{eh}^\alpha u - M_e^+]_+)\big).$$

Here we have cancelled the factor $\varepsilon h^{2\alpha}$.

By partial summation we obtain

$$(2.9)\quad \big(\nabla D_{eh}^\alpha u, \nabla(\zeta^2 G_\varrho[D_{eh}^\alpha u - M_e^+]_+)\big) \leqslant (D_{eh}^\alpha f, \zeta^2 G_\varrho[D_{eh}^\alpha u - M_e^+]_+)$$

for $0 < h < h_0(\partial\Omega, \zeta)$. We rewrite the left hand side of (2.9) and estimate the right hand side. This gives

$$(2.10)\quad \int |\nabla D_{eh}^\alpha u|_+^2 G_\varrho \zeta^2 dx + \tfrac{1}{2}\big(\nabla|D_{eh}^\alpha u - M_e^+|_+^2, \nabla(\zeta^2 G_\varrho)\big) \leqslant \|f\|_{1,\infty}\|G_\varrho\|_1\|\zeta^2|D_{eh}^\alpha u - M_e^+|_+\|_\infty,$$

where $|z(x)|_+ = |z(x)|$ if $D_{eh}^\alpha u(x) \geqslant M_e^+$ and $|z(x)|_+ = 0$ otherwise. We now use the identity

$$(2.11)\quad \big(\nabla|D_{eh}^\alpha u - M_e^+|_+^2, \nabla(\zeta^2 G_\varrho)\big) = \big(\nabla(\zeta^2|D_{eh}^\alpha u - M_e^+|_+^2), \nabla G_\varrho\big) + \tfrac{1}{2}A + \tfrac{1}{2}B,$$

where

$$(2.12)\quad A = (\nabla|D_{eh}^\alpha u - M_e^+|_+^2, G_\varrho \nabla\zeta^2) \geqslant -\tfrac{1}{2}\int |\nabla D_{eh}^\alpha u|_+^2 G_\varrho \zeta^2 dx - 2\int |D_{eh}^\alpha u - M_e^+|^2 G_\varrho |\nabla\zeta|^2 dx$$

and

$$(2.13)\quad B = -(|D_{eh}^\alpha u - M_e^+|_+^2, \nabla\zeta^2 \nabla G_\varrho).$$

From the definition of G_ϱ we obtain

$$(2.14)\quad |B_\varrho|^{-1}\int_\varrho |D_{eh}^\alpha u - M_e^+|_+^2 dx = \big(\nabla(\zeta^2|D_{eh} u - M_e^+|_+^2), \nabla G_\varrho\big),$$

where $\int_\varrho$ denotes integration over $B_\varrho(x_0)$. From (2.10)–(2.13) we get

$$\tfrac{3}{4}\int |\nabla D_{eh}^\alpha u|_+^2 G_\varrho \zeta^2 dx + |B_\varrho|^{-1}\int_\varrho |D_{eh}^\alpha u - M_e^+|_+^2 dx \leqslant \|f\|_{1,\infty}\|G_\varrho\|_1\|\zeta^2|D_{eh}^\alpha u - M_e^+|_+\|_\infty + \int |\nabla\zeta|^2|D_{eh}^\alpha u - M_e^+|_+^2 G_\varrho dx + \tfrac{1}{2}(|D_{eh}^\alpha u - M_e^+|_+^2, \nabla\zeta^2 \nabla G_\varrho).$$

We now pass to the limit $\varrho \to 0$ and observe that

$$G_\varrho \to G \text{ weakly in } H^{1,1}(\Omega) \quad (\varrho \to 0).$$

Here G is the Green function of $-\Delta$, i.e.,

$$(\nabla G, \nabla\varphi) = \varphi(x_0), \quad \varphi \in C_0^\infty(\Omega), \ G \in \mathring{H}^{1,1}(\Omega).$$

Taking into account that $D_{eh}^\alpha u \in L^\infty(\Omega_0)$, $\Omega_0 \subset\subset \Omega$, $\nabla\zeta \in L^\infty$, we obtain for points $x_0 \in \Omega_0$

$$(2.15)\quad \tfrac{3}{4}\int |\nabla D_{eh}^\alpha u|_+^2 \zeta^2 G dx + |D_{eh}^\alpha u(x_0) - M_e^+|_+^2 \zeta^2(x_0) \leqslant K\|\zeta^2|D_{eh}^\alpha u - M_e^+|_+\|_\infty + \int |\nabla\zeta|^2|D_{eh}^\alpha u - M_e^+|_+^2 G dx + \tfrac{1}{2}(|D_{eh}^\alpha u - M_e^+|_+^2, \nabla\zeta^2 \nabla G).$$

From Lemma 1.3 we conclude that

$$(2.16)\quad \int |D_{eh}^\alpha u - M_e^+|_+^2 G dx \leqslant K$$

uniformly for $0 < h < h_0$, where $\int$ denotes integration over $\operatorname{supp}\zeta$. We proceed by estimating

$$(|D_{eh}^\alpha u - M_e^+|_+^2, \nabla\zeta^2 \nabla G) \leqslant \|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty \int |D_{eh}^\alpha u - M_e^+|_+ |\nabla\zeta|\,|\nabla G| dx$$

$$\leqslant \|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty \|\nabla\zeta\|_\infty \Big(\int |\nabla G|^2 G^{-1-\sigma} dx\Big)^{1/2}\Big(\int |D_{eh}^\alpha u - M_e^+|_+^2 G^{1+\sigma} dx\Big)^{1/2}$$

$$\leqslant K\|\nabla\zeta\|_\infty \|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty.$$

We used the fact that $\int |\nabla G|^2 G^{-1-\sigma} dx \leqslant K_0$ and $\int_\Omega |D_{eh}^\alpha u - M_e^+|_+^2 G^{1+\sigma} dx \leqslant K_0$, according to Theorem 1.4, for some $\sigma \in]0, 1[$. Choosing a point $x_0 \in \Omega$ such that

$$\zeta(x_0)|D_{eh}^\alpha u(x_0) - M_e^+|_+ = \|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty,$$

we arrive at the inequality

$$\tfrac{1}{4}\int |\nabla D_{eh}^\alpha u|_+^2 \zeta^2 G dx + \tfrac{1}{2}\|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty^2 \leqslant K\|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty + K\|\nabla\zeta\|_\infty \|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty,$$

from which we infer that

$$\|\zeta|D_{eh}^\alpha u - M_e^+|_+\|_\infty \leqslant K$$

uniformly for $h \to 0$, $|e| = 1$. Similarly we obtain

$$\|\zeta|D_{eh}^\alpha u - M_e^-|_-\|_\infty \leqslant K.$$

The theorem (in case of (2.8)) follows.

The corollary follows by the following consideration. Taking into account that $\alpha = 1$ and that Theorem 2.1 implies $\nabla u \in L_{\mathrm{loc}}^\infty(\Omega)$, we obtain from (2.10)

$$\int_0 |\nabla D_{eh}^1 u|_+^2 dx \leqslant K$$

uniformly for $h \to 0$, $|e| = 1$. The symbol $\int_0$ denotes integration over $\Omega_0 \subset\subset \Omega$. Thus

$$\sup\Big\{\int [D^1_{eh}u - M^+_e]_+ \partial_i\varphi\, dx \,\Big|\, \varphi \in C^\infty_0(\Omega_0),\ \|\nabla\varphi\|_2 \leqslant 1\Big\} \leqslant K, \quad i = 1, \ldots, n,$$

and

$$\sup\Big\{\int [(e\nabla)u - M^+_e]_+ \partial_i\varphi\, dx \,\Big|\, \varphi \in C^\infty_0(\Omega_0),\ \|\nabla\varphi\|_2 \leqslant 1\Big\} \leqslant K, \quad i = 1, \ldots, n.$$

The corollary follows.

Proof of Theorem 2.1 *in the general case.* We use a similar variation $u^\pm_h$ as before, namely

$$u^\pm_h := u + \varepsilon h^\alpha D^\alpha_{-eh}(\zeta^2 G_\varrho[h^\alpha D^\alpha_{eh}u - h^\alpha M^+_e]_\pm) \in \boldsymbol{K},$$

$\zeta \in H^{1,\infty}(\Omega)$, $\operatorname{supp}\zeta \subset\subset \Omega$, $0 \leqslant \zeta \leqslant 1$; however, this time $G_\varrho = G_\varrho(\cdot, z)$ is the Green function of the operator

$$L = -\sum_{i=1}^n \partial_i(a_{ik}\partial_k),$$

where

$$(2.17)\quad a_{ik}(x) = \int_0^1 a_{ik}\big(x + hte,\ tu(x+hte) + (1-t)u(x),\ t\nabla u(x+hte) + (1-t)\nabla u(x)\big)\, dt,$$

$$a_{ik}(x, u, p) = (\partial/\partial p_k)\, a_i(x, u, p).$$

We have $G_\varrho \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and

$$\sum_{i,k=1}^n (a_{ik}\partial_k G_\varrho, \partial_i\varphi) = |B_\varrho|^{-1} \int_{B_\varrho(z)} \varphi(x)\, dx, \quad \varphi \in C^\infty_0(\Omega).$$

We insert the above function u^+_h into the variational inequality (2.1) (we do not treat the case u^-_h, which is analogous) and we obtain

$$\sum_{i=0} \big(D^\alpha_{eh} a_i(\cdot, u, \nabla u),\ \partial_i(\zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+)\big) \leqslant 0,$$

$\partial_0 =$ identity. This yields

$$(2.18)\quad \sum_{i,k=1}^n \big(a_{ik}\partial_k D^\alpha_{eh}u,\ \partial_i(\zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+)\big) + A_\varrho + B_\varrho + C_\varrho + D_\varrho + E_\varrho \leqslant 0,$$

where

$$A_\varrho = \sum_{i=1}^n \big(a_{i0}D^\alpha_{eh}u,\ \partial_i(\zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+)\big),$$

$$B_\varrho = h^{1-\alpha}\sum_{i=1}^n \big(a_{ix}e,\ \partial_i(\zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+)\big),$$

$$C_\varrho = \sum_{k=1}^n (a_{0k}\partial_k D^\alpha_{eh}u,\ \zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+),$$

$$D_\varrho = (a_{00}D^\alpha_{eh}u,\ \zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+),$$

$$E_\varrho = h^{1-\alpha}(a_{0x}e,\ \zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+).$$

Here $a_{i0}(x)$, $a_{00}(x)$ is defined as in (2.17),

$$a_{i0}(x, u, p) = (\partial/\partial u)\, a_i(x, u, p), \quad i = 0, \ldots, n.$$

We rewrite the first summand in (2.18) and estimate it from below via ellipticity:

$$\sum_{i,k=1}^n \big(a_{ik}\partial_k D^\alpha_{eh}u,\ \partial_i(\zeta^2 G_\varrho[D^\alpha_{eh}u - M^+_e]_+)\big)$$

$$\geqslant c\int |\nabla D^\alpha_{eh}u|^2_+ G_\varrho\zeta^2 dx + \tfrac{1}{2}\sum_{i,k=1}^n \big(a_{ik}\partial_k(\zeta[D_{eh}u - M^+_e]_+)^2,\ \partial_i G_\varrho\big) + F_1 + F_2,$$

where

$$F_1 = \sum_{i,k=1}^n (a_{ik}\partial_k D^\alpha_{eh}u,\ G_\varrho[D_{eh}u - M^+_e]_+ \partial_i\zeta^2),$$

$$F_2 = -\sum_{i,k=1}^n (a_{ik}|D^\alpha_{eh}u - M^+_e|^2_+ \zeta\partial_k\zeta,\ \partial_i G_\varrho).$$

Here $|D^\alpha_{eh}u|^2_+$ is defined as in the proof before.

We take into account that $G_\varrho \in L^\infty\big(\Omega_0 - U(z)\big)$, where $\Omega_0 \subset\subset \Omega$ and $U(z)$ is a neighbourhood of z, and that the L^∞-bound of G_ϱ taken over $\Omega_0 - U(z)$ remains bounded as $\varrho \to 0$. Thus we may estimate the term F_1 by

$$F_1 \geqslant -\frac{c}{2}\int |\nabla D^\alpha_{eh}u|^2_+ G_\varrho\zeta^2 dx - K_0\int |D^\alpha_{eh}u - M^+_e|^2_+ |\nabla\zeta|^2 dx$$

$$\geqslant -\frac{c}{2}\int |\nabla D^\alpha_{eh}u|^2_+ G_\varrho\zeta^2 dx - K_1,$$

where K_1 remains bounded for $\varrho \to 0$, $0 < h \leqslant h_0$. Recall that $\nabla\zeta = 0$ in $U(z)$. Since $\|\nabla G_\varrho\|_{2;\,\Omega - U(z)} \leqslant K$, we may also estimate the term F_2 by

$$F_2 \geqslant -K_0\|\zeta|D^\alpha_{eh}u - M^+_e|_+\|_\infty \Big(\int |D^\alpha_{eh}u - M^+_e|^2_+ |\nabla\zeta|^2 dx\Big)^{1/2}$$

$$\geqslant -K_1\|\zeta|D_{eh}u - M^+_e|_+\|_\infty.$$

Thus we arrive at the inequality

$$(2.19)\qquad \sum_{i,k=1}^{n}\big(a_{ik}\partial_k D^{\alpha}_{eh}u,\ \partial_i(\zeta^2 G_\varrho[D^{\alpha}_{eh}u-M_e^+]_+)\big)$$

$$\geqslant \frac{c}{2}\int |\nabla D^{\alpha}_{eh}u|_+^2 G_\varrho^2 dx+|B_\varrho|^{-1}\int_\varrho \zeta^2|D^{\alpha}_{eh}u-M_e^+|_+^2\,dx-$$

$$-K_1\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty-K_1.$$

Here $\int_\varrho$ denotes integration over $B_\varrho(z)$.

We now estimate the terms $A_\varrho, \ldots, E_\varrho$, uniformly for $\varrho\to 0$, $h\to 0$, $z\in\Omega_0$. To this end, we use the growth conditions (2.5) for the functions a_{ik}, a_{ix} etc., and also the C^μ-regularity results of § 1, which establish that

$$(2.20)\qquad \int|\nabla u|^2|x-z|^{2-n-\mu}dx\leqslant K_0$$

(at least locally) with $\mu>0$ and K_0 denoting some constants.

Estimation of A_ϱ. By (2.5) and Hölder's inequality,

$$|A_\varrho|\leqslant \varepsilon_0\int|\nabla D^{\alpha}_{eh}u|_+^2 G_\varrho\zeta^2dx+K(\varepsilon_0)\|\zeta|D^{\alpha}_{eh}u|_+\|_\infty^2\int_*(|\nabla u|^2+1)G_\varrho dx+$$

$$+\|\zeta|D^{\alpha}_{eh}u|_+\|_\infty\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty\int_*(|\nabla u|+1)|\nabla G_\varrho|dx+$$

$$+K_0\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty\int_*(|\nabla u|+1)|D^{\alpha}_{eh}u||\nabla\zeta|G_\varrho dx.$$

Here $\int_*$ denotes integration over $\operatorname{supp}\zeta$.

By (2.20) the quantities $\int_*(|\nabla u|^2+1)G_\varrho dx$ and $\int_*(|\nabla u|+1)|\nabla G_\varrho|dx$ are small if the support of ζ is contained in a ball with radius sufficiently small. (Estimate $\big(\int_*|\nabla u|\,|\nabla G_\varrho|dx\big)^2\leqslant\int_*|\nabla u|^2G^{1+\sigma}dx\int_*|\nabla G_\varrho|^2G^{-1-\sigma}dx$ and $\int_*|\nabla G_\varrho|^2G^{-1-\sigma}dx$ via the definition of G_ϱ.)

Thus we obtain for every $\varepsilon_0, \varepsilon_1>0$

$$(2.21)\qquad |A_\varrho|\leqslant\varepsilon_0\int|\nabla D^{\alpha}_{eh}u|_+^2G_\varrho\zeta^2dx+\varepsilon_1\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2+K(\varepsilon_0,\varepsilon_1),$$

provided that the support of ζ is chosen sufficiently small (in accordance with $\varepsilon_0, \varepsilon_1$). Note that we estimate $|\nabla\zeta|G_\varrho<K(\varepsilon_1)$, since $\nabla\zeta=0$ in $U(z)$. The term $\int_*|\nabla u|\,|D^{\alpha}_{eh}u|dx$ can be estimated by $K\int|\nabla u|^2dx$.

Estimation of B_ϱ. This term behaves slightly better than the term A_ϱ and can be estimated similarly. This yields

$$(2.22)\qquad |B_\varrho|\leqslant\varepsilon_0\int|\nabla D^{\alpha}_{eh}u|_+^2G_\varrho\zeta^2dx+\varepsilon_1\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2+K(\varepsilon_0,\varepsilon_1).$$

Estimation of C_ϱ, D_ϱ, E_ϱ. By (2.5) and Hölder's inequality we obtain

$$|C_\varrho|+|D_\varrho|+|E_\varrho|\leqslant\varepsilon_0\int|\nabla D^{\alpha}_{eh}u|_+^2G_\varrho\zeta^2dx+$$

$$+K(\varepsilon_0)\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2\int_*(|\nabla u|^2+1)G_\varrho dx+$$

$$+K_0(\|\zeta|D^{\alpha}_{eh}u|_+\|_\infty+1)\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty\int_*(|\nabla u|^2+1)G_\varrho dx.$$

As before we conclude, for $\zeta\in C_0^\infty(\Omega)$ with support sufficiently small, that

$$(2.23)\qquad |C_\varrho|+|D_\varrho|+|E_\varrho|$$

$$\leqslant\varepsilon_0\int|\nabla D^{\alpha}_{eh}u|_+^2G_\varrho\zeta^2dx+\varepsilon_1\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2+K(\varepsilon_0,\varepsilon_1).$$

From (2.18), (2.19), (2.21), (2.22) and (2.23) we see that

$$(c/2-3\varepsilon_0)\int|\nabla D^{\alpha}_{eh}u|_+^2G_\varrho\zeta^2dx+|B_\varrho|^{-1}\int_\varrho\zeta^2|D^{\alpha}_{eh}u-M_e^+|_+^2dx-$$

$$-4\varepsilon_1\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2\leqslant K(\varepsilon_1).$$

We choose ε_0 and ε_1 small enough (say, $\varepsilon_0=c/12$, $\varepsilon_1=1/8$) and pass to the limit $\varrho\to 0$. This yields

$$(2.24)\qquad \int|\nabla D^{\alpha}_{eh}u|_+^2G\zeta^2dx+(\zeta|D^{\alpha}_{eh}u-M_e^+|_+)^2(z)-\tfrac{1}{2}\|\zeta|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2\leqslant K_0.$$

We choose $z\in\Omega_0$ such that

$$(|D^{\alpha}_{eh}u-M_e^+|_+)^2(z)=\|\,|D^{\alpha}_{eh}u-M_e^+|_+\|_{\infty;\Omega_0}^2$$

and we obtain from (2.24)

$$\|\,|D^{\alpha}_{eh}u-M_e^+|_+\|_\infty^2\leqslant 2K_0.$$

(Recall that $\zeta=1$ in $U(z)$.) In a similar fashion we get

$$\|\,|D^{\alpha}_{eh}u-M_e^-|_-\|_\infty^2\leqslant 2K_0.$$

This completes the proof of Theorem 2.1.

The corollary follows from (2.24), which implies also an estimate for $\int|\nabla D^{\alpha}_{eh}u|_\pm^2G\zeta^2dx$.

Remarks. (i) The case of two Hölder continuous obstacles ψ_1,ψ_2, $\psi_1\leqslant u\leqslant\psi_2$ can be reduced by a partition of unity to the case of one obstacle if $\psi_1<\psi_2$ and u is already known to be continuous. For the latter question, cf. § 1.

(ii) If the obstacle is only Hölder continuous with exponent α in the direction of some unit vector e and if $u\in C^\mu$ for some $\mu\in\,]0,1[$, then u is Hölder continuous with exponent α in the direction of e. This holds also for thin obstacles and "boundary obstacles"

$$u\geqslant\psi\quad\text{ on }\partial\Omega.$$

In the latter case, if $\psi\colon \partial\Omega \to \mathbf{R}$ is Lipschitz, the method of proof shows that the derivatives tangential to $\partial\Omega$ are bounded. (Here we assume that $\partial\Omega$ is smooth, say, $\partial\Omega \in C^2$.)

(iii) If $\psi \geqslant 0$ on $\partial\Omega$, $\psi \in H^{1,\infty}(\Omega)$, $\partial\Omega \in C^2$, and $\mathbf{K} = \{v \in \mathring{H}^1_0(\Omega) | v \geqslant \psi\}$ is not empty, then the method of the proof of Theorem 2.1 shows that the derivatives of u tangential to $\partial\Omega$ are bounded. To obtain the regularity of the normal derivatives, one has to establish certain differential inequalities for the solution u as it has been done in [20], [31].

(iv) If we require the obstacle ψ to be merely continuous, then one can still prove that u is continuous, provided that the lower order terms which are quadratic in ∇u (cf. the proof of Theorem 2.1) do not occur. (Clearly, Theorem 1.3 is much stronger.)

For the simple case of (2.8) we obtain

THEOREM 2.2. *Let* $u \in \mathbf{K} = \{v \in \mathring{H}^1_0(\Omega) |\ v \geqslant \psi\}$ *be a solution of*

$$(\nabla u,\ \nabla u - \nabla v) \leqslant (f,\ u-v), \qquad v \in \mathbf{K},$$

where $f \in H^{1,\infty}(\Omega)$ *and* $\psi \in C(\Omega)$. *Then* $u \in C(\Omega)$.

For the proof, let $\Omega_0 \subset\subset \Omega$ and let

$$\omega(h) = \omega(h, \Omega_0) = \sup\{|\psi(x+se) - \psi(x)| \mid x, x+se \in \Omega_0, 0 < s < h, |e| = 1, e \in \mathbf{R}^n\}.$$

Define

$$D^{\omega}_{eh} z(x) = \omega(h)^{-1} |z(x+he) - z(x)|.$$

Then the method of the proof of Theorem 2.1 can be easily adapted to the new situation and it yields an estimate for the modulus of continuity of u; namely we get

$$|D^{\omega}_{eh} u(x)| \leqslant K, \quad x, x+he \in \Omega_0, \ |e| = 1, \ e \in \mathbf{R}^n.$$

3. $C^{1+\alpha}$-regularity results for obstacles in $C^{1+\alpha}$, $0 \leqslant \alpha \leqslant 1$

In this section we consider the simplest elliptic variational inequality over a bounded domain $\Omega \subset \mathbf{R}^n$:

Find $u \in \mathbf{K} = \{u \in \mathring{H}^{1,2}_0(\Omega) |\ u \geqslant \psi$ *a.e. in* $\Omega\}$ *such that*

$$(3.1) \qquad (\nabla u,\ \nabla u - \nabla v) \leqslant (f,\ u - v)$$

for all $v \in \mathbf{K}$.

Here and in the following $(w, z) = \int_\Omega wz\,dx$.

Inequality (3.1) implies that u minimizes

$$\tfrac{1}{2}(\nabla u,\ \nabla u) - (f,\ u)$$

on $\mathbf{K}$ and vice versa.

We shall impose the following conditions on the data:

$$(3.2) \qquad \partial\Omega \text{ satisfies the Wiener condition } (1.8),$$

$$(3.3) \qquad f \in L^{n/(1-\alpha)}(\Omega) \quad \text{for some } \alpha \in]0, 1[,$$

which implies $\Delta^{-1} f \in C^{1+\alpha}(\Omega)$,

$$(3.4) \qquad \psi \leqslant -\delta < 0 \text{ in a neighbourhood of } \partial\Omega,$$

$$(3.5') \qquad \psi \in C^{1+\alpha}(\Omega).$$

Condition (3.5′) may be replaced by

$$(3.5) \quad m = \inf\{\delta^{\alpha}_{eh}\psi(x) |\ h > 0,\ e \in \mathbf{R}^n,\ |e| = 1;\ x, x \pm he \in \Omega\} > -\infty.$$

Here $\delta^{\alpha}_{eh} w(x) = h^{-1-\alpha}\big(w(x+he) - 2w(x) + w(x-he)\big)$.

If we leave out (3.5′), we have to assume some regularity of ψ, say

$$(3.6') \qquad \psi \in C(\Omega),$$

or, which is weaker,

$$(3.6) \qquad \psi \in L^1(\Omega) \quad \text{and} \quad \mathbf{K} \neq \emptyset.$$

Under these assumptions we prove the following

THEOREM 3.1. *Let u be the solution of the variational inequality* (3.1), *whose data satisfy* (3.2)–(3.6). *Then* $u \in C^{1+\alpha}(\Omega)$.

We remind the reader that in our notation $u \in C^{1+\alpha}(\Omega)$ does not imply $u \in C^{1+\alpha}(\bar{\Omega})$. If $\partial\Omega$ is smooth enough, then $u \in C^{1+\alpha}(\bar{\Omega})$, since the solution u of the variational inequality does not touch the obstacle in a neighbourhood of $\partial\Omega$.

Proof. We first prove the theorem under the assumption (3.6′) instead of (3.6). From the results of § 1 it follows that $u \in C(\bar{\Omega})$ and $u > \psi$ in $U \cap \Omega$, where U is an open neighbourhood of $\partial\Omega$. Hence the set $I = \{x \in \Omega |\ u(x) = \psi(x)\}$ is closed and contained in an open subset $\Omega_0 \subset\subset \Omega$ with smooth boundary $\partial\Omega_0 \subset U \cap \Omega$. The assumptions on the data imply $u \in C^{1+\alpha}(U \cap \Omega)$ for the restriction of u to $U \cap \Omega$. The proof consists in showing that the quantities

$$\delta^{\alpha}_{eh} u(x) = h^{-1-\alpha}\big(u(x+he) - 2u(x) + u(x-he)\big)$$

are uniformly bounded for $x \in \Omega_0$, $e \in \mathbf{R}^n$, $|e| = 1$, $0 < h < h_0 := \frac{1}{2}\inf\{|y_0 - y_1| \mid y_0 \in \partial\Omega,\ y_1 \in \partial\Omega_0\}$. From this the theorem follows via a result of the theory of Sobolev–Besov-spaces. The proof of the boundedness of $\delta^{\alpha}_{eh} u$ is similar to the author's proof [15] of the boundedness of the second derivatives of u in the case where $\psi \in C^{1,1}$ and $f \in C^{\beta}$; now, however, additional technical difficulties arise.

From (3.1) and the closedness of I we obtain that in the sense of distributions

(3.7) $$-\Delta u = f \quad \text{on } \Omega - I$$

and

(3.8) $$\Delta u \leqslant -f \quad \text{on } \Omega.$$

To see this, choose $v = u \pm \varepsilon\varphi$, $\varphi \in C_0^\infty(\Omega - I)$, ε small, or $v = u+\varphi$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$.

From (3.7) and (3.8) we conclude that

(3.9) $$\Delta \delta_{eh}^a u \leqslant -\delta_{eh}^a f \quad \text{on } \Omega_0 - I, \quad 0 < h < h_0,\ e \in \boldsymbol{R}^n,\ |e| = 1.$$

We solve the equation

$$-\Delta z = f \quad \text{in } \Omega$$

and observe that (3.3) implies $z \in C^{1+a}(\Omega)$. From (3.9) we obtain

$$\Delta \delta_{eh}^a (u-z) \leqslant 0 \quad \text{on } \Omega_0 - I$$

and from the maximum principle (which is proved by truncation methods here, cf. [36])

(3.10) $$\delta_{eh}^a (u-z) \geqslant \mu_0 \quad \text{on } \Omega_0, \quad 0 < h < h_1,\ e \in \boldsymbol{R}^n,\ |e| = 1,$$

where

$$\mu_0 = \inf\{\delta_{eh}^a(u-z)(x) |\ x \in \partial\Omega_0 \cup I,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_1\},$$
$$h_1 = \tfrac{1}{2}\inf\{|y_0 - y_1|\ |\ y_0 \in \partial\Omega \cup I,\ y_1 \in \partial\Omega_0\}.$$

Since $u \geqslant \psi$ on Ω, we have

(3.11) $$\delta_{eh}^a u \geqslant \delta_{eh}^a \psi \geqslant m \quad \text{on } I, \quad 0 < h < h_1,\ e \in \boldsymbol{R}^n,\ |e| = 1.$$

Since $-\Delta z = f$ in Ω and $\Delta u = f$ in $\Omega - I$, $I \subset \Omega_0$, we conclude that $z \in C^{1+a}(\Omega)$, $u \in C^{1+a}(\Omega - I)$ and that

$$\zeta_{\max} := \sup\{\delta_{eh}^a z(x) |\ x \in \Omega_0,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_1\} < \infty,$$
$$\zeta_{\min} := \inf\{\delta_{eh}^a z(x) |\ x \in \Omega_0,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_1\} > -\infty,$$
$$\nu_0 := \inf\{\delta_{eh}^a u(x) |\ x \in \partial\Omega_0,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_1\} > -\infty.$$

Recall that $h_1 = \frac{1}{2}\inf\{|y_0 - y_1|\ |\ y_0 \in \partial\Omega \cup I,\ y_1 \in \partial\Omega_0\} > 0$. Thus

(3.12) $$\mu_0 \geqslant \bar{\mu} := \min\{m - \zeta_{\max},\ \nu_0 - \zeta_{\max}\} > -\infty,$$

where m is defined in (3.5). From (3.10) and Lemma 3.1 applied to $v := u - z$ we obtain

$$\sum_{i=1}^{n} \delta_{ih}^a (u-z) \leqslant -C(n, a)\mu_0 \quad \text{in } \Omega_0,$$

where $C(n, a) > 0$ and $\delta_{ih}^a v(x) = h^{-1-a}\big(v(x+he_i) - 2v(x) + v(x-he_i)\big)$.

Hence

$$\delta_{ih}^a (u-z) \leqslant -C(n, a)\mu_0 - \sum_j \delta_{jh}^a (u-z) \quad (j \neq i,\ j = 1, \ldots, n)$$

and in view of (3.10) and (3.12)

$$\bar{\mu} \leqslant \delta_{ih}^a (u-z) \leqslant -[n-1+C(n, a)]\bar{\mu} \quad \text{in } \Omega_0, \quad i = 1, \ldots, n.$$

Since $z \in C^{1+a}$, we see that $\delta_{ih}^a z$ is bounded uniformly on Ω_0 as $h \to 0$. The theorem then follows from a result of the theory of Besov-spaces, cf. [10], p. 229, Theorem 4.1.4. In the case of (3.6′) being replaced by (3.6) we argue via an approximation argument: Let $\psi \in L^1$, $v_0 \in \boldsymbol{K}$. Since $\psi < 0$ in $U(\partial\Omega)$, there is a test function $\varphi \in C_0^\infty(\Omega)$ such that $v := \varphi v_0 \in \boldsymbol{K}$. We extend ψ outside $\bar{\Omega}$ by $-\delta$, and denote by J_k the usual modification operator which convolutes a function with nonnegative mean functions converging to the Dirac measure ($k \to 0$), cf. [1]. Then $J_k\psi < 0$ in a neighbourhood U_0 of $\partial\Omega$ uniformly for $0 < k < k'$. Furthermore $J_k v \in \mathring{H}^{1,2}(\Omega)$ and $J_k v \geqslant J_k \psi$.

We consider the variational inequality (3.1) with the obstacle $J_k\psi$ instead of ψ. Since $J_k v \geqslant J_k\psi$, $J_k v \in \mathring{H}^{1,2}(\Omega)$, the corresponding admissible set is non-empty and the variational inequality has a continuous solution $u_k \geqslant J_k\psi$. Since $J_k\psi < -\delta/2$ in a neighbourhood U_0 of $\partial\Omega$, $0 < k < k'$, the functions u_k are uniformly Hölder continuous in a neighbourhood U_1 of $\partial\Omega$. This was proved in § 1, Lemma 1.2. Hence there exist open sets $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$ such that the sets of coincidence $I_k = \{x \in \Omega |\ u_k(x) = J_k\psi(x)\}$ are contained in Ω_1 and that $|u_k - J_k\psi| \geqslant \varepsilon_0 > 0$ on $\Omega - \Omega_1$.

Since, further,

$$\delta_{eh}^a J_k\psi(x) \geqslant m, \quad x \in \Omega_0,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_1,\ 0 < k < k',$$

we may apply the method of proof used in the case of $\psi \in C(\Omega)$ in order to obtain a uniform bound for $|\delta_{ih}^a u_k|$ on Ω_0.

By well known perturbation theorems on variational inequalities (cf. e.g. [4], [28], [30]) it follows that the sequence $(u_k)_{k\to 0}$ tends to the unique solution u of (3.1) in the weak (or strong) topology of $H^{1,2}(\Omega)$. This can be proven easily using theorems on weak compactness in $H^{1,2}$. Hence also $|\delta_{ih}^a u|$ is bounded uniformly on Ω_0, $0 < h < h_0$. Applying Theorem 4.1.4 of [10] once more, we infer that $u \in C^{1+a}(\Omega_0)$. Since $|u - \psi| \geqslant \varepsilon_0 > 0$ a.e. on $\Omega - \Omega_1$, we obtain $u \in C^{1+a}(\Omega - \Omega_1)$, and thus the theorem.

It remains to prove

Lemma 3.1. *Let $v \in C(\Omega) \cap H^{1,2}(\Omega)$ and suppose that $\Delta v \leqslant 0$ in Ω in the sense of distributions. Assume further that for an open subset $\Omega_0 \subset\subset \Omega$ we have*

$$\mu_0 := \inf\{\delta_{eh}^a v(x) |\ x \in \Omega_0,\ e \in \boldsymbol{R}^n,\ |e| = 1,\ 0 < h < h_0\} > -\infty,$$

where $h_0 = \frac{1}{2}\inf\{|y_0 - y_2|\ |\ y_0 \in \partial\Omega,\ y_1 \in \partial\Omega_0\}$.

Then

$$\sum_{i=1}^{n} \delta_{ih}^{a} v \leqslant -C(n, a)\mu_0 \quad \text{on } \Omega_0, \quad 0 < h < h_0,$$

with $C(n, a)$ being a constant.

Proof. Let $Q \subset \Omega$ be the cube with centre $x_0 \in \Omega_0$ and with edges of length $2h$ parallel to the coordinate axes. Let G_ϱ be the discrete Green function defined by

$$-\Delta G_\varrho = \delta_\varrho, \quad G_\varrho \in \mathring{H}^{1,2}_0(Q),$$

where $\delta_\varrho = |B_\varrho|^{-1}$ on $B_\varrho(x_0)$, and $\delta_\varrho = 0$ otherwise, $0 < \varrho < h$. The function G_ϱ has the following properties:

(3.13) $$G_\varrho \geqslant 0 \quad \text{on } Q,$$

(3.14) $$G_\varrho \in C(\bar{\Omega}),$$

(3.15) $$\nabla G_\varrho|_{\partial Q} \in L^2(\partial Q),$$

(3.16) $$\nu \nabla G_\varrho \geqslant 0 \text{ a.e.} \quad \text{on } \partial Q,$$

(3.17) $$\int_{\partial Q} \nu \nabla G_\varrho ds = 1,$$

where $\nu(x)$ is the inner normal at $x \in \partial Q$.

From the hypothesis that $\Delta v \leqslant 0$ in the sense of distributions we obtain

$$-(\nabla v, \nabla G_\varrho) \leqslant 0$$

and using the definition of G_ϱ we conclude that

(3.18) $$\int_{\partial Q} v\nu \nabla G_\varrho dx - \bar{v}_\varrho(x_0) \leqslant 0, \quad \bar{v}_\varrho(x_0) = (\delta_\varrho, v).$$

We split the integral over $\partial\Omega$ into $2n$ integrals over the $(n-1)$-dimensional faces of Q and denote by $\int_{+i}$, resp. $\int_{-i}$, the integration over the face $Q_{\pm i}$ through $x_0 + he_i$, resp. $x_0 - he_i$ (e_i = ith unit vector). We want to examine the term

$$D_i = \int_{+i} v\nu \nabla G_\varrho dy - \frac{1}{n}\bar{v}_\varrho(x_0) + \int_{-i} v\nu \nabla G_\varrho dy.$$

Note that (3.18) implies

(3.19) $$\sum_{i=1}^{n} D_i \leqslant 0.$$

On account of the symmetry properties of G_ϱ we have $\int_{\pm i} \nu \nabla G_\varrho dy = 1/2n$ and hence

(3.20) $$D_i = \frac{1}{2n}[v(x_0 + he_i) - 2\bar{v}_\varrho(x_0) + v(x_0 - he_i)] + \int_{+i}[v - v(x_0 + he_i)]\nu \nabla G_\varrho dy + \int_{-i}[v - v(x_0 - he_i)]\nu \nabla G_\varrho dy.$$

We may split each Q_{+i} and Q_{-i} into two congruent rectangles. We thus get $(n-1)$-dimensional rectangular parallelepipeds R_i such that

$$Q_{\pm i} = (x_0 \pm he_i + R_i) \cup (x_0 \pm he_i - R_i).$$

On account of symmetry

$$\int_{\pm i/2} \nu \nabla G_\varrho dx = \frac{1}{4n}, \quad i = 1, \ldots, n,$$

where $\int_{\pm i/2}$ denotes integration over $x_0 \pm he_i + R_i$. Thus we may write

$$\begin{aligned}\int_{\pm i} [v - v(x_0 \pm he_i)]\nu \nabla G_\varrho dy &= \int_{R_i}[v(x_0 \pm he_i + y) - 2v(x_0 \pm he_i) + v(x_0 \pm he_i - y)]\nu \nabla G_\varrho dy \\ &= \int_{R_i} |y|^{1+a}\delta_y^a v(x_0 \pm he_i)\nu \nabla G_\varrho dy \\ &\geqslant \int_{R_i} |y|^{1+a}\mu_0 \nu \nabla G_\varrho dy = h^{1+a}\mu_0 c(n, a).\end{aligned}$$

Here we used the definition of μ_0 and the notation $\delta_y^a = \delta_{e|y|}^a$, $e = |y|^{-1}y$.

From the last inequality and (3.20) we obtain

$$D_i \geqslant \frac{1}{2n}[v(x_0 + he_i) - 2\bar{v}_\varrho(x_0) + v(x_0 + he_i)] + 2\mu_0 c(n, a)h^{1+a}$$

and in view of (3.19)

$$\frac{1}{2n}\sum_{i=1}^{n}[v(x_0 + he_i) - 2\bar{v}_\varrho(x_0) + v(x_0 + he_i)] \leqslant -2\mu_0 c(n, a)h^{1+a}.$$

Passing to the limit $\varrho \to 0$ we obtain

$$\frac{1}{2n}\sum_{i=1}^{n} h^{1+a}\delta_{ih}^a v(x_0) \leqslant -2m_0 c(n, a)h^{1+a}.$$

The lemma follows.

Remarks. (i) D. Kinderlehrer and L. Caffarelli have announced a result similar to Theorem 3.1, which they obtained independently.

(ii) The generalization of the proof of Theorem 3.1 to operators with variable coefficients or non-linear operators causes greater difficulties than in the case $a = 1$, cf. [16], [21], since the symmetry properties of the Green function which we have used do not hold any more in the general case. One has to be more careful while splitting the integral over ∂Q.

(iii) Lemma 3.1 and the inequalities (3.10) and (3.12) yield an a priori bound for the $C^{1+a}(\bar{\Omega}_0)$-norm of u.

4. Higher order variational inequalities with obstacles

We restrict the discussion to the polyharmonic variational inequality, since this case seems to be sufficiently characteristic to indicate what regularity properties can be expected in the general case, and not many results are known anyhow so far. Let Ω be a bounded domain in $\boldsymbol{R}^n$ and $V = H_0^{m,2}(\Omega)$ the closure of the test functions $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{m,2}$ of the Sobolev space $H^{m,2}(\Omega)$, where

$$\|w\|_{m,2} = \sum_{j=0}^{n} \|\nabla^j u\|_2, \qquad \|w\|_2 = \Big(\int_\Omega w^2 dx\Big)^{1/2}.$$

We denote the natural pairing of elements $v \in V$ and $l \in V^*$ (= dual of V) by $\langle l, v\rangle$.

Let $g \in H^{m,2}(\Omega)$ be a function which represents the boundary condition and let $\psi\colon \Omega \to \boldsymbol{R}$ be a function.

We define

$$\boldsymbol{K} = \{v \in g+V \mid v \geqslant \psi \text{ in } \Omega\}.$$

The inequality sign in the definition of $\boldsymbol{K}$ can be understood in the sense of $H^{m,2}(\Omega)$, cf. the definition in [24], p. 155 or in the sense "almost everywhere in Ω". If ψ is not smooth, the set $\boldsymbol{K}$ and the solution of the variational inequality (4.1) can of course depend quite strongly on the choice between these two possibilities.

Finally, let $l \in V^*$ and $m \in \boldsymbol{N}$, $m \geqslant 2$, be given. We consider the variational inequality:

(4.1) *Find $u \in \boldsymbol{K}$ such that*

$$\langle (-1)^m \Delta^m u, u-v\rangle \leqslant \langle l, u-v\rangle$$

for all $v \in \boldsymbol{K}$.

It is well known that a solution of (4.1) exists if $\boldsymbol{K} \neq \emptyset$. Setting $v = u+\varphi$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, we have $v \in \boldsymbol{K}$ and we deduce from (4.1) that

$$\langle (-1)^m \Delta^m u - l, \varphi\rangle \geqslant 0.$$

Hence $(-1)^m \Delta^m u - l$ is a measure, cf. [34]. If l is a measure, i.e., if l satisfies some weak regularity assumption, we have

$$\sup\{\langle l, \varphi\rangle \mid \|\varphi\|_\infty = 1,\ \varphi \in C_0^\infty(\Omega)\} < \infty,$$

and hence

$$\sup\{\langle (-1)^m \Delta^m u, \varphi\rangle \mid \|\varphi\|_\infty = 1, \varphi \in C_0^\infty(\Omega)\} < \infty.$$

By Sobolev's inequality also

$$\sup\Big\{\sum_{i=1}^{n} \langle (-1)^m \partial_i \Delta^{m-1} u, \partial_i\varphi\rangle \Big| \int |\nabla\varphi|^{n+\sigma} dx = 1, \varphi \in C_0^\infty(\Omega)\Big\} < \infty$$

if $\sigma > 0$, and by Garding's inequality in L^p, cf. [35], we obtain that $\nabla \Delta^{m-1} u \in L_{\text{loc}}^{n/(n-1)-\delta}$ for $0 < \delta < 1$.

This holds without any restriction on the obstacle ψ. We state this result (which we consider to be well known) as

THEOREM 4.1. *Let $u \in \boldsymbol{K}$ be the solution of* (4.1). *Then*

$$\nabla^{2m-1} u \in L_{\text{loc}}^{n/(n-1)-\delta} \quad \text{for all } \delta \in]0, 1[.$$

Remarks. (i) An a priori bound of the $L_{\text{loc}}^{n/(n-1)-\delta}$-norms of $\nabla^{2m-1}u$ in terms of the $H^{m,2}$-norm of u can be easily given.

(ii) Consider the penalty approximation to problem (4.1):

$$(-1)^m \Delta^m u + \varepsilon^{-1}(u-\psi)_- = l$$

where $\varepsilon > 0$, and $(u-\psi)_- = \min(0, u-\psi)$. If, say, $\psi \in L^2$, $l \in L^2$, then u_ε exists and one can prove the existence of a uniform bound for the L_{loc}^1-norms $\|\Delta^m u_\varepsilon\|_1$ as $\varepsilon \to 0$, cf. [19].

(iii) The statement in Theorem 4.1 gives rise to the question whether $\nabla^{2m-1} u \in L_{\text{loc}}^2$ for regular obstacles. The answer is affirmative for $m = 1, 2$, cf. Theorem 4.2.

Another simple method of obtaining a stronger assertion concerning the differentiability of the solutions to elliptic variational inequalities with obstacles consists in the classical finite difference procedure; cf. [1], [29] for PDE's, [26] for variational inequalities. Here we impose the following condition on the obstacle, which allows a one-sided irregularity.

(4.2) *There exist functions $g_i \in H_{\text{loc}}^{m,2}(\Omega)$, $i = 0, \ldots, n$, such that for all $\Omega_0 \subset\subset \Omega$*

$$\Delta_h \psi \geqslant g_0 + \sum_{i=1}^{n} D_i^h g_i \quad \text{in } \Omega_0, \quad 0 < h < h_0,$$

where $h_0 = \frac{1}{2}\inf\{|y_1 - y_2| \mid y_1 \in \partial\Omega_0,\ y_2 \in \partial\Omega\}$ *and*

$$D_i^{\pm h} w(x) = \pm h^{-1}\big(w(x \pm he_i) - w(x)\big) \quad (e_i = i\text{-th unit vector})$$

$$\Delta_h = \sum_{i=1}^{n} D_i^{-h} D_i^{+h}.$$

Condition (4.2) is satisfied if $\psi \in H^{2,\infty}(\Omega)$.

As regards the right hand side of (4.1), we assume:

$$(4.3) \qquad \langle l, \varphi\rangle = \sum_{i=1}^{n} (f_i, \partial_i\varphi), \quad \varphi \in C_0^\infty(\Omega), \quad \text{where } f_i \in L^2(\Omega).$$

Under these additional assumption we obtain

THEOREM 4.2. *Under the assumptions* (4.2) *and* (4.3) *the solution $u \in \boldsymbol{K}$ to the variational inequality* (4.1) *satisfies $u \in H_{\text{loc}}^{m+1,2}(\Omega)$.*

Remarks. For obstacles $\psi \in H^{m+1,2}$ this theorem can be found in [26]. In the case where the set $\boldsymbol{K}$ is defined by a two-sided restriction $\psi_1 \leqslant u \leqslant \psi_2$ a technique different from that applied in [26] and the one described here has to be used, cf. [14]. The proof of the differentiability of u on the boundary gives additional difficulties, which were attacked in [33] for the boundary obstacle case.

If one wants to prove the analogue of Theorem 4.2 for elliptic operators with variable coefficients, one has to replace the inequality (4.2) for $\Delta_h\psi$ by a corresponding one for $D_i^h D_i^{-h}\psi$.

Proof of Theorem 4.2. Since $u \geqslant \psi$ in Ω (in the sense of $H^{m,2}$ or in the sense "almost everywhere"), we conclude that

$$u+\frac{1}{2n}h^2\tau^2\Delta_h(u-\psi)\geqslant\psi \quad \text{in } \Omega_0, \quad \Omega_0\subset\subset\Omega,\ 0<h<h_0,$$

$$\tau\in C_0^\infty(\Omega),\ 0\leqslant\tau\leqslant 1.$$

Hence

$$u+\frac{1}{2n}h^2\tau^2\Delta_h u\geqslant\psi+\frac{1}{2n}h^2\tau^2\Big(g_0+\sum_{i=1}^n D_i^h g_i\Big)$$

and

$$u_h:=u+\frac{1}{2n}h^2\tau^2\Big(\Delta_h u-g_0-\sum_{i=1}^n D_i^h g_i\Big)\geqslant\psi,$$

and thus $u_h\in\boldsymbol{K}$.

We insert $v=u_h$ into the variational inequality (4.1), cancel the factor $(1/2n)h^2$ and obtain

$$\Big\langle(-1)^m\Delta^m u-l,\ \tau^2\Delta_h u-g_0-\sum_{i=1}^n D_i^h g_i\Big\rangle\leqslant 0.$$

By routine estimates, this yields a uniform bound for the L^2-norms $\|\tau D_i^h\nabla^m u\|_2$ as $h\to 0$ and thus the theorem.

In the case of the biharmonic variational inequality ($m=2$) a more detailed analysis of the regularity properties of the solution of the variational inequality can be given. It can be easily seen that the solution of the biharmonic variational inequality cannot have continuous third order derivatives — even in the case of dimension one ($n=1$).

In [17] we proved that the second order derivatives are bounded. A simplification of the proof which extends it to the case of irregular obstacles is given below. The experts suspect that the second order derivatives of the solution of the biharmonic variational inequality are continuous if ψ is smooth. This was recently proved by A. Friedman and L. Caffarelli in the case of two dimensions. They established a logarithmic

estimate for the modulus of continuity. For the proof of the boundedness of the second order derivatives of the solution we need only the following one-sided condition on the obstacle ψ.

(4.4) *In every subdomain $\Omega_0\subset\subset\Omega$ we have*

$$\Delta_h\psi\geqslant -c_0,\qquad 0<h<h_0$$

where $h_0=\frac{1}{2}\inf\{|y_1-y_2|\ y_1\in\partial\Omega_0,\ y_2\in\partial\Omega\}$ *and* $c_0=c_0(\Omega_0)$ *is some constant.*

Regarding the right hand side of (4.1) we assume:

$$\text{(4.5)}\qquad \langle l,\varphi\rangle=\sum_{i=1}^n(f_i,\partial_i\varphi),\qquad \varphi\in C_0^\infty(\Omega),$$

where $f_i\in L^{n+\delta}(\Omega)$ *with some* $\delta>0$.

THEOREM 4.3. *Under the assumptions* (4.4) *and* (4.5) *the solution* u *of the biharmonic variational inequality* (4.1), $m=2$, *has second order derivatives in* $L^\infty_{\mathrm{loc}}(\Omega)$. *Furthermore, the third order derivatives of* u *satisfy*

$$\text{(4.6)}\qquad \int_{\Omega_0}|\nabla^3 u|^2 G_\zeta\,dx\leqslant K=K(\Omega_0),\qquad \zeta\in\boldsymbol{R}^n,\ \Omega_0\subset\subset\Omega,$$

where $G_\zeta(x)=|x-\zeta|^{2-n}$ *for* $n\geqslant 2$, $G_\zeta(x)=\big|\ln|x-\zeta|\big|$ *for* $n=2$.

Proof. By Theorem 4.2 we know already that $u\in H^3_{\mathrm{loc}}(\Omega)$. Let $\omega_\varrho *$ denote the usual modification operation which convolutes a function f with a non-negative mean function $\omega_\varrho\in C^\infty$ with support in $B_\varrho(0)$. Let $G_\zeta^\varrho=\omega_\varrho * G_\zeta$. Then $G_\zeta^\varrho\geqslant 0$, $G_\zeta^\varrho\in C^\infty$, and $(\nabla f,\nabla G_\zeta^\varrho)=\omega_\varrho * f(\cdot-\zeta)$ for $f\in H^{1,2}$, $\operatorname{supp} f\subset B_s(\zeta)$ ($s\leqslant\exp 1-\varrho/2$ for $n=2$).

Finally, let $\tau\in C_0^\infty(\Omega)$, and $\tau=1$ in a neighbourhood of ζ. Then we get for $0<h<h_0=h_0(\tau)$

$$u+\varepsilon_0\tau^2 G_\zeta^\varrho\Delta_h(u-\psi)\geqslant\psi$$

if $\varepsilon_0=\varepsilon_0(h,\varrho,\text{etc.})>0$ is chosen to be small enough. Hence and from (4.5) we obtain

$$u_h=u+\varepsilon_0\tau^2 G_\zeta^\varrho(\Delta_h u+c_0)\geqslant\psi$$

and so $u_h\in\boldsymbol{K}$.

We insert this function u_h into the variational inequality (4.1), which yields, after cancelling $\varepsilon_0>0$,

$$\text{(4.7)}\qquad -\langle\Delta\Delta u-l,\ \tau^2 G_\zeta^\varrho(\Delta_h u+c_0)\rangle\geqslant 0.$$

We may solve the equation

$$\Delta\Delta u_0=l=\sum_{i=1}^n\partial_i f_i$$

in a ball containing Ω in its interior. The functions f_i are equal to zero outside Ω. Since $f_i \in L^{n+\delta}$, we have

$$u_0 \in C^{2+\alpha} \cap H^{3,n+\delta}, \quad \alpha = \alpha(\delta),$$

and setting $z = u + u_0$ we obtain from (4.7) that

$$-\langle \Delta\Delta z, \tau^2 G_\zeta^\varrho (\Delta_h z + c_0 - \Delta_h u_0)\rangle \leqslant 0.$$

Since u, u_0 and z lie in $H^{3,2}_{\mathrm{loc}}(\Omega)$, we may write

$$\big(\nabla\Delta z,\ \nabla\big(\tau^2 G_\zeta^\varrho(\Delta_h z + c_0 - \Delta_h u_0)\big)\big) \leqslant 0$$

and passing to the limit $h \to 0$ we conclude that

$$\big(\nabla\Delta z,\ \nabla\big(\tau^2 G_\zeta^\varrho(\Delta z + c_0 - \Delta u_0)\big)\big) \leqslant 0.$$

We rewrite this in the form

$$\int |\nabla\Delta z|^2 G_\zeta^\varrho \tau^2\, dx + A_\varrho \leqslant B_\varrho + C_\varrho, \tag{4.8}$$

where

$$A_\varrho = \big(\nabla\Delta z,\ \tau^2(\Delta z + c_0 - \Delta u_0)\nabla G_\zeta^\varrho\big),$$

$$B_\varrho = 2\int |\nabla\Delta z| G_\zeta^\varrho |\Delta z + c_0 - \Delta u_0| |\tau| |\nabla\tau|\, dx,$$

$$C_\varrho = \int |\nabla\Delta z| |\nabla\Delta u_0| G_\zeta^\varrho \tau^2\, dx.$$

Since $\nabla\tau = 0$ in a neighbourhood of the singularity ζ, we conclude that

$$B_\varrho \leqslant K \quad \textit{uniformly as } \varrho \to 0,\ \zeta \in \Omega_0. \tag{4.9}$$

The term C_ϱ is estimated by Young's inequality

$$C_\varrho \leqslant \tfrac{1}{2}\int |\nabla\Delta z|^2 G_\zeta^\varrho \tau^2\, dx + K, \tag{4.10}$$

where K is an upper bound for

$$\int |\nabla\Delta u_0|^2 G_\zeta^\varrho \tau^2\, dx.$$

(Recall that $u_0 \in H^{3,n+\delta}$.) Thus we obtain from (4.8), (4.9), (4.10),

$$\int |\nabla\Delta z|^2 G_\zeta^\varrho \tau^2\, dx + 2A_\varrho \leqslant K \quad \textit{uniformly for } \varrho \to 0,\ \zeta \in \Omega_0. \tag{4.11}$$

The term A_ϱ is rewritten in the following form:

$$A_\varrho = \tfrac{1}{2}\big(\nabla|\Delta z + c_0 - \mu_0|^2, \tau^2\nabla G_\zeta^\varrho\big) + \big(\nabla\Delta z, \tau^2(\mu_0 - \Delta u_0)\nabla G_\zeta^\varrho\big),$$
$$\mu_0 = \Delta u_0(\zeta).$$

Thus

$$A_\varrho \geqslant \tfrac{1}{2}\big(\nabla(\tau^2|\Delta z + c_0 - \mu_0|^2), \nabla G_\zeta^\varrho\big) - \int |\Delta z + c_0 - \mu_0|^2 |\nabla\tau| |\tau| |\nabla G_\zeta^\varrho|\, dx -$$
$$-\tfrac{1}{4}\int |\nabla\Delta z|^2 G_\zeta^\varrho \tau^2\, dx - \int |\mu_0 - \Delta u_0|^2 \tau^2 (G_\zeta^\varrho)^{-1} |\nabla G_\zeta^\varrho|^2\, dx.$$

(If $n = 2$, we have to choose the support of τ to be small enough for $(G_\zeta^\varrho)^{-1} \leqslant K'$ to hold there.) From the definition of G_ζ^ϱ we obtain, for $\operatorname{supp}\tau \subset B_s(\zeta)$, $s < \frac{1}{2}\exp(1)$, $\varrho \to 0$,

$$\tfrac{1}{2}\omega_\varrho * \tau^2 |\Delta z + c_0 - \mu_0|^2(\zeta) = \tfrac{1}{2}\big(\nabla(\tau^2|\Delta z + c_0 - \mu_0|^2, \nabla G_\zeta^\varrho)\big).$$

Hence, the inequality (4.11) results in

$$\int |\nabla\Delta z|^2 G_\zeta^\varrho\, dx + \omega_\varrho * \tau^2 |\Delta z + c_0 - \mu_0|^2 \leqslant 2K + D_\varrho + E_\varrho, \tag{4.12}$$

where K is given by (4.11) and

$$D_\varrho = 2\int |\Delta z + c_0 - \mu_0|^2 |\nabla\tau| |\tau| |\nabla G_\zeta^\varrho|\, dx,$$

$$E_\varrho = 2\int |\mu_0 - \Delta u_0|^2 \tau^2 (G_\zeta^\varrho)^{-1} |\nabla G_\zeta^\varrho|^2\, dx.$$

Since $\nabla\tau = 0$ in a neighbourhood of the singularity ζ, the term D_ϱ remains bounded as $\varrho \to 0$. Finally, since $\Delta u_0 \in C^\alpha$, $\mu_0 = \Delta u_0(\zeta)$, the term $|\mu_0 - \Delta u_0|^2(x)$ behaves like $k_0|x-\zeta|^{2\alpha}$. The factor $(G_\zeta^\varrho)^{-1}|\nabla G_\zeta^\varrho|^2$ behaves like $|x-\zeta|^{-n}$. Hence the term E_ϱ remains bounded as $\varrho \to 0$ and we obtain the uniform bound

$$\int |\nabla\Delta z|^2 G_\zeta^\varrho\, dx + \omega_\varrho * \tau^2 |\Delta z + c_0 - \mu_0|^2(\zeta) \leqslant \overline{K} \tag{4.13}$$

as $\varrho \to 0$, $\zeta \in \Omega_0$. We pass to the limit $\varrho \to 0$ and conclude that

$$\int_{\Omega_0} |\nabla\Delta z|^2 G_\zeta\, dx \leqslant K, \quad \zeta \in \Omega_0, \tag{4.14}$$

$$\Delta z \in L^\infty_{\mathrm{loc}}(\Omega), \quad \Delta u \in L^\infty_{\mathrm{loc}}(\Omega). \tag{4.15}$$

From (4.14) we obtain the last statement of the theorem.

The last part of the proof consists in showing that $\Delta u \in L^\infty_{\mathrm{loc}}$ implies $\Delta^2 u \in L^\infty_{\mathrm{loc}}$. This is done with the help of an idea from the author's paper [17].

Let

$$g(x) = \begin{cases} |x-x_0|^{-n+4}, & n = 5, 6, \ldots, \\ -\ln|x-x_0|, & n = 4, \\ -|x-x_0|, & n = 3, \\ |x-x_0|^2(\ln|x-x_0| - 1), & n = 2. \end{cases}$$

By elementary calculations

$$g = c_n\delta(\cdot - x_0)$$

in the sense of distributions, where $\delta(\cdot - x_0)$ is the Dirac functional at the point x_0 and $c_n > 0$ a constant depending on the dimension.

In [17] we observed that, for $j = 1, \ldots, n$,

$$(\partial_j^2 - \tfrac{1}{2}\Delta)g(x) \geqslant 0, \quad x \neq x_0,\ n = 3, 4, \ldots,$$
$$(\partial_j^2 - \tfrac{1}{2}\Delta)g(x) \geqslant -c, \quad x \neq x_0,\ n = 2,\ x \in \Omega,\ c = \mathrm{const},$$

which can be checked by simple computations. Thus, for $\tau \in C_0^\infty(\Omega)$, $\tau \geqslant 0$, $\tau = 1$ in $U(x_0)$,

$$\Phi_\varrho := \tau\omega_\varrho * [(\partial_j^2 - \tfrac{1}{2}\Delta)g + c] \in C_0^\infty \qquad 0 < \varrho < \varrho_0(\tau),$$

and $\Phi_\varrho \geqslant 0$.

Therefore, $u + \Phi_\varrho \in \boldsymbol{K}$, and we obtain

$$-\langle \Delta\Delta u - l, \Phi_\varrho \rangle \leqslant 0;$$

using the function u_0 and z defined above we get

$$\langle \Delta\Delta z, \Phi_\varrho \rangle \geqslant 0$$

and

$$\Big(\nabla\Delta z, \nabla\big(\tau\omega_\varrho * [(\partial_j^2 - \tfrac{1}{2}\Delta)g + c]\big)\Big) \leqslant 0.$$

By partial integrations, we may move the operator $\partial_j^2 - \frac{1}{2}\Delta$ to the left factor and the operation $\nabla\Delta$ to the right factor. This yields

$$\big((\partial_j^2 - \tfrac{1}{2}\Delta)z, \tau\omega_\varrho * \Delta\Delta g\big) \geqslant -E_\varrho,$$

where E_ϱ contains several error terms which arise while performing the partial integrations, according to Leibniz rule. However, these error terms remain bounded as $\varrho \to 0$, cf. [17]. Thus we obtain

$$[\omega_\varrho * \tau(\partial_j^2 - \tfrac{1}{2})z](x_0) \geqslant -E_0$$

uniformly as $\varrho \to 0$. Hence

$$\tau(\partial_j^2 - \tfrac{1}{2}\Delta)z \geqslant -E_0 \quad \text{a.e. in } \Omega,$$

i.e., the functions $\partial_j^2 z - \frac{1}{2}\Delta z$ are locally uniformly bounded from below.

Since $\Delta = \sum_{j=1}^{n} \partial_j^2$ and $\Delta z \in L_{\mathrm{loc}}^\infty(\Omega)$, we conclude that $\partial_j^2 z \in L_{\mathrm{loc}}^\infty(\Omega)$ and, finally, $\partial_j^2 u \in L_{\mathrm{loc}}^\infty(\Omega)$.

The boundedness of the mixed derivatives $\partial_i\partial_k u$ is shown by an orthogonal transformation of the problem, which yields the boundedness of $(\partial_i \pm \partial_k)^2 u$. This is possible, since Δ^2 is invariant under orthogonal transformations and the lower order term $\sum_i \big(f_i, \partial_i(u-v)\big)$ is transformed into a similar one.

Remark. Theorem 4.3 can be extended to operators whose principal part is the product of two second order operators with smooth coefficients; cf. [13].

For applications to engineering problems cf. [12], where also further references can be found.

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ON EXISTENCE AND NONEXISTENCE RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Introduction

In this paper we shall speak about existence and nonexistence results for initial value problems for equations of the form

$$iu_t + \Delta u + f(|u|^2)u = 0, \quad i^2 = -1, \; u_t = \frac{\partial u}{\partial t}, \tag{1}$$

where Δ is the n-dimensional Laplacian and f is a continuous real function. In the special case $f(s) = qs$, $q = \bar{q} = \text{const.}$, (1) is the dimensionless standard form of the nonlinear Schrödinger equation which has been sometimes called Ginsburg–Landau equation or recently also Zakharov–Shabat equation. The latter notation is due to the fact that Zakharov and Shabat [18] were the first to see that Cauchy's problem for the spatially one-dimensional Schrödinger equation can be solved globally by means of the inverse scattering method. This famous method was discovered by Gardner, Greene, Kruskal and Miura [4] and firstly applied to Cauchy's problem for the Korteweg–de Vries equation. Unfortunately the approach of Zakharov–Shabat does not seem to generalize neither to higher space dimensions nor to other functions f than $f(s) = qs$. Since we are interested in more general cases we do not go into details of the inverse scattering method here.

In the last decade, existence and nonexistence results for initial value problems for (1) have been published by many authors. In this paper we take into account existence results of Shabat [13], Strauss [15], Baillon, Cacenave & Figueira [1] and Ginibre & Velo [5] as well as nonexistence results of Talanov [16], Shabat [13], Zakharov, Sobolev & Synach [19], Kudrashov [8] and Glassey [6].