The operator $\mathscr{A}_\theta$ in $L_2^2(S^{n-1})$ has the eigenfunctions

$$\begin{pmatrix} \lambda_k^{\pm} \Lambda^{-1} P_{k,a}(\theta) \\ P_{k,a}(\theta) \end{pmatrix}, \quad 1 \leqslant a \leqslant a(k) = (2k+n-2)(n-k+3)!\,[(n-2)!k!]^{-1},$$

which correspond to the eigenvalues

$$\lambda_k^{\pm} = \tfrac{1}{2}\{(\mu+2-n) \pm [(n+\mu+2)^2 + 4k(k+n-2) - 4(2n+\mu+\lambda)]^{1/2}\}, \quad k \in \boldsymbol{Z},$$

$P_{k,a}(\theta)$ denoting the spherical harmonics.

The algebraic multiplicities of $\lambda_k^{\pm}$ are equal to $a(k)$, if $\lambda_k^{+} \neq \lambda_k^{-}$. For $\lambda_k^{\pm} = \lambda_k$ the algebraic multiplicity of λ_k is $2a(k)$.

Now one can derive most of the results of [5] for arbitrary $s \geqslant 0$ from Theorems 2.1, 3.1 and 4.1.

References

[1] L. A. Bagirov and V. A. Kondratiev, *On elliptic equations in R^n* (Russian), Differenc. Urawn. 11 (1975), 948–504.

[2] M. S. Baouendi, C. Goulaouic and J. Lipkin, *On the operator $\Delta r^2 + \mu\partial/\partial r \cdot r + \lambda$*, J. Differential Equations 15 (1974), 499–509.

[3] M. S. Baouendi and J. Sjöstrand, *Régularité analytique pour des opérateurs elliptiques singuliers en une point*, Arkiv för Mat. 14 (1976), 9–33.

[4] —, —, *Non hypoellipticité d'opérateurs elliptiques singuliers*, Séminaire Goulaouic–Schwartz 1975–76, Exposé XXIV.

[5] J. Elschner and M. Lorenz, *On the normal solvability of the operator $\Delta r^2 + \mu\partial/\partial r \cdot r + \lambda$*, J. Differential Equations 36 (1980), 408–424.

[6] L. V. Kantorovich and G. P. Akilov, *Functional analysis* (Russian), Moscow 1977.

[7] V. A. Kondratiev, *Boundary value problems for elliptic equations in domains with conical or angular points* (Russian), Trudy Moskov. Mat. Obšč. 16 (1967), 209–292.

[8] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications I*, Berlin, Heidelberg, New York 1972.

[9] S. Prößdorf, *Einige Klassen singulärer Gleichungen*, Berlin 1974.

[10] E. V. Radkevich, *A theorem on a removable singularity for solutions of a class of linear partial differential equations* (Russian), Uspehi Mat. Nauk 32 (1977), 265–266.

[11] R. T. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure Math. A.M.S. 10 (1967), 288–307.

[12] D. V. Widder, *The Laplace transform*, Princeton 1946.

Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978

PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

MONOTONE OPERATORS WITH LINEAR RANGE

JENS FREHSE

Institut für Angewandte Mathematik und Informatik
der Universität Bonn, Bonn, F.R.G.

The following theorem is a standard tool for existence proofs in the theory of non-linear elliptic boundary value problems, cf. [2], [13]:

THEOREM 0. *Let B a reflexive Banach space with dual B^* and let $T: B \to B^*$ be a continuous mapping which satisfies the monotonicity condition*

$$(1) \qquad (Tu - Tv, u - v) \geqslant 0, \qquad u, v \in B$$

and the coerciveness condition

$$(2) \qquad (Tu, u)/\|u\| \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Then T is surjective.

Many generalizations of this theorem have been given, cf. [1], [2], [11], [14]. In applications to elliptic equations, the space B is a closed subspace of the usual Sobolev-space $H^{1,p}(\Omega)$ over a domain Ω of $\boldsymbol{R}^n$, containing the space $C_0^\infty(\Omega)$ of testfunctions. The mapping $T: B \to B^*$ then is defined by

$$(3) \qquad (Tu, v) = \sum_a \int_\Omega A_a(x, u, \ldots, \nabla^m u)\,\partial^a v\,dx \quad (|a| \leqslant m), \qquad v \in B.$$

The notation uses multi-indices $a = (a_1, \ldots, a_n)$ and the abbreviation $\partial^a = \prod_i (\partial/\partial x_i)^{a_i}$ $(i = 1, \ldots, n)$. If one assumes only an ellipticity condition for the principal part of the operator $\sum_a (-1)^{|a|}\partial^a A_a$, and corresponding growth conditions for the A_a $(|a| \leqslant m)$, one cannot derive the "full" coerciveness of T as required in Theorem 0. With appropriate growth and ellipticity conditions it is possible to derive only a "Gårding-type inequality"

$$(4) \qquad (Tu, u) \geqslant c\|u\|_{m,p}^p - K\|u\|_p^p - K$$

with positive constants c and K. Here

$$\|u\|_p = \left(\int |u|^p dx\right)^{1/p}, \qquad \|u\|_{m,p} = \sum_i \|\nabla^i u\|_p \qquad (i = 1, \ldots, m).$$

If T comes from a linear operator, inequality (4) is the key for proving Fredholm alternative theorems for T, i.e. the range $R(T)$ of T has a finite co-dimension and the equation $Tu = f$ is solvable if and only if

$$f \perp R(T)^{\perp}.$$

It is a surprising fact that the Fredholm alternative theorem holds for monotone (or asymptotic monotone) operators which are of *polynomial type* (cf. condition (7)) and satisfy (4) or some abstract analogue of semi-coerciveness. This was first discovered by the author in 1977, cf. [4], [6], and also [9]. The Fredholm property is equivalent to the linearity of $R(T)$.

The purpose of this paper is to present the precise statement and proof of our alternative theorem from [6], however, under slightly different — more elegant — conditions.

We shall assume the following properties of T:

(5) *Normalization*: $T(0) = 0$.

(6) *Asymptotic monotonicity.* For every fixed $v \in B$

$$\liminf \|u\|^{-1}(Tu - Tv, u - v) \geqslant 0 \qquad (\|u\| \to \infty).$$

(7) *Polynomial type condition.* If for some pair $v, w \in B$

$$\limsup |(T(w+tv), v)| < \infty \qquad (t \to \infty)$$

then $(T(w+tv), v)$ is constant in $t \in \mathbf{R}$.

(8) *Weak differentiability at* 0. The function ψ defined by

$$\psi(t) = (T(tw), v)$$

is differentiable at $t = 0$ for all $v \in B$, $w \in D_0$ where D_0 is a dense subset in B.

(9) *Regularity.* For every bounded closed convex set $K \subset B$ and every $f \in B^*$ the variational inequality

$$u \in K: \ (Tu - f, u - v) \leqslant 0 \quad \text{for all } v \in K$$

has a solution.

(10) *Semi-coerciveness.* There is a continuous linear projection $Q: B \to B$ such that $\dim QB < \infty$ and for all $K \geqslant K_0$

$$\sup\{\|u\|/(\|Qu\|+1) \mid (Tu, u)/\|u\| < K, \ u \in B\} < \infty.$$

Remark. If T is pseudo-monotone in the sense of [1], then (9) is satisfied.

With these assumptions we have the following

THEOREM 1. *Let $T: B \to B^*$ be a continuous mapping from a reflexive Banach space B into its dual which satisfies* (5)–(10). *Then the range $R(T)$ is a closed linear subspace of B^* with finite co-dimension.*

Remarks. (i) If $B = \mathbf{R}^n$, the condition (10) of semi-coerciveness follows from (5) and (6). Hence, if $\dim B = \infty$, the range of the approximate mapping P^*TP is a linear subspace of B^* for every finite-dimensional projection P and dual P^*. The question arises whether our condition (10) of semi-coerciveness of T can be relaxed. F. E. Browder [3] generalized the above theorem and obtained without assuming (10) that the closure of $R(T)$ is a linear subspace of B^*. Under the additional assumption that $R(T)$ has an interior point in $\overline{R(T)}$ with respect to the weak topology, he obtained that $R(T)$ itself is linear. For applications to partial differential equation, condition (10) seems to have a wider applicability. The results of [3] use stronger "polynomial"-type conditions than here or [6]; our key lemma was not published yet.

(ii) A variational analogue of Theorem 1 together with a variational type of proof was presented in [5]. The theorem asserts that the mapping $F_f: B \to \mathbf{R}$ defined by

$$F_f(u) = F(u) - (f, u)$$

has a minimum on B if and only if

$$f \perp D = \{v \in B \mid F(w+tv) \text{ is constant in } t \text{ for all } v \in B\}.$$

Here $F: B \to \mathbf{R}$ is a lower semi-continuous mapping satisfying a certain "polynomial"-type condition.

(iii) The results of Theorem 1 and (ii) give rise to Landesman–Lazer-alternative theorems, cf. [10], [12], for elliptic equations with non-linear principal part. This was done originally in [4], [7], [8].

For the proof of Theorem 1 we need the following key lemma.

LEMMA 1. *Let $v \in B$ and $K \in \mathbf{R}$ be such that*

$$(Tw, v) \leqslant K, \qquad w \in B, \tag{11}$$

and assume that $T: B \to B^*$ is continuous and satisfies (5)–(8). *Then $v \perp R(T)$.*

Remark. The idea behind Lemma 1 is that $(T(tw), v)$ behaves like a monotone polynomial in t. If a monotone polynomial is bounded from above, it is constant, and hence zero if it vanishes at zero.

Proof. By (11)

$$g(t) := \big(T(w+tv), v\big) \leqslant K, \quad t \in \boldsymbol{R}, w \in B,$$

and by (6)

$$\liminf t^{-1}\big(T(w+tv)-Tw, tv\big) \geqslant 0 \quad (t \to \infty).$$

Thus $-C(w) \leqslant \big(T(w+tv), v\big) \leqslant K$, $t \geqslant t_0$, and we may apply (7) to obtain

(12) $$g(t) = \text{const} = (Tw, v), \quad t \in \boldsymbol{R}, w \in B.$$

By (6)

$$\liminf |t|^{-1}\big(T(w+tv)-T(aw), (1-a)w+tv\big) \geqslant 0 \quad (t \to \pm\infty).$$

Hence

$$\liminf \big[(1-a)|t|^{-1}\big(T(w+tv), w+tv\big)+a|t|^{-1}\big(T(w+tv), tv\big)- -|t|^{-1}\big(T(aw), tv\big)\big] \geqslant 0 \quad (t \to \pm\infty).$$

Using (12), we obtain from the last inequality

$$a\sigma(Tw, v)-\sigma\big(T(aw), v\big) \geqslant (a-1)\liminf \big[|t|^{-1}\big(T(w+tv), w+tv\big)\big] \quad (t \to \pm\infty)$$

where $\sigma = 1$ if $t \to \infty$ and $\sigma = -1$ if $t \to -\infty$.

For $a \geqslant 1$ the right-hand side of the last inequality is nonnegative on account of (6). This yields

$$a\sigma(Tw, v)-\sigma\big(T(aw), v\big) \geqslant 0, \quad \sigma = \pm 1, a \geqslant 1,$$

and

(13) $$a(Tw, v) = \big(T(aw), v\big), \quad a \geqslant 1.$$

Using hypothesis (11) again, we obtain

$$a(Tw, v) \leqslant K, \quad a \geqslant 1,$$

and thus by passing to the limit $a \to \infty$

(14) $$(Tw, v) \leqslant 0, \quad w \in B.$$

Setting $\beta = 1/a$, $z = (1/a)w$, $a \geqslant 1$, we obtain from (13) that

$$\beta(Tz, v) = \big(T(\beta z), v\big), \quad 0 < \beta \leqslant 1.$$

Hence

(15) $$\lim \beta^{-1}\big(T(\beta z), v\big) = (Tz, v) \quad (\beta \to +0).$$

On the other hand, the function ψ defined for $\beta \in \boldsymbol{R}$ by

$$\psi(\beta) = \big(T(\beta z), v\big)$$

is differentiable at $\beta = 0$ if $z \in D_0$; cf. hypothesis (8).

Since $\psi(0) = 0$, on account of (5) we conclude from (14) that ψ has a maximum at $\beta = 0$. Hence the derivative $\psi'(0)$ vanishes at 0 and we obtain

$$\lim \beta^{-1}\big(T(\beta z), v\big) = 0 \quad (\beta \to 0).$$

From (15) we then conclude that

$$(Tz, v) = 0, \quad z \in D_0,$$

and since D_0 is dense in B and T is continuous, we obtain $v \perp R(T)$. ■

LEMMA 2. *Let $T\colon B \to B^*$ be semi-coercive in the sense of* (10). *Then*

$$\dim R(T)^{\perp} \leqslant \dim QB.$$

Proof. Let $n = \dim QB$ and $z_i \in R(T)^{\perp}$, $i = 1, \ldots, n+1$. We shall prove that the z_i are linearly dependent. Since $\dim QB = n$, there exist numbers λ_i such that

$$\sum_{i=1}^{n+1} \lambda_i Q z_i = 0, \quad \sum_{i=1}^{n+1} |\lambda_i| \neq 0.$$

Set $z = \sum_{i=1}^{n+1} \lambda_i z_i$. Since $z \perp R(T)$, we have

$$\big(T(tz), tz\big) = 0, \quad t \in \boldsymbol{R}$$

and by semi-coerciveness

$$\sup\big\{\|tz\|/\big(1+\|Q(tz)\|\big)\big|\ t \in \boldsymbol{R}\big\} < \infty.$$

Since

$$Q(tz) = t\sum_{i=1}^{n+1} \lambda_i Q z_i = 0,$$

we conclude that $z = 0$. Hence there cannot exist more than n linearly independent elements in $R(T)^{\perp}$. The lemma follows.

Proof of Theorem 1 (from [6]). The finiteness of $\dim R(T)^{\perp}$ follows from Lemma 2. (In fact, we have $\dim R(T)^{\perp} \leqslant \dim QB < \infty$.) Linearity and closedness of $R(T)$ are proved in the following formulation:

The equation $Tu = f$ is solvable if and only if $f \perp R(T)^{\perp}$.

The "only if" part of the above statement is trivial. For the "if" part, we assume that $f \perp R(T)^{\perp}$ and assume that the equation $Tu = f$ were *not* solvable. By induction we then shall construct linearly independent elements $z_i \perp R(T)$, $i = 1, 2, \ldots, m$, $m = 1+\dim QB$, which will contradict Lemma 2.

Let $i \in \{1, 2, \ldots\}$ and if $i \geqslant 2$, assume that linearly independent elements $z_j \perp R(T)$, $j = 1, 2, \ldots, i-1$, have been constructed.

Set $V_1 = 0$ and $V_i = \text{linear hull } \{z_1, \ldots, z_{i-1}\}$ if $i \geqslant 2$.

Let W be a closed linear complement to the space V_i. By (9), for all $r > 0$, there exists an $u_r \in B_r \cap W$ such that

(16) $$(Tu_r - f, u_r - x) \leqslant 0, \quad x \in B_r \cap W$$

where $B_r = \{x \in B \mid \|x\| \leqslant r\}$.

If $\|u_r\| < r$ for some $r > 0$, then $Tu_r - f \perp W$; hence $Tu_r - f \perp W + V_i$ since $R(T) - f \perp V_i$, by induction hypothesis and $f \perp V_i$ by hypothesis. Thus $Tu_r - f \perp B$ or $Tu_r = f$ which contradicts our assumption that the equation $Tu = f$ is not solvable. Therefore we need to consider only the case $\|u_r\| = r$, $r \to \infty$.

Since B is reflexive, there is an element $z \in B$ such that $\|u_r\|^{-1} u_r \to z$ weakly for a subsequence $(r \to \infty)$. Setting $x = 0$ in (16), we obtain

$$\limsup \|u_r\|^{-1} (Tu_r, u_r) < \infty \quad (r \to \infty)$$

and, from the condition of semi-coerciveness (10),

$$\|u_r\| \leqslant C\|Qu_r\| + C \quad (r \to \infty)$$

with some constant C. Hence

$$C^{-1} - \|u_r\|^{-1} \leqslant \|u_r\|^{-1} \|Qu_r\| = \left\| Q(\|u_r\|^{-1} u_r) \right\|$$

and by passing to the limit $r \to \infty$ we have

$$C^{-1} \leqslant \|Qz\|$$

since Q is completely continuous. Thus $z \neq 0$.

From the asymptotic monotonicity condition we conclude that

$$\liminf \|u_r\|^{-1} (Tu_r - Tw, u_r - w) \geqslant 0 \quad (r \to \infty).$$

From the variational inequality (16) and the orthogonality $V_i \perp R(T)$, $V_i \perp f$, we conclude for $w = w_1 + w_2 \in B$, $w_1 \in W \cap B_r$, $w_2 \in V_i$ that

(17) $$(Tu_r - f, u_r - w) \leqslant 0.$$

From (17) we obtain

$$\liminf \|u_r\|^{-1} (f - Tw, u_r - w) \geqslant 0 \quad (r \to \infty)$$

and thus

$$(f - Tw, z) \geqslant 0, \quad w \in B.$$

Hence from Lemma 1 we conclude that $z \perp R(T)$ and, by hypothesis, $f \perp z$. Since $z \in W$, $z \neq 0$, we have $z \notin V_i$, i.e. z does not depend linearly on $z_1, \ldots, z_{i-1}$. Setting $z_i = z$ completes the construction of the z_j and we obtain the contradiction. The theorem is proved.

EXAMPLES. The hypotheses of Theorem 1 are satisfied in each of the following examples:

(i) $\langle Tu, v\rangle := (|\nabla u|^{p-2} \nabla u, \nabla v)$, $u, v \in H^{1,p}(\Omega)$, $p > 2$;

(ii) $\langle Tu, v\rangle := \big((2 + \cos u)|\nabla u|^{p-2} \nabla u, \nabla v\big)$, $u, v \in H^{1,p}(\Omega)$, $p > 2$;

(iii) $\langle Tu, v\rangle := (|\nabla u_1|^{p-2} \nabla u_1, \nabla v_1) + (|u_1|^p u_1, v_1) + (\sin u_1 \nabla u_2, \nabla v_1 + \nabla v_2) + (|\nabla u_2|^{p-2} \nabla u_2, \nabla v_2)$, $u = (u_1, u_2)$, $v = (v_1, v_2) \in H^{1,p}(\Omega) \times H^{1,p}(\Omega)$, $p > 2$;

(iv) $\langle Tu, v\rangle := \big(|\Delta u - \lambda u|^{p-2} (\Delta u - \lambda u), \Delta u - \lambda v\big)$, $u, v \in H_0^{1,p}(\Omega) \times H^{2,p}(\Omega)$, where $p > 1$ and λ may be an eigenvalue of Δ;

(v) $\langle Tu, v\rangle := \sum_{j=1}^{s} \big(P_j(L_1 u, \ldots, L_s u), L_j v\big)$, $u, v \in H_0^{1,p} \cap H^{2,p}(\Omega)$,

where $P_j\colon \mathbf{R}^s \to \mathbf{R}$, $j = 1, \ldots, s$, are polynomials such that

$$|P_j(\xi)| \leqslant K + K|\xi|^{p-1},$$

$$\sum_j P_j(\xi)\, \xi_j \geqslant c|\xi|^p - K,$$

$$\sum_j \big(P_j(\xi) - P_j(\zeta)\big)(\xi_j - \zeta_j) \geqslant 0, \quad j = 1, \ldots, s,$$

$$P_j(0) = 0, \quad j = 1, \ldots s,$$

with constants $K, c > 0$, $p > 1$. The operators L_j are second order uniformly elliptic operators defined by

$$L_j u = \sum_{i,k=0}^{n} a_{ik}^{(j)} \partial_i \partial_k u$$

$(\partial_0 = \text{identity})$.

In all the examples, $H_0^{1,p}$, $H^{1,p}$, $H^{2,p}$ denote the usual Sobolev spaces of a bounded domain Ω of $\mathbf{R}^n$.

We have used the notation $(w, z) = \int_\Omega wz\,dx$ for functions $w, z\colon \Omega \to \mathbf{R}$ or $\mathbf{R}^n$.

The condition of semi-coerciveness (10) follows from Rellich's lemma in L^p, i.e. the inequality

$$\|u\|_p \leqslant \varepsilon \|u\|_{1,p} + K_\varepsilon \|Qu\|_{1,p}$$

with a finite dimensional projection $Q = Q(\varepsilon)\colon H^{1,p} \to H^{1,p}$.

Appendix

The following conjecture is of interest in algebraic geometry:

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a mapping whose components are polynomials in n variables. Assume for the Jacobian ∇T that

(18) $$\det\big(\nabla T(x)\big) \neq 0 \quad \text{for all } x \in \mathbf{R}^n.$$

Does it follow that T is a homeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$?

We are interested in the question whether T is a mapping *onto* $\boldsymbol{R}^n$ if (18) holds.

A simple consequence of our theorem is

THEOREM A. *Let* $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ *be a polynomial mapping such that for all* x, $\nabla T(x)$ *is positively definite. Then* T *is a mapping onto* $\boldsymbol{R}^n$.

Proof. By Theorem 1 the range of T is linear and hence closed. By (18), it is open and, therefore, all $\boldsymbol{R}^n$.

References

[1] H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Sci. Inst. Fourier (Grenoble) 19 (1968), 115–176.

[2] F. E. Browder, *Existence and uniqueness theorems for solutions of non-linear boundary value problems*, Proceedings of symposia in applied mathematics 17 (1965), 24–49.

[3] —, *Sur l'image d'opérateurs maximaux monotones*, Submitted to Comptes Rendus.

[4] J. Frehse, *Existence and alternative theorems for semi-coercive problems and applications to non-linear elliptic equations*, Proceedings Ferrara, Sept. 1977. SFB-Preprint 175, Universtiy of Bonn.

[5] —, *An existence theorem for a class of non-coercive optimization and variational problems*, Math. Zeitschr. 159 (1978), 51–63.

[6] —, *Solvability and alternative theorems for a class of non-linear functional equations in Banach spaces*. Arkiv för Matematik.

[7] —, *Landesman–Lazer alternative theorem for a class of non-linear functional equations*, Math. Ann. 238 (1978), 59–65.

[8] —, *A Landesman–Lazer alternative theorem for a class of optimization problems*, Commentarii Helvetici.

[9] —, *On the range of non-linear semi-coercive operators*, Proceedings Rome, May 1978.

[10] E. M. Landesman and A. C. Lazer, *Non-linear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. 19 (1970), 609–623.

[11] J. Leray et J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptique non linéaire par le méthode de Minty–Browder*, Bull. Soc. Math. France 93 (1965), 97–107.

[12] J. Mawhin, *Landesman–Lazer's type problems for non-linear equations*, Conferenze del Seminario die Matematica dell' Università di Bari, No. 147 (1977).

[13] G. J. Minty, *Monotone (non-linear) operators in Hilbert space*, Duke Math. Journal 29 (1962), 341–346.

[14] M. I. Visik, *Solution of a system of quasi-linear equations having divergence form under periodic boundary conditions*, Dokl. Acad. Nauk 137 (1961), 502–505.

Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978

PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

ON THE SMOOTHNESS OF SOLUTIONS OF VARIATIONAL INEQUALITIES WITH OBSTACLES

JENS FREHSE

Institut für Angewandte Mathematik der Universität Bonn, Bonn, F.R.G.

0. Introduction

The first three sections of this contribution are devoted to the question of the regularity of solutions of scalar variational inequalities with obstacles, that is, to problems of the type:

(0.1) *Find* $u \in \boldsymbol{K} = \{v \in H_0^1(\Omega) |\ v \geqslant \psi \text{ in } \Omega\}$ *such that*

$$\sum_{i=0}^{n} \big(a_i(x, u, \nabla u),\ \partial_i u - \partial_i v\big) \leqslant 0$$

for all $v \in \boldsymbol{K}$.

Here Ω is a bounded open subset of $\boldsymbol{R}^n$, $H_0^1(\Omega)$ the usual Sobolev space of functions u which have a generalized gradient in $L^2(\Omega)$ and vanish on $\partial\Omega$ in the generalized sense. The scalar product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, i.e. $(w, z) = \int wz\,dx$; ∂_0 denotes the identity map. The inequality sign $v \geqslant \psi$ in the definition of $\boldsymbol{K}$ is to be understood in the sense of H^1, cf. [25] or [37], or in the sense "almost everywhere" (which may be quite different).

We shall assume natural growth and ellipticity conditions for the functions a_i, cf. § 1 and § 2. For a sufficiently smooth obstacle ψ, say, for $\psi \in H^{2\infty}(\Omega)$ (i.e. ψ having bounded second derivatives), the question of the regularity of solutions of (0.1) has been essentially solved. From the general regularity theory due to Brézis–Stampacchia [8] one obtains that $u \in H^{2,p}(\Omega)$ for all $p < \infty$, and the final step yielding $u \in H^{2,\infty}(\Omega)$ was performed in [15], [16], [21], [9]. It is well known that the further regularity condition $u \in C^2(\Omega)$ is false, in general. Cf. also [2], [24], [26], [27], [37] for many other results on regularity and historical remarks.