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ON A CLASS OF ELLIPTIC DIFFERENTIAL OPERATORS DEGENERATING AT ONE POINT

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Introduction

We consider the differential operator

$$(0.1)\qquad P_0(x, D) = \sum_{|\alpha|=|\beta|\leqslant m} c_{\alpha\beta} x^\beta D^\alpha,$$

$$c_{\alpha\beta} \in \boldsymbol{C}, \quad D^\alpha = \partial^{\alpha_1+\ldots+\alpha_n}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n},$$
$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \beta = (\beta_1, \ldots, \beta_n), \quad x^\beta = x_1^{\beta_1}\ldots x_n^{\beta_n},$$

in $\boldsymbol{R}^n$, $n \geqslant 2$, and assume that P_0 is elliptic in $\boldsymbol{R}^n\setminus\{0\}$:

$$(0.2)\qquad p(x, \xi) = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0, \quad \forall (x, \xi) \in (\boldsymbol{R}^n\setminus\{0\})\times(\boldsymbol{R}^n\setminus\{0\}).$$

Baouendi and Sjöstrand [3] studied analytic regularity of the operator P_0 at $x = 0$. Furthermore, it was proved in [4] that the operator P_0 is not hypoelliptic at the origin (under a certain additional condition). More general results of this kind were announced in [10]. The special case

$$(0.3)\qquad L_0(x, D) = \Delta r^2 + \mu\partial/\partial r\cdot r + \lambda,$$
$$\lambda, \mu \in \boldsymbol{C}, \quad r^2 = \sum x_i^2, \quad \Delta = \sum \partial^2/\partial x_i^2,$$

of the operator (0.1) was studied by Baouendi, Goulaouic and Lipkin [2] in the space of germs of analytic functions at the origin. They gave a complete description of the kernel and the range of the operator L_0. The same operator was considered in Sobolev spaces $H_{2s}(\Omega)$, $s \in \boldsymbol{N}$, by the authors [5]. The aim of this paper is to investigate normal solvability and index of the operator (0.1) in the spaces $H_s(\boldsymbol{R}^n)$ and $H_s(\Omega)$, $s \geqslant 0$, Ω denoting a bounded domain in $\boldsymbol{R}^n$ with $0 \in \Omega$ (for the definition of normally solvable and Fredholm operators we refer to [9]; the index of an operator A is defined by $\operatorname{ind} A = \dim\ker A - \operatorname{codim}\operatorname{im} A$).

We also consider the more general operator (cf. [3], [4], [10])

$$P_0(x, D) + \sum_{|\alpha|=|\beta|\leqslant m} d_{\alpha\beta}(x)x^\beta D^\alpha, \qquad d_{\alpha\beta}(0) = 0,$$

in a small neighbourhood of the origin.

As in [3], using the Mellin transform with respect to the radial variable, we reduce the equation $P_0 u = f$ to an elliptic system of pseudo-differential equations on the sphere S^{n-1} which depends on a complex parameter. We mention that similar methods have been used by Bagirov and Kondratiev [1], [7] in the study of elliptic equations in unbounded domains.

1. Preliminaries

1.1. From now on, we denote by Ω the whole space $\boldsymbol{R}^n$ or a bounded domain with the following property:

The boundary Γ of Ω is a $(n-1)$-dimensional infinitely differentiable variety, Ω being locally on one side of Γ.

Let $H_s(\Omega)$, $s \geqslant 0$, be the usual Sobolev space with the norm

$$\|u\|_s^2 = \sum_{|\alpha|\leqslant[s]} \int_\Omega \|D^\alpha u\|^2 dx + \sum_{|\alpha|=[s]} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} dx dy \tag{1.1}$$

(but without the second term for $s \in \boldsymbol{N}$). Furthermore, we introduce the spaces $H_{s,m}(\Omega) = \{u \in H_s(\Omega)\colon x^\beta D^\alpha u \in H_s(\Omega),\ |\alpha| = |\beta| \leqslant m\}$ with the canonical norm

$$\|u\|_{s,m} = \sum_{|\alpha|=|\beta|\leqslant m} \|x^\beta D^\alpha u\|_s. \tag{1.2}$$

When Ω is bounded and $0 \notin \overline{\Omega}$, the spaces $H_{s,m}(\Omega)$ and $H_{s+m}(\Omega)$ coincide (algebraically and topologically). $H_{s,m}(\Omega)$ is a Hilbert space and we have $P_0 \in \mathscr{L}(H_{s,m}(\Omega), H_s(\Omega))$, i.e. the operator $P_0\colon H_{s,m} \to H_s$ is linear and continuous.

We set $\mathring{C}_0^\infty = \{u \in C_0^\infty(\boldsymbol{R}^n)\colon 0 \notin \operatorname{supp} u\}$ and $\mathring{H}_s(\Omega) = \{u \in H_s(\Omega)\colon D^\alpha u(0) = 0,\ |\alpha| < s - n/2\}$, $0 \in \Omega$. In virtue of Sobolev's embedding theorem, $\mathring{H}_s$ is a closed subspace of finite codimension in H_s.

LEMMA 1.1. *$\mathring{C}_0^\infty$ is dense in $\mathring{H}_s(\boldsymbol{R}^n)$, $s \geqslant 0$.*

The proof is similar to the proof of Lemma 11.1 in [8], Chap. 1 (but simpler).

We obviously have $\mathring{H}_0 = H_0 = L_2$. Now, let $\Omega \subset \boldsymbol{R}^n$, $n \geqslant 1$, $r^2 = \sum_i x_i^2$ and $\varrho_{t,s} = r^t(1+r)^{-s}$, $t, s \in \boldsymbol{R}$. The linear set $L_2(\Omega; \varrho_{t,s}) = \varrho_{t,s}\cdot L_2(\Omega)$ is a Hilbert space with the norm $\|u\| = \|\varrho_{t,s}^{-1} u\|_0$. Furthermore,

$$H_{0,m}(\Omega; \varrho_{t,s}) = \{u \in L_2(\Omega; \varrho_{t,s})\colon x^\alpha D^\beta u \in L_2(\Omega; \varrho_{t,s}),\ |\alpha| = |\beta| \leqslant m\},$$

$t \geqslant 0$, is a Hilbert space with the canonical norm and the relation $H_{0,m}(\Omega; \varrho_{t,s}) = \varrho_{t,s}\cdot H_{0,m}(\Omega)$ holds.

Denoting by $\mathscr{F}$ the (n-dimensional) Fourier transform, we get the commutative diagram

$$\begin{array}{ccc} P_0\colon H_{s,m}(\boldsymbol{R}^n) & \longrightarrow & H_s(\boldsymbol{R}^n) \\ \downarrow\mathscr{F} & & \downarrow\mathscr{F} \\ \hat{P}_0\colon H_{0,m}(\boldsymbol{R}^n; \varrho_{0,s}) & \longrightarrow & L_2(\boldsymbol{R}^n; \varrho_{0,s}) \end{array} \tag{1.3}$$

with the operator $\hat{P}_0 = \sum_{|\alpha|=|\beta|\leqslant m} c_{\alpha\beta}(-1)^{|\alpha|} D^\beta x^\alpha$, which is also elliptic in $\boldsymbol{R}^n\setminus\{0\}$ (cf. (0.2)).

1.2. In analogy to [3] we reduce the equation

$$\hat{P}_0 u = f \tag{1.4}$$

to a system of pseudo-differential equations of first order. The operator $\hat{P}_0$ can be written in spherical coordinates (r, θ)

$$P_0(\theta, D_\theta, rD_r) = \sum_{j=0}^m A_j(\theta, D_\theta)(r\partial/\partial r)^{m-j},$$

where $A_j(\theta, D_\theta)$ are differential operators of order $\leqslant j$ on S^{n-1} with analytic coefficients. It follows from (0.2) that $A_0(\theta) \neq 0$, $\theta \in S^{n-1}$. Setting $B_j(\theta, D_\theta) = -A_0(\theta)^{-1}A_j(\theta, D_\theta)$, $U = (u_j)_1^m$, $u_j = \Lambda^{m-j}(r\partial/\partial r)^{j-1}u$, $j = 1, \ldots, m$, $\Lambda = (1+\delta)^{1/2}$, where δ is the Beltrami operator on S^{n-1}, $F = (f_j)_1^m$, $f_j = 0$, $j < m$, $f_m = A_0(\theta)^{-1}f$, and

$$\mathscr{A}_\theta = \begin{bmatrix} 0 & \Lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \Lambda \\ B_m\Lambda^{-m+1} & \ldots & B_2\Lambda^{-1} & B_1 \end{bmatrix},$$

we obtain from equation (1.4) the system

$$\mathfrak{A}_0 U = r\partial/\partial r\cdot U - \mathscr{A}_\theta U = F. \tag{1.5}$$

In view of (0.2), $\mathscr{A}_\theta$ is an elliptic pseudo-differential operator on S^{n-1} (cf. [3]). The operator $\mathscr{A}_\theta$ has the following properties ([3], [11]):

(i) There is only a discrete set of different eigenvalues λ_k, $k \in J$, of the operator $\mathscr{A}_\theta$ in the space $L_2^m(S^{n-1})$. (1) Each eigenvalue λ_k has finite algebraic multiplicity m_k.

(1) By X^m we denote the space $X \times \ldots \times X$ (m factors).

(ii) The operator $zI-\mathscr{A}_\theta$ has an inverse $\mathscr{R}_z \in \mathscr{L}(L_2^m(S^{n-1}))$ for all $z \neq \lambda_k$. Furthermore, there exist a cone $K = \{z \in \boldsymbol{C}:\ \varepsilon < |\arg z| < \pi-\varepsilon\}$, $\varepsilon \in (0, \pi/2)$, and numbers c, $M > 0$ such that

$$\|\mathscr{R}_z\|_{\mathscr{L}(L_2^m(S^{n-1}))} \leqslant c|z|^{-1}, \quad \forall\, z \in K,\ |z| \geqslant M. \tag{1.6}$$

In order to investigate the equivalence of equations (1.4) and (1.5), we state two lemmas. The first lemma is a generalization of Lemmas 1.1 and 1.2 in [5]; the method of the proof is the same.

LEMMA 1.2. *$\mathring{C}_0^\infty$ is dense in $H_{0,m}(\boldsymbol{R}^n)$ and the norms* (1.2) *and*

$$|u|_{0,m} = \sum_{i+j\leqslant m} \|\Lambda^i (r\partial/\partial r)^j u\|_0$$

are equivalent in the space $H_{0,m}(\boldsymbol{R}^n)$.

LEMMA 1.3. *With the above notations we have*

$$|u|_{0,m} \sim |U|_{0,1} = \|U\|_0 + \|r\partial/\partial r \cdot U\|_0 + \|\Lambda U\|_0 \sim \|U\|_0 + \|r\partial/\partial r\cdot U\|_0 + \|\mathscr{A}_\theta U\|_0,$$
$$\forall\, u \in H_{0,m}(\boldsymbol{R}^n).\ (^2)$$

Proof. The relation $|u|_{0,m} \sim |U|_{0,1}$ is obvious. Furthermore, thanks to the ellipticity of $\mathscr{A}_\theta$ on S^{n-1}, we obtain for all $u \in \mathring{C}_0^\infty$ and fixed $r_0 > 0$

$$\|\mathscr{A}_\theta U(r_0, \theta)\|_{L_2^m(S^{n-1})} + \|U(r_0, \theta)\|_{L_2^m(S^{n-1})} \sim \|U(r_0, \theta)\|_{H_1^m(S^{n-1})}$$

and

$$\|U(r_0, \theta)\|_{H_1^m(S^{n-1})} \sim \|\Lambda U(r_0, \theta)\|_{L_2^m(S^{n-1})}$$

(cf. [8], Chap. 1) uniformly with respect to r_0. Since $\mathring{C}_0^\infty$ is dense in $H_{0,m}(\boldsymbol{R}^n)$, the result follows by integration with respect to r_0.

COROLLARY 1.4. *The equation* (1.4) *with $f \in L_2(\boldsymbol{R}^n)$ has the solution $u \in H_{0,m}(\boldsymbol{R}^n)$ if and only if the system* (1.5) *admits the solution $U \in H_{0,1}^m(\boldsymbol{R}^n)$.*

Analogous assertions to Lemmas 1.2 and 1.3 and Corollary 1.4 are true for the spaces with weights $\varrho_{t,s}$, $t \geqslant 0$.

1.3. The *Mellin transform*

$$\tilde{u}(z, \theta) = (Mu)(z, \theta) = \int_0^\infty r^{-iz-1} u(r, \theta)\, dr$$

is an isomorphic map from $L_2(\boldsymbol{R}^n; \varrho_{n/2,0}) = L_2((0, \infty); r^{1/2}) \otimes L_2(S^{n-1})$ onto the space $\tilde{L}_2(\boldsymbol{R}^n; \varrho_{n/2,0}) = L_2(-\infty, \infty) \otimes L_2(S^{n-1})$. In virtue of

(2) $\|x\| \sim |x|$, $x \in X$, means that there exist constants $c_1, c_2 > 0$ such that $c_1\|x\| \leqslant |x| \leqslant c_2\|x\|$, $x \in X$.

the relation $M(r\partial/\partial r \cdot u) = izMu$ the operator M is also an isomorphism of $H_{0,1}(\boldsymbol{R}^n; \varrho_{n/2,0})$ onto the space $\tilde{H}_{0,1}(\boldsymbol{R}^n; \varrho_{n/2,0}) = \{\tilde{u} \in L_2(\boldsymbol{R}^n; \varrho_{n/2,0}):\ z\tilde{u},\ \Lambda\tilde{u} \in \tilde{L}_2(\boldsymbol{R}^n; \varrho_{n/2,0})\}$ (with the canonical norm).

Let $t \geqslant n/2$, $s > 0$. We denote by $\tilde{L}_2(\Pi_{t,s})$ the Banach space of the functions which are analytic in the strip $\Pi_{t,s} = \{z \in \boldsymbol{C}:\ -t+n/2 < \operatorname{Im} z < -t+s+n/2\}$ with the norm

$$\|\tilde{u}(z)\|^2 = \sup_{\tau-n/2\in(-t,s-t)} \int_{-\infty}^{\infty} |\tilde{u}(\sigma+i\tau)|^2\, d\sigma, \quad z = \sigma+i\tau.$$

Then the Mellin transform is an isomorphic map from $L_2(\boldsymbol{R}^n; \varrho_{t,s})$ onto the space $\tilde{L}_2(\boldsymbol{R}^n; \varrho_{t,s}) = \tilde{L}_2(\Pi_{t,s}) \otimes L_2(S^{n-1})$ and we have

$$\|u\|^2_{L_2(\boldsymbol{R}^n; \varrho_{\tau,0})} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{S^{n-1}} |(Mu)(\sigma+i\tau, \theta)|^2\, d\sigma\, d\theta,$$
$$\forall u \in L_2(\boldsymbol{R}^n; \varrho_{t,s}) \forall \tau \in (t-s, t)$$

(cf. [12], p. 247, for $n = 1$).

Furthermore, $M\colon H_{0,1}(\boldsymbol{R}^n; \varrho_{t,s}) \to \tilde{H}_{0,1}(\boldsymbol{R}^n; \varrho_{t,s}) = \{\tilde{u} \in \tilde{L}_2(\boldsymbol{R}^n; \varrho_{t,s}):\ z\tilde{u},\ \Lambda\tilde{u} \in \tilde{L}_2(\boldsymbol{R}^n; \varrho_{t,s})\}$ is an isomorphism.

2. The operator P_0 in $L_2(\boldsymbol{R}^n)$

Let P_0 be the operator defined in (0.1) and λ_k, $k \in J$, the eigenvalues of the operator $\mathscr{A}_\theta$ (cf. 1.2).

THEOREM 2.1. *Under the hypothesis*

$$\operatorname{Re} \lambda_k \neq -n/2, \quad k \in J, \tag{2.1}$$

the operator $P_0 \in \mathscr{L}(H_{0,m}(\boldsymbol{R}^n), L_2(\boldsymbol{R}^n))$ is invertible.

Proof. Thanks to (1.3) and Corollary 1.4 it is sufficient to show that the operator $\mathfrak{A}_0 = r\partial/\partial r - \mathscr{A}_\theta$ is invertible in $\mathscr{L}(H_{0,1}^m(\boldsymbol{R}^n), L_2^m(\boldsymbol{R}^n))$ under hypothesis (2.1). Now we consider the operator

$$\mathfrak{B}_0 = r^{n/2}\mathfrak{A}_0 r^{-n/2} = r\partial/\partial r - \mathscr{A}_\theta - (n/2)I \in \mathscr{L}(H_{0,1}^m(\boldsymbol{R}^n; \varrho_{n/2,0}), L_2(\boldsymbol{R}^n; \varrho_{n/2,0}))$$

and the commutative diagram

$$\begin{array}{ccc} \mathfrak{B}_0\colon\ H_{0,1}^m(\boldsymbol{R}^n; \varrho_{n/2,0}) & \longrightarrow & L_2^m(\boldsymbol{R}^n; \varrho_{n/2,0}) \\ \downarrow M & & \downarrow M \\ \tilde{\mathfrak{B}}_0\colon\ \tilde{H}_{0,1}^m(\boldsymbol{R}^n; \varrho_{n/2,0}) & \longrightarrow & \tilde{L}_2^m(\boldsymbol{R}^n; \varrho_{n/2,0}) \end{array} \tag{2.2}$$

with $\tilde{\mathfrak{B}}_0\tilde{u}(z, \theta) = (iz-(n/2)I-\mathscr{A}_\theta)\tilde{u}(z, \theta)$ (cf. 1.3). We have to prove that $\tilde{\mathfrak{B}}_0$ is invertible under hypothesis (2.1). It follows from (2.1) and (1.6) that the estimate

$$\|\mathscr{R}_{iz-n/2}\|_{\mathscr{L}(L_2^m(S^{n-1}))} \leqslant c(1+z)^{-1}, \quad \forall z \in \boldsymbol{R},$$

holds. Therefore, we obtain

$$\|\mathscr{R}_{iz-n/2}\tilde{F}\|_{\tilde{L}_2^m} + \|z\mathscr{R}_{iz-n/2}\tilde{F}\|_{\tilde{L}_2^m} \leqslant c\|\tilde{F}\|_{\tilde{L}_2^m}, \quad \forall \tilde{F} \in \tilde{L}_2^m(\boldsymbol{R}^n; \varrho_{n/2,0}),$$

and $\mathscr{R}_{iz-n/2}$ is the desired inverse of $\tilde{\mathfrak{B}}_0$.

Remark 2.2. One can prove that the operator $P_0 \in \mathscr{L}(H_{0,m}(\boldsymbol{R}^n), L_2(\boldsymbol{R}^n))$ is not normally solvable if the condition (2.1) is violated.

Applying Theorem 2.1 to the operator $r^{-\mu}P_0 r^{\mu}$, we get

COROLLARY 2.3. *Let* $\mu \geqslant 0$. *If* $\operatorname{Re}\lambda_k \neq -n/2-\mu$, $k \in J$, *then the operator* $P_0 \in \mathscr{L}(H_{0,m}(\boldsymbol{R}^n; \varrho_{\mu,0}), L_2(\boldsymbol{R}^n; \varrho_{\mu,0}))$ *is invertible.*

It follows from Corollary 2.3 that the operator P_0 is always locally solvable at the origin.

3. The operator P_0 in $H_s(\boldsymbol{R}^n)$, $s > 0$

We denote by $z(s)$ the sum of the algebraic multiplicities m_k of the eigenvalues λ_k, $k \in J' \subset J$, which satisfy $-n/2-s < \operatorname{Re}\lambda_k < -n/2$. The set J' is finite (cf. 1.2).

THEOREM 3.1. *Under the hypothesis*

$$\operatorname{Re}\lambda_k \neq -n/2, \quad -n/2-s, \; k \in J \tag{3.1}$$

the operator $P_0 \in \mathscr{L}(H_{s,m}(\boldsymbol{R}^n), H_s(\boldsymbol{R}^n))$ *is a Fredholm operator with* $\dim\ker P_0 = 0$ *and* $\operatorname{codim}\operatorname{im} P_0 = z(s)$.

Proof. As in the proof of Theorem 2.1, we consider the diagram (2.2) with the spaces with the weight $\varrho_{n/2,s}$ instead of $\varrho_{n/2,0}$. First, we prove the relation $\operatorname{codim}\operatorname{im}\tilde{\mathfrak{B}}_0 = z(s)$ for the operator $\tilde{\mathfrak{B}}_0 \in \mathscr{L}(\tilde{H}_{0,1}^m(\boldsymbol{R}^n; \varrho_{n/2,s}), \tilde{L}_2^m(\boldsymbol{R}^n; \varrho_{n/2,s}))$. We choose mutually disjoint neighbourhoods $U(z_k) \subset \Pi_s = \{z \in \boldsymbol{C}\colon 0 < \operatorname{Im} z < s\}$ of the points $z_k = -i\lambda_k - i(n/2)$, $k \in J'$. Then it follows from (3.1) and (1.6) that the estimate

$$\|\mathscr{R}_{iz-n/2}\|_{\mathscr{L}(L_2^m(S^{n-1}))} \leqslant c(1+|z|)^{-1}, \quad \forall z \in \Pi_s \setminus \bigcup_{k \in J'} U(z_k), \tag{3.2}$$

holds. Further, there are Laurent expansions

$$\mathscr{R}_{iz-n/2} = \sum_{l=-n_k}^{\infty} (z-z_k)^l A_l^{(k)}, \quad A_l^{(k)} \in \mathscr{L}(L_2^m(S^{n-1})), \; n_k \in \boldsymbol{N}, \tag{3.3}$$
$$z \in U(z_k)\setminus\{z_k\}, \; k \in J',$$

where $A_{-1}^{(k)}$ and $I - A_{-1}^{(k)}$ are projections on the subspaces $\ker \mathscr{T}_{z_k}^{n_k}$ and $\operatorname{im}\mathscr{T}_{z_k}^{n_k}$, $\mathscr{T}_{z_k} = iz_k - (n/2)I - \mathscr{A}_0$, and the relations $A_{-l-1}^{(k)} = \mathscr{T}_{z_k}^l A_{-1}^{(k)}$, $l \geqslant 1$, hold (this follows from Theorem 5 in [6], Chap. 13, since $\mathscr{R}_{iz-n/2}$ is compact in $L_2^m(S^{n-1})$ for some $z \in \boldsymbol{C}$).

Now we consider the equation

$$\tilde{\mathfrak{B}}_0 \tilde{U} = \tilde{F}, \quad \tilde{F} \in \tilde{L}_2^m(\boldsymbol{R}^n; \varrho_{n/2,s}). \tag{3.4}$$

In virtue of (3.2) there exists a solution $\tilde{U} \in \tilde{H}_{0,1}^m(\boldsymbol{R}^n; \varrho_{n/2,s})$ of (3.4) if and only if the function $\mathscr{R}_{iz-n/2}\tilde{F}(z, \theta)$ is analytic in all sets $U(z_k)$, $k \in J'$, with respect to z. The representations (3.3) imply that $F \in \operatorname{im}\tilde{\mathfrak{B}}_0$ if and only if

$$\sum_{l=0}^{j} \frac{1}{(j-l)!} A_{-n_k+l}^{(k)}\big((\partial^{j-l}/\partial z^{j-l})\tilde{F}(z_k, \theta)\big) = 0, \tag{3.5}$$
$$j = 0, \ldots, n_k - 1, \; k \in J'.$$

It can be shown that conditions (3.5) determine m_k linearly independent continuous functionals on $\tilde{L}_2^m(\boldsymbol{R}^n; \varrho_{n/2,s})$ for each $k \in J'$. Therefore we have $\operatorname{codim}\operatorname{im}\tilde{\mathfrak{B}}_0 = z(s)$.

Next, the inequality $\operatorname{codim}\operatorname{im} P_0 = \operatorname{codim}\operatorname{im}\hat{P}_0 \leqslant \operatorname{codim}\operatorname{im}\mathfrak{A}_0 = \operatorname{codim}\operatorname{im}\tilde{\mathfrak{B}}_0 = z(s)$ holds for the operators $P_0 \in \mathscr{L}(H_{s,m}(\boldsymbol{R}^n), H_s(\boldsymbol{R}^n))$, $\hat{P}_0 \in \mathscr{L}(H_{0,m}(\boldsymbol{R}^n; \varrho_{0,s}), L_2(\boldsymbol{R}^n; \varrho_{0,s}))$ and $\mathfrak{A}_0 \in \mathscr{L}(H_{0,1}^m(\boldsymbol{R}^n; \varrho_{0,s}), L_2^m(\boldsymbol{R}^n; \varrho_{0,s}))$ (cf. 1.2). For the proof of the converse inequality, we note that the system $\mathfrak{A}_0 U = F$, $F = (f_j)_1^m \in \mathring{C}_0^{\infty,m}$, has the solution $U = (u_j)_1^m \in H_{0,1}^m(\boldsymbol{R}^n; \varrho_{0,s})$ with

$$u_j = \Lambda^{m-j}(r\partial/\partial r)^{j-1}u - \sum_{l=1}^{j-1} \Lambda^{l-j}(r\partial/\partial r)^{j-1-l}f_l, \quad j = 1, \ldots, m,$$

if the equation $\hat{P}_0 u = f$ with

$$f = A_0(\theta)f_m - \sum_{j=1}^{m}\sum_{l=1}^{j-1} A_{m-j+1}(\theta, D_\theta)(r\partial/\partial r)^{j-1-l}\Lambda^{l-j}f_l \in \mathring{C}_0^{\infty},$$

admits the solution $u \in H_{0,m}(\boldsymbol{R}^n; \varrho_{0,s})$ and that $\mathring{C}_0^{\infty}$ is dense in $L_2(\boldsymbol{R}^n; \varrho_{0,s})$.

Finally, Theorem 2.1 implies $\dim\ker P_0 = 0$. The proof is complete.

4. The operator P_0 in $H_s(\Omega)$

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with $0 \in \Omega$ and $\mathring{H}_{s,m}(\Omega) = \{u \in H_{s,m}(\Omega)\colon x^\beta D^\alpha u \in \mathring{H}_s(\Omega), \, |\alpha| = |\beta| \leqslant m\}$. We note that the relation $\mathring{H}_{s,m}(\Omega) = H_{s,m}(\Omega) \cap \mathring{H}_s(\Omega)$ holds.

THEOREM 4.1. *Under the hypothesis*

(4.1) $$\operatorname{Re}\lambda_k \neq -n/2-s, \quad k \in J,$$

the operator $P_0 \in \mathscr{L}\big(\mathring{H}_{s,m}(\Omega), \mathring{H}_s(\Omega)\big)$ *is right invertible.*

COROLLARY 4.2. *The equation* $P_0u = f$, $f \in H_s(\Omega)$, *has a solution* $u \in H_{s,m}(\Omega)$, *if and only if the algebraic system*

$$(D^\alpha P_0 u)(0) = (D^\alpha f)(0), \quad |\alpha| < s-n/2,$$

with the unknowns $D^\alpha u(0)$, $|\alpha| < s-n/2$, *is solvable.*

Proof of Theorem 4.1. The assertion for $s = 0$ easily follows from Theorem 2.1. Let $s > 0$. First, we prove the relation

(4.2) $$P_0\big(H_{s,m}(\Omega)\big) + N = H_s(\Omega),$$

N being a finite-dimensional space. We consider the equation $P_0u = f$, $f \in H_s(\Omega)$, and extend f to a function $g \in H_s(\boldsymbol{R}^n)$ with compact support and $\int_{\boldsymbol{R}^n} g\,dx = 0$. Applying the Fourier transform $\mathscr{F}$ to $P_0u = g$, we obtain the equation

(4.3) $$\hat{P}_0\hat{u} = \hat{g}, \quad \hat{g} = \mathscr{F}g.$$

Since $\hat{g}$ is an entire function with $\hat{g}(0) = 0$, we have $\hat{g} \in L_2(\boldsymbol{R}^n; \varrho_{\varepsilon,s+\varepsilon})$ for some $\varepsilon > 0$. It is possible to choose ε such that $\operatorname{Re}\lambda_k \neq -n/2+\varepsilon$, $-n/2-s$, $k \in J$. Analogous to the proof of Theorem 3.1 we get the relation

(4.4) $$\dim L_2(\boldsymbol{R}^n; \varrho_{\varepsilon,\varepsilon+s})/\hat{P}_0\big(H_{0,m}(\boldsymbol{R}^n; \varrho_{\varepsilon,\varepsilon+s})\big) < \infty.$$

But each solution $\hat{u} \in H_{0,m}(\boldsymbol{R}^n; \varrho_{\varepsilon,\varepsilon+s})$ of (4.3) belongs to $H_{0,m}(\boldsymbol{R}^n; \varrho_{0,s})$ and thus (4.4) implies (4.2).

Further, the operator P_0 is invariant under the transformation $x \to cx$, $c > 0$. Thus it is sufficient to prove Theorem 4.1 for small balls $\Omega = \Omega_r = \{x \in \boldsymbol{R}^n\colon |x| < r\}$.

From (4.2) we obtain the relation

(4.5) $$P_0\big(H_{s,m}(\Omega_1)\big) \cap \mathring{H}_s(\Omega_1) + N_1 = \mathring{H}_s(\Omega_1), \quad \dim N_1 < \infty.$$

Now, thanks to Lemma 1.1 $\mathring{C}_0^\infty$ is dense in $\mathring{H}_s(\boldsymbol{R}^n)$. Hence, $\mathring{C}^\infty(\bar{\Omega}_1) = \{u \in C^\infty(\bar{\Omega}_1)\colon 0 \notin \operatorname{supp} u\}$ is dense in $\mathring{H}_s(\Omega_1)$. Therefore, we can assume that the space N_1 in (4.5) is the linear hull of functions $f_i \in \mathring{C}^\infty(\bar{\Omega}_1)$, $i = 1, \ldots, q = \dim N_1$ (cf. [9], Chap. 1, Lemma 2.2).

Now we choose r, $0 < r < 1$, such that $\Omega_r \cap \operatorname{supp} f_i = \emptyset$, $i = 1, \ldots, q$, and we prove that the equation

(4.6) $$P_0 u = f, \quad f \in \mathring{H}_s(\Omega_r),$$

has always a solution $u \in \mathring{H}_{s,m}(\Omega_r)$. In order to do so, we extend f to a function $h \in \mathring{H}_s(\Omega_1)$. In virtue of (4.5) there exists an element $h_1 = \sum c_i f_i \in N_1$, $c_i \in \boldsymbol{C}$, such that $h + h_1 \in P_0\big(H_{s,m}(\Omega_1)\big) \cap \mathring{H}_s(\Omega_1)$. Thus the equation $P_0 u = h + h_1$ admits a solution $u_1 \in H_{s,m}(\Omega_1)$ and (4.6) has the solution $v = u_1|_{\Omega_r} \in H_{s,m}(\Omega_r)$. Finally, setting

$$p(x) = \sum_{|\alpha| < s-n/2} \frac{1}{\alpha!}(D^\alpha v)(0)x^\alpha$$

we obtain $P_0(v-p) = f - P_0 p \in \mathring{H}_s(\Omega_r)$. Since $f \in \mathring{H}_s(\Omega_r)$, we have $P_0 p = 0$ and $u = v-p \in \mathring{H}_{s,m}(\Omega_r)$ is a solution of (4.6). This finishes the proof.

We now consider the operator

$$P_x = \sum_{|\alpha| = |\beta| \leqslant m} c_{\alpha\beta}(x)x^\beta D^\alpha, \quad c_{\alpha\beta}(x) \in C^\infty(\bar{\Omega}),\ 0 \in \Omega.$$

THEOREM 4.3. *Assume that the operator* $P_0 = \sum_{|\alpha| = |\beta| \leqslant m} c_{\alpha\beta}(0)x^\beta D^\alpha$ *satisfies condition* (4.1). *Then the assertion of Theorem* 4.1 *is true for the operator* $P_x \in \mathscr{L}\big(\mathring{H}_{s,m}(\Omega), \mathring{H}_s(\Omega)\big)$ *for all sufficiently small domains* Ω.

Proof. It is sufficient to prove the theorem for small balls Ω_r. The map Ψ_n: $u(x) \to u(x/n)$ is an isomorphism of $\mathring{H}_s(\Omega_{1/n})$ onto $\mathring{H}_s(\Omega_1)$ and of $\mathring{H}_{s,m}(\Omega_{1/n})$ onto $\mathring{H}_{s,m}(\Omega_1)$. Furthermore, the relations

$$P_0 = \Psi_n^{-1} P_0 \Psi_n \quad \text{and} \quad P_x = P_0 + T = \Psi_n^{-1}(P_0 + T_n')\Psi_n$$

hold, where

$$T = \sum_{|\alpha| = |\beta| \leqslant m} [c_{\alpha\beta}(x) - c_{\alpha\beta}(0)]x^\beta D^\alpha, \quad T_n' = \sum_{|\alpha| = |\beta| \leqslant m} [c_{\alpha\beta}(x/n) - c_{\alpha\beta}(0)]x^\beta D^\alpha.$$

Using the norm (1.1) we get $\|T_n'\|_{\mathscr{L}(H_{s,m}(\Omega_1), H_s(\Omega_1))} \to 0$, $n \to \infty$. Therefore $P_0 + T_n' \in \mathscr{L}\big(\mathring{H}_{s,m}(\Omega_1), \mathring{H}_s(\Omega_1)\big)$, $\forall n \geqslant n_0$, is right invertible. Hence, $P_x \in \mathscr{L}\big(\mathring{H}_{s,m}(\Omega_{1/n}), \mathring{H}_s(\Omega_{1/n})\big)$ is right invertible for $n \geqslant n_0$.

5. An example

We consider the operator (0.3) $P_0 = L_0 = \Delta r^2 + \mu\partial/\partial r \cdot r + \lambda = (r\partial/\partial r)^2 + \mu_1 r\partial/\partial r + \lambda_1 - \delta$, where $\mu_1 = \mu + n + 2$, $\lambda_1 = \lambda + \mu + 2n$. Then we have $\hat{P}_0 = (r\partial/\partial r)^2 + \mu_2 r\partial/\partial r + \lambda_2 - \delta$, $\mu_2 = 2n - \mu_1$, $\lambda_2 = \lambda_1 + n^2 - n\mu_1$, and

$$\mathscr{A}_\theta = \begin{pmatrix} 0 & \Lambda \\ (\delta - \lambda_2)\Lambda^{-1} & -\mu_2 \end{pmatrix} \quad \text{(cf. 1.1, 1.2).}$$

The operator $\mathscr{A}_\theta$ in $L_2^2(S^{n-1})$ has the eigenfunctions

$$\begin{pmatrix} \lambda_k^{\pm} \Lambda^{-1} P_{k,a}(\theta) \\ P_{k,a}(\theta) \end{pmatrix}, \quad 1 \leqslant a \leqslant a(k) = (2k+n-2)(n-k+3)!\,[(n-2)!k!]^{-1},$$

which correspond to the eigenvalues

$$\lambda_k^{\pm} = \tfrac{1}{2}\{(\mu+2-n) \pm [(n+\mu+2)^2 + 4k(k+n-2) - 4(2n+\mu+\lambda)]^{1/2}\}, \quad k \in \mathbf{Z},$$

$P_{k,a}(\theta)$ denoting the spherical harmonics.

The algebraic multiplicities of $\lambda_k^{\pm}$ are equal to $a(k)$, if $\lambda_k^+ \neq \lambda_k^-$. For $\lambda_k^{\pm} = \lambda_k$ the algebraic multiplicity of λ_k is $2a(k)$.

Now one can derive most of the results of [5] for arbitrary $s \geqslant 0$ from Theorems 2.1, 3.1 and 4.1.

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MONOTONE OPERATORS WITH LINEAR RANGE

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The following theorem is a standard tool for existence proofs in the theory of non-linear elliptic boundary value problems, cf. [2], [13]:

THEOREM 0. *Let* B *a reflexive Banach space with dual* B^* *and let* $T: B \to B^*$ *be a continuous mapping which satisfies the monotonicity condition*

$$(1) \qquad (Tu - Tv, u - v) \geqslant 0, \quad u, v \in B$$

and the coerciveness condition

$$(2) \qquad (Tu, u)/\|u\| \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Then T *is surjective.*

Many generalizations of this theorem have been given, cf. [1], [2], [11], [14]. In applications to elliptic equations, the space B is a closed subspace of the usual Sobolev-space $H^{1,p}(\Omega)$ over a domain Ω of $\mathbf{R}^n$, containing the space $C_0^\infty(\Omega)$ of testfunctions. The mapping $T: B \to B^*$ then is defined by

$$(3) \qquad (Tu, v) = \sum_a \int_\Omega A_a(x, u, \ldots, \nabla^m u)\, \partial^a v\, dx \quad (|a| \leqslant m), \quad v \in B.$$

The notation uses multi-indices $a = (a_1, \ldots, a_n)$ and the abbreviation $\partial^a = \prod_i (\partial/\partial x_i)^{a_i}$ $(i = 1, \ldots, n)$. If one assumes only an ellipticity condition for the principal part of the operator $\sum_a (-1)^{|a|}\partial^a A_a$, and corresponding growth conditions for the A_a $(|a| \leqslant m)$, one cannot derive the "full" coerciveness of T as required in Theorem 0. With appropriate growth and ellipticity conditions it is possible to derive only a "Gårding-type inequality"

$$(4) \qquad (Tu, u) \geqslant c\|u\|_{m,p}^p - K\|u\|_p^p - K$$