PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

THE BERNOULLI MANIFOLDS FOR NONELLIPTIC QUASILINEAR SYSTEMS OF FIRST ORDER AND APPLICATIONS IN CONTINUUM MECHANICS

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1. Introduction

We consider a system of the form

$$(1)\qquad a_j^{si}(u^1, \ldots, u^l)u^j_{x^i} = 0, \qquad i = 1, \ldots, n;\ j = 1, \ldots, l;\ s = 1, \ldots, m \geqslant l,$$

where the summation is performed over repeated indices. Our considerations may be extended to the general nonhomogeneous quasilinear systems (see [6], [7], [8]) but in the case of (1) the geometrical features of the method are more clear. By assumption, $m \geqslant l$, the system (1) may be overdetermined. It allows one to apply our considerations to quasilinear systems of higher order, which may be reduced to a system of first order.

Our chief examples in the present lectures are systems describing inviscid, isentropie, compressible gas flow.

$$(2)\qquad \begin{cases} c_t + u^\delta c_{x^\delta} + \dfrac{c}{k}\operatorname{div} u = 0, \\ u^s_t + u^\delta u^s_{x^\delta} + kc\cdot c_{x^s} = 0, \qquad \delta = 1, 2, 3;\ s = 1, 2, 3 \end{cases}$$

and

$$(3)\qquad \begin{cases} u^\delta c_{x^\delta} + \dfrac{c}{k}\operatorname{div} u = 0, \\ u^\delta u^s_{x^\delta} + kc\cdot c_{x^s} = 0, \qquad \delta = 1, 2, 3; \quad s = 1, 2, 3 \end{cases}$$

where $u = (u^1, u^2, u^3)$ is the flow velocity, c the sound speed, and $0 < k = \text{const}$. The unknown functions $U = (c, u^1, u^2, u^3)$ depend in (2) on

(t, x^1, x^2, x^3) — the nonsteady flow and in (3) on (x^1, x^2, x^3) — the steady flow.

The other example is the system describing the steady two-dimensional flow of ideal plastic material:

$$(4)\quad \begin{cases} \sigma_{x^1} - k(\theta_{x^1}\cos 2\theta + \theta_{x^2}\sin 2\theta) = \varrho(u^1 u^1_{x^1} + u^2 u^1_{x^2}), \\ \sigma_{x^2} - k(\theta_{x^1}\sin 2\theta - \theta_{x^2}\cos 2\theta) = \varrho(u^1 u^2_{x^1} + u^2 u^2_{x^2}), \\ (u^1_{x^2} + u^2_{x^1})\sin 2\theta + (u^1_{x^1} - u^2_{x^2})\cos 2\theta = 0, \\ u^1_{x^1} + u^2_{x^2} = 0, \end{cases}$$

where $\varrho, k = \text{const}$, $u = (u^1, u^2)$ is the flow velocity; σ, θ are functions defining the stress tensor. The unknown functions depend on (x^1, x^2).

We assume that the solutions of (1) $u\colon G \to R^l$ are defined in a domain $G \subset R^n$ and $u \in C^1(G)$. The coefficients a^{si}_j have to be continuously differentiable.

We introduce two cones of characteristic vectors:

$$R^n \supset \Lambda(u) = \{\lambda = (\lambda_1, \ldots, \lambda_n)\colon \operatorname{rank}\|a^{si}_i(u)\lambda_i\| < l\},$$

$$R^l \supset \Gamma(u) = \{\gamma = (\gamma^1, \ldots, \gamma^l)\colon \operatorname{rank}\|a^{si}_j(u)\gamma^j\| < n\};$$

Λ — the characteristic cone of parametrization.
Γ — the characteristic cone of the sets of values.

For $\lambda \in \Lambda, \gamma \in \Gamma$ we write $\lambda \rightleftharpoons \gamma$ iff:

$$a^{si}_j \lambda_i \gamma^j = 0, \qquad s = 1, \ldots, m,$$

where the summation is performed over repeated indices. The adjointness $\lambda \rightleftharpoons \gamma$ means that γ is the right null vector of the characteristic vector λ.

We call the system (1) *nonelliptic* if $\Gamma, \Lambda \neq 0$.

For example, in the case of the system (3) we have:

$$R^3 \supset \Lambda(U)\colon (u, \lambda)^2[(u, \lambda)^2 - c^2|\lambda|^2] = 0,$$

where $U = (c, u^1, u^2, u^3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $(u, \lambda) = u^i\lambda_i$,

$$R^4 \supset \Gamma(U) = \Gamma^1(U) \cup \Gamma^2(U),$$

$$(5)\qquad \Gamma^1(U)\colon \begin{cases} (u, \bar\gamma)^2 - c^2|\bar\gamma|^2 = 0, \\ k\gamma^0 c + (u, \bar\gamma) = 0, \end{cases} \qquad \Gamma^2(U)\colon \gamma^0 = 0,$$

where $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, $\bar\gamma = (\gamma^1, \gamma^2, \gamma^3)$.

Moreover, for $\gamma \in \Gamma^1$ we have

$$\gamma \rightleftharpoons \lambda = \bar\gamma.$$

A system of independent functions

$$F_a(u), \qquad a = 1, \ldots, r < l$$

is called a *system of independent first integrals of the system* (1) *for the infinite class K of solutions* iff:

$$(u\colon G \to R^l,\ u \in K) \Rightarrow \big(F(u(x))_{\substack{x\in G\\ a=1,\ldots,r}} = 0\big).$$

Hence, denoting by H_{l-r} the following $(l-r)$-dimensional manifold:

$$R^l \supset H_{l-r}\colon F_a(u) = 0, \qquad a = 1, \ldots, r,$$

we may write: $u\colon G \to H_{l-r}$. That means, H_{l-r} represents the set of values of the infinite class of solutions considered.

In nonlinear mathematical physics (gas dynamics) one encounters very interesting and important first integrals of this form. The example is the classical Bernoulli manifold for the system (3) given by two first integrals:

$$R^4 \supset H_2(M)\colon \begin{cases} |u|^2 + kc^2 = M, \\ u^3 = 0. \end{cases} \qquad M = \text{const},$$

The corresponding class $K(H_2(M))$ of solutions of the system (3) $U(x^1, x^2, x^3) = (c, u^1, u^2, u^3)$ consists of functions satisfying:

$$U_{x^3} = 0, \qquad c = \left(\frac{M - |u|^2}{k}\right)^{1/2}$$

and

$$(6)\qquad \begin{cases} \left(\dfrac{M - |u|^2}{k}\delta^{ij} - u^i u^j\right) u^j_{x^i} = 0, \qquad i, j = 1, 2, \\ u^1_{x^2} - u^2_{x^1} = 0, \end{cases}$$

where δ^{ij} is the Kronecker symbol.

The system (6) is elliptic if $k^{-1}(M - |u|^2) > |u|^2$ and hyperbolic if $k^{-1}(M - |u|^2) < |u|^2$, and the class $K(H_2(M))$ is infinite. The class $K(H_2(M))$ was of great importance in the development of supersonic gas dynamics because it is broad enough to include solutions of interesting boundary value problems and it allows a quite simple qualitative analysis (see [1]). This is based on the following geometrical method of construction of the solution $U \in K(H_2(M))$.

It may be shown that the tangential space of H_2 is spanned by two characteristic vectors γ:

$$T_U(H_2(M)) = [\underset{1}{\gamma}(U), \underset{2}{\gamma}(U)], \qquad \underset{\mu}{\gamma} \in \Gamma^1(U), \qquad U \in H_2(M).$$

Hence $H_2(M)$ contains two families of curves $\underset{1}{h}, \underset{2}{h} \subset H_2(M)$, $\underset{i}{h}$ tangent to the field $\underset{i}{\gamma}(U)$.

Moreover, there exist two linear independent vectors $\overset{i}{\lambda}(U) \in \Lambda(U)$, $U \in H_2(M)$ such that

$$(\overset{i}{\lambda}_1, \overset{i}{\lambda}_2, 0) = \overset{i}{\lambda}(U) \Rightarrow \underset{i}{\gamma}(U).$$

Now, for each C^1-mapping $V\colon \{x^3 = 0\} \to H_2(M)$ we have in $\{x^3 = 0\}$ two families of curves $\overset{i}{C}$ whose normal spaces satisfy the following condition:

$$N_x(\overset{i}{C}) = [\overset{j}{\lambda}(V(x))], \quad i \neq j, i, j = 1, 2,$$

where $\overset{j}{\lambda} = (\overset{j}{\lambda}_1, \overset{j}{\lambda}_2)$. The required method of solution is formulated in the following theorem:

THEOREM 1. *Every solution* $U \in K(H_2(M))$ *may be constructed in the following two steps*:

1. *We find the* C^1*-mapping* $\underset{0}{U}\colon \{x^3 = 0\} \to H_2(M)$ *such that for each curve* $\overset{i}{C} \subset \{x^3 = 0\}$ *there exists a curve* $\overset{i}{h} \subset H_2(M)$ *satisfying*:

$$\underset{0}{U}\colon \overset{i}{C} \to \overset{i}{h},$$

2. *We extend* $\underset{0}{U}$ *on* R^3 *by putting*

$$U(x) = \underset{0}{U}(p), \quad x = s\sigma + p,$$

where $\sigma = (0, 0, 1)$, $-\infty < s < +\infty$.

The above theorem enables one to construct solutions of boundary value problems in the class $K(H_2(M))$ in the way as follows: first we obtain the set of values of the required solution and then we construct in a quite simple way the appropriate parametrization.

The importance of this theorem may be illustrated by the fact that the book [1] is based on Theorem 1 and an analogous one for the system (2) concerning the Bernoulli manifold $H_2 = \{u^2 = u^3 = 0\} \subset R^4$.

There arise the following questions. Do there exist other "Bernoulli manifolds" for the system (3)? Do there exist general methods allowing the construction of all "Bernoulli manifolds" for the system (1)?

The chief purpose of the present lectures is to give answers to these questions.

2. The Bernoulli manifolds

Consider the r-dimensional manifolds $H_r \subset R^l$ together with its tangent space of the form:

$$T_u(H_r) = [\underset{1}{\gamma}(u), \ldots, \underset{r}{\gamma}(u)],$$

where $\underset{i}{\gamma}(u)$ are C^1-functions and $\underset{i}{\gamma}(u) \in \Gamma(u)$. Moreover, assume that for each point $u \in H_r$ there exist r independent vectors $\overset{i}{\lambda}(u) \in \Lambda(u)$ such that

$$\overset{i}{\lambda}(u) \Rightarrow \underset{i}{\gamma}(u), \quad i = 1, \ldots, r,$$

and $\overset{i}{\lambda}(u)$ are C^1-functions. Hence H_r contains r families of curves $\underset{i}{h} \subset H_r$ tangent to the fields $\underset{i}{\gamma}(u)$.

We introduce the following notation:

$$[\lambda](u) = [\overset{1}{\lambda}(u), \ldots, \overset{r}{\lambda}(u)], \quad [\lambda/i](u) = [\overset{1}{\lambda}, \ldots, \overset{i-1}{\lambda}, \overset{i+1}{\lambda}, \ldots, \overset{r}{\lambda}].$$

Let $\Sigma_{n-r}(u)$ denote the $(n-r)$-dimensional linear space such that for $u \in H_r$:

$$R^n \supset \Sigma_{n-r}(u) \perp [\lambda](u).$$

There exists an r-dimensional manifold $L_r \subset R^n$ such that for $x \in L_r$, $u \in H_r$:(1)

$$\dim[\Sigma_{n-r}(u), T_x(L_r)] = n,$$

$$T_x(L_r) = [\underset{1}{c}(x, u), \ldots, \underset{r}{c}(x, u)] \quad \text{where} \quad \underset{i}{c}(x, u) \perp [\lambda/i](u).$$

If $M \subset R^n$ is a linear space, then by $M^p, p \in R^n$, we shall denote the plane such that $p \in M_p$ and $T(M^p) = M$.

In order to obtain a general formulation of Theorem 1 for the system (1) let us observe that each C^1-mapping $\nu\colon L_r \to H_r$ defines on L_r the C^1-vector fields $\underset{i}{c}(x, \nu(x)) \neq 0$, $x \in L_r$ $(i = 1, \ldots, r)$. Hence L_r contains r families of curves $\underset{i}{C} \subset L_r$ (L_r plays the role of the manifold $\{x^3 = 0\}$ in Theorem 1). Consider now for some neighbourhood $D(L_r) \subset R^n$ of L_r the C^1-mapping

$$u\colon D(L_r) \to H_r$$

constructed in the following two steps:

1. We find a mapping $\underset{0}{u}\colon L_r \to H_r$ such that for each curve $\overset{i}{C} \subset L_r$ corresponding to $\underset{0}{u}$ there exists a curve $\underset{i}{h} \subset H_r$ such that

$$\underset{0}{u}\colon \overset{i}{C} \to \underset{i}{h}.$$

2. We extend the mapping $\underset{0}{u}$ to the neighbourhood $D(L_r)$ by putting:

$$u(x) \overset{\text{df}}{=} \underset{0}{u}(p) \quad \text{for} \quad x \in \Sigma^p_{n-r}(\underset{0}{u}(p)).$$

(1) The existence is understood in the local sense. That is to say, L_r may be small and it may exist only for small enough pieces of H_r.

Our question is, when does the mapping $u(x)$ represent a solution of (1)? To answer it we introduce in R^n the following $(n-r+1)$-dimensional manifolds:

$$\overset{i}{C}_{n-r+1} = u^{-1}(\underset{i}{h}) \quad \text{or} \quad \overset{i}{C}_{n-r+1} = \bigcup_{p\in \overset{i}{C}} \Sigma_{n-r}^{p}\big(\underset{0}{u}(p)\big).$$

We shall say that the mapping u has the *Δ-property* if the manifolds $\overset{i}{C}_{n-r+1}$ are developable, i.e.

$$N_x(\overset{i}{C}_{n-r+1}) = \text{const} \quad \text{for} \quad x\in \Sigma_{n-r}^{p}\big(\underset{0}{u}(p)\big), \quad i = 1, \ldots, r.$$

The following generalization of Theorem 1 is valid.

THEOREM 2. *The mapping $u\colon D(L_r)\to H_r$ is a solution of the system* (1) *if u has the Δ-property.*

THEOREM 3. *If for manifold H_r the condition*

$$\text{(7)} \qquad \frac{\partial \overset{i}{\lambda}(u)}{\partial \underset{j}{\gamma}(u)} = \overset{i}{\lambda}_{u_j^a}\gamma^a \in [\lambda](u)$$

is fulfiled for $u\in H_r$, $i\neq j$, $i,j = 1, \ldots, r$, then every mapping $u\colon D(L_r)\to H_r$ obtained by the procedure described in steps 1, 2 *has the Δ-property.*

We shall call H_r a *Bernoulli manifold* if (7) is fulfiled. The manifolds $H_2(M)$ are obviously Bernoulli manifolds. The class of solutions of (1) obtained for a Bernoulli manifold H_r by proceeding as described in steps 1, 2 will be denoted by $K(H_r)$.

Bernoulli manifolds have the so-called conic property, which in some cases enables one to find a simple parametrization of subsets of H_r, thus leading to the solution.

Let $y\in R^n$ be a fixed point and $D\subset R^n$ be a region having the property that through each point of D there passes only one plane $\Sigma_{n-r}^{y}(u)$, $u\in H_r$. Then for the conical mapping

$$u_{\text{con}}(x) \overset{\text{df}}{=} u \quad \text{for} \quad x\in \Sigma_{n-r}^{y}(u),\ u\in H_r,$$
$$u_{\text{con}}\colon D\to H_r,$$

the following theorem holds.

THEOREM 4. *If there exists at least one solution $u\in K(H_r)$ such that the mapping $u\colon D\to H_r$ is one-to-one, then u_{con} is a solution and $u_{\text{con}}\in K(H_r)$.*

3. Existence of Bernoulli manifolds

(a) *One-dimensional Bernoulli manifolds H_1.* H_1-manifolds are curves in R^l tangent to vectors $\gamma\in\Gamma(u)$. We have for $u\in H_1$

$$\Gamma(u)\ni\gamma(u) \rightleftharpoons \lambda(u)\in\Lambda(u), \qquad R^n\supset \Sigma_{n-1}(u)\perp\lambda(u).$$

The condition (7) is fulfiled, as we have only one $\lambda(u)$. $L_1\subset R^n$ is an arbitrary curve satisfying the condition

$$\dim[\Sigma_{n-1}(u), T_x(L_1)] = n.$$

The construction of the solutions $u\in K(H_1)$ may be performed in the following two steps:

1. We take an arbitrary mapping $u_0\colon L_1\to H_1$.
2. We extend u_0 to the neighbourhood of L_1 as follows:

$$u(x) \overset{\text{df}}{=} \underset{0}{u}(p) \quad \text{for } x\in\Sigma_{n-1}^{p}\big(\underset{0}{u}(p)\big).$$

The assumption of Theorem 4 is fulfiled and $u_{\text{con}}\in K(H_1)$.

(b) *$H_2\subset R^l$ for two independent variables* $(n = 2)$:

$$\text{(8)} \qquad T_u(H_2) = [\underset{1}{\gamma}(u), \underset{2}{\gamma}(u)], \quad \underset{i}{\gamma}\in\Gamma(u),$$

$$\Sigma_{n-r}(u) = \Sigma_0 \text{ (points)}, \quad L_2 = R^2, \quad \overset{i}{C}_{n-r+1} = \overset{i}{C}_1 = \overset{i}{C}.$$

Hence the problem of existence of H_2 for $n = 2$ reduces to the problem of existence of manifolds satisfying (8). The construction of $u\in K(H_2)$ reduces to the step 2, which is equivalent to the solution of a hyperbolic system of two equations with two dependent and two independent variables.

EXAMPLE 1. For the system

$$\text{(9)} \qquad \begin{aligned} \sigma_{x^1} - k(\theta_{x^1}\cos 2\theta + \theta_{x^2}\sin 2\theta) &= 0,\\ \sigma_{x^2} - k(\theta_{x^1}\sin 2\theta - \theta_{x^2}\cos 2\theta) &= 0,\\ (u^1_{x^2} + u^2_{x^1})\sin 2\theta + (u^1_{x^1} - u^2_{x^2})\cos 2\theta &= 0,\\ u^1_{x^1} + u^2_{x^2} &= 0, \end{aligned}$$

describing the steady two-dimensional slow flows of ideal plastic material (see (4)), putting $U = (u^1, u^2, \sigma, \theta)$ we have:

$$(\cos\theta, \sin\theta) = \overset{1}{\lambda}(U) \rightleftharpoons \underset{1}{\gamma}(U, a^1, a^2) = (-a^1\tan\theta, a^1, a^2k, a^2)$$

where $-\infty < a^1, a^2 < +\infty$,

$$(-\sin\theta, \cos\theta) = \overset{2}{\lambda}(U) \rightleftharpoons \underset{2}{\gamma}(U, \beta^1, \beta^2) = (\beta^1\cot\theta, \beta^1, -\beta^2k, \beta^2)$$

where $-\infty < \beta^1, \beta^2 < +\infty$.

The construction of the manifold

$$H_2\colon\ U = U(\tau^1, \tau^2), \quad U_{\tau^i} = \underset{i}{\gamma}$$

leads to the linear hyperbolic system of equations

$$u^1_{\tau^1} = -u^2_{\tau^1}\tan\theta, \qquad \sigma_{\tau^1} = k\theta_{\tau^1},$$

$$u^1_{\tau^2} = u^2_{\tau^2}\cot\theta, \qquad \sigma_{\tau^2} = -k\theta_{\tau^2}.$$

Each solution may be obtained by Theorem 2.

In the case of fast plastic flows described by (4) there do not exist H_2 manifolds for which $u^1(\tau^1, \tau^2) \neq \text{const}$, $u^2(\tau^1, \tau^2) \neq \text{const}$.

EXAMPLE 2. For the system describing the inviscid isentropic compresible, steady plane flow:

$$u^\delta c_{x^\delta} + \frac{c}{k}\operatorname{div} u = 0,$$

$$u^\delta u^s_{x^\delta} + kc\cdot c_{x^s} = 0,$$

where $\delta = 1, 2$, $s = 1, 2$, the solutions $U = (c, u^1, u^2)$ are mappings $U\colon R^2 \to R^3$; only the classical Bernoulli manifolds do exist:

$$R^3 \supset H_2\colon |u|^2 + kc^2 = M.$$

(c) *H_2-manifolds for the system* (3). There exist an infinite set of Bernoulli manifolds H_2 for the system (3) such that

$$(10)\qquad H_2 \neq H_2(M)\colon \begin{cases} |u|^2 + kc^2 = M, \\ u^3 = 0. \end{cases}$$

The construction of H_2: $U = U(\tau^1, \tau^2)$ such that

$$U_{\tau^i} \in \Gamma^1\big(U(\tau)\big) \qquad (i = 1, 2)$$

leads to a hyperbolic system of the form

$$U_{\tau^1\tau^2} = f(U, U_{\tau^1}, U_{\tau^2}),$$

which gives the existence of manifolds satisfying (10). For those H_2-manifolds we have $u_{\text{con}} \in K(H_2)$.

4. The problem of a nozzle changing one uniform flow into an other without shocks

In this part we give an application of Theorems 2 and 3 to the solution of a certain boundary value problem for the system (3).

A *nozzle* is a surface $W \subset R^3$ (wall of the nozzle) cutting R^3 into two unbounded sets, one of which is of the form

$$M = N_1 \cup N \cup N_2$$

where N is a bounded domain and N_1, N_2 are unbounded cylinders. We seek such nozzles for which there exist solutions of (3) $U(x) = \big(c(x), u(x)\big) \in C^1_s(M)$ satisfying the following conditions:

$$(u, n) = 0, \qquad x \in W,$$

$$u(x) = \underset{i}{u} = \text{const}, \qquad x \in N_i,$$

where n is the normal vector to the surface W. $C^1_s(M)$ denotes the class of C^1-functions with weak discontinuities.

This is a global nonlinear problem important in practice. Because of the nonellipticity of (3), for an arbitrary given nozzle there does not exist, in general, a solution U satisfying the required conditions (shocks).

Using the cone property of the H_2-manifolds for the system (3), we obtain a great class of nozzles and corresponding solutions of our boundary value problem.

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Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978