Step 3. "Translation" of the singular values of A to those of B by a diffeomorphism a_3.

Step 4. Construction of the final homeomorphism.

Indeed, after Step 3 we find

$$a_3 A a_1 a_2 = \big(a(t, \omega), \omega\big). \tag{13}$$

Using Step 4 of Part I, we represent the right-hand side of (13) as the composition $B\varrho$, where ϱ is a diffeomorphism $H \to H$. Thus

$$a_3 A a_1 a_2 = B\varrho, \tag{14}$$

which is the desired equation.

4. Stability of the methods introduced under a perturbation of A

Under a C^1-perturbation of A in the sense of the metric in H, our analytical results of Step 1 carry over to study perturbation problems.

Indeed, we prove

THEOREM. *Under a suitably restricted C^1-perturbation of A, the number of solutions of the perturbed problem is exactly the same as in* (2), *away from a neighbourhood of the singular values.*

Moreover, in this case, the solutions of (2) *are accurate approximations to the perturbed problem.*

The proof is based on a careful analysis of the steps in Part I above.

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INDEX THEORY AND ELLIPTIC BOUNDARY VALUE PROBLEMS REMARKS AND OPEN PROBLEMS

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1. Index theory for elliptic differential equations

Let M be an oriented Riemannian manifold of dimension n, and SM the *covariant sphere bundle* of differential forms of "length" 1. A linear differ-

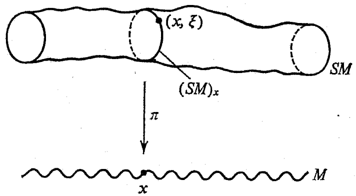

Fig. 1

ential operator of order m, operating between smooth sections of C^∞-vector bundles E and F over M of fibre dimension k can be written in local coordinates in the form

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha,$$

where $x \in M$, $D^\alpha = D^{\alpha_1 \dots \alpha_n} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, $|\alpha| := \alpha_1 + \dots + \alpha_n$ and a_α a matrix valued C^∞-function. A is called *elliptic* if

$$\sigma(A)(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x)\, \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \in \mathrm{GL}(k, C)$$

for all $(x, \xi) \in SM$.

If M is closed, it is well known that

$$\ker A := \{u \in C^\infty(M, E);\, Au = 0\}$$

and

$$\operatorname{coker} A := C^\infty(M, F)/A\big(C^\infty(M, E)\big)$$

have finite dimensions and that their difference, the *index* $\operatorname{ind} A$ of A is stable under small perturbations of the coefficients of A. The same holds for compact manifolds with boundary, if additional *elliptic boundary conditions* are posed (cf. § 3).

Bounded operators on topological vector spaces say complex separable Hilbert spaces with finite dimensional kernel and cokernel are called *Fredholm operators*. Their index belongs to a series of integer valued topological invariants which have been largely examined in the context of geometry, complex analysis and "analysis situs", as

— the *Euler characteristic* of stereometry, i.e. the alternating sum

$$\chi(M) := \sum_j (-1)^j v_j$$

where v_j is the number of j-simplices for a given triangulation of a compact topological space M;

— the *Betti numbers* $\beta_q(M) := \dim H_q(M, C)$ in homology, counting in some sense the numbers of "holes of different dimensions" in a topological space with the relation

$$\chi(M) = \sum (-1)^q \beta_q(M);$$

— the *genus* $g(M)$ of a Riemann surface M in the geometry of complex algebraic curves, which can be interpreted as the number of "handles" attached to a sphere S^2 in order to obtain the surface M and which satisfies the relation

$$\chi(M) = 2 - 2g(M);$$

— the *winding number* $\deg(f)$ of a continuous map $f\colon S^1 \to C\setminus\{0\}$ counting the number of times the path "goes round" the origin;

— the *local index* $I_v(x)$ of a vector field $v\colon M \to TM$ in the theory of ordinary differential equations, expressing the local mapping degree of v in points $x \in M$ where $v(x) = 0$, thus giving a characterization of different types of singularities of dynamical systems;

— the *number of fixed points* $L(f)$ of a continuous map $f\colon M \to M$ of a topological space M in itself.

In partial differential equations, it was F. Noether who found in 1921 that a certain singular integral equation, arising from the classical

Riemann–Hilbert transmission problem (cf. § 7), has non-vanishing index. Since the early 50ies when Hellwig discovered that some differential operators of mathematical physics have non-vanishing index and Vekua was able to express the index of the regular oblique boundary value problem in the plane by purely topological means, there has been continued interest in the index of elliptic differential problems.

The key result in that field is the formula obtained by Atiyah and Singer [10], [17], which takes the following form in the "system case", when the above mentioned bundles E and F are trivial:

$$\operatorname{ind} A = (-1)^n \int_{SM} \sigma(A)^* \omega \wedge \pi^* \tau.$$

Here, ω is a differential form on the Lie group $GL(k, C)$, which is canonically defined like a "world constant"; ω has to be lifted to a differential form on SM along the map

$$\sigma(A)\colon SM \to GL(k, C).$$

The "Todd form" τ is a differential form on M defined in terms of the curvature of the Riemannian manifold M which has to be lifted to a differential form on SM, too, along the natural projection $\pi\colon SM \to M$. One integrates over SM the $(2n-1)$-dimensional component of their alternating product.

More refined formulas were obtained in the bundle case. As to elliptic boundary value problems over the compact manifold X with boundary Y, similar formulas hold when extending the domain of integration from SX to the closed manifold $SX \cup BX|Y$ and lifting ω back to that manifold by a continuation of the symbol defined explicitly by the boundary conditions (cf. § 3).

In fact, the above mentioned, and roughly speaking *all* integer valued topological invariants can be interpreted as indices of the Cauchy–Riemann operator $\partial/\partial\bar{z}$, the Laplace operator Δ or of other standard elliptic differential operators associated with the underlying complex or Riemannian structure of the spaces involved. Moreover, the famous classical theorems on these invariants can be obtained as special cases of the index theorem. E.g. the index theory reproves and partly generalizes

— the *Gauss–Bonnet theorem* of differential geometry

$$(2\pi)^{-1}\Big(\int_X K + \int_Y s\Big) = \chi(X),$$

where X is a compact Riemannian surface with boundary Y, K the total Gauss curvature of X, and s the geodesic curvature of Y in X;

— the *residues theorem* of complex analysis which in simple situations takes the form $(2\pi i)^{-1}\int \hat{f}^{-1} d\hat{f} = \deg f = N(\hat{f}) - P(\hat{f})$, if $\hat{f}\colon S^1 \to C\setminus\{0\}$

is a differentiable approximation for the path f, and if $\hat{f}$ is a finite Laurent series approximating f, and $N(\hat{f})$ is the number of zeros and $P(\hat{f})$ the number of poles of $\hat{f}$ in the interior of the unit circle, counted with their respective multiplicities;

— the *Riemann–Roch–Hirzebruch theorem* of algebraic geometry which in its most elementary form provides the relation

$$l(\vartheta)-l'(\vartheta) = |\vartheta|-g(M)+1,$$

where ϑ is a "divisor" of degree $|\vartheta|$ on a Riemann surface M, i.e. a finite set of points on M with positive or negative multiplicities adding up to a total of $|\vartheta|$. $g(M)$ is the genus of M, and $l(\vartheta)$ is the dimension of the linear space of meromorphic functions on M with zeros and poles as prescribed by ϑ, $l'(\vartheta)$ being defined in a similar way;

— the *Poincaré–Hopf formula*

$$\sum_{x\in M} I_v(x) = \chi(M)$$

for the number of singularities of a vector field v on a manifold M weighted by the "local indices";

— the *Lefschetz fixed point formula*

$$L(f) = \sum(-1)^q \operatorname{trace} H^q f$$

for a continuous map $f\colon M\to M$, if

$$H^q f\colon H^q(M, \boldsymbol{C})\to H^q(M, \boldsymbol{C})$$

are the induced homomorphisms in the cohomology;

— *de Rham's theorem*

$$\dim H^q\big(\Omega(M)\big) = \beta_q(M)$$

in the "Hodge theory" of Riemannian manifolds, which gives an isomorphy between the space of "harmonic" q-forms on M which can be identified with the qth "cohomology"

$$H^q\big(\Omega(M)\big) := \ker d_q/\operatorname{im} d_{q-1}$$

of the complex

$$\Omega(M)\colon 0\to\Omega^0(M)\xrightarrow{d_0}\Omega^1(M)\xrightarrow{d_1}\dots\to\Omega^n(M)\to 0$$

of exterior differential forms, and the qth singular cohomology of the underlying topological space;

— the *Noether–Muskhelišvili index formula* for singular integral operators,

e.g. of convolution type over the half axis

$$W_f u(t) := \int_0^\infty f(t-\tau)u(\tau)\,d\tau, \quad u\in L^2(\boldsymbol{R}_+),\ t\geqslant 0,$$

with $f\in L^1(\boldsymbol{R})$ for which one has

$$\operatorname{ind}(\mathrm{Id}+W_f) = \deg(\hat{f}+1),$$

where $\hat{f}$ is the Fourier transform of f;

— the *Vekua formula* for the Laplace equation on the disc with boundary conditions given by a non-vanishing vector field $\partial/\partial v$ with winding number p:

$$\operatorname{ind}\begin{pmatrix}\Delta\\ \partial/\partial v\end{pmatrix} = 2(1-p).$$

Moreover, index theory has been applied to a wide range of problems reaching from number-theoretical problems [37] to the determination of the number of parameters entering the general self-dual Euclidean Yang–Mills configuration with a given topological charge in non-commutative gauge theory of quantum field dynamics [10].

Many of these applications yield examples for "using the Atiyah–Singer theorem, one of the deepest and hardest results in mathematics, to prove a series of perfectly elementary identities, which can be proved much more easily by direct means. This may seem to be a rather pointless course requiring justification", as noted by Hirzebruch and Zagier [37].

They explain why many aspects of index theory are still presenting an "enigma to the members of the mathematical community, for their puzzlement or entertainement as the case may be" and merit methodological attention as follow:

"Of course, one can defend it simply by saying that both the number-theoretical and the topological ideas involved are important and far-reaching ... The Atiyah–Singer index theorem probably has wider ramifications in topology and analysis than any other single result — and therefore any relationship between them, however nebulous, cannot fail to be of interest. Nevertheless, it would be nice, and would possibly have important consequences, if one could understand the real reasons for the relationship."

2. Comparison of the various proofs

Opposite the widely diversified and hardly comparable approaches to the different above mentioned special and earlier index formulas, we can distinguish two major directions in proving the general index theorem, namely roughly speaking, whether K-theory is essentially used or not.

K-theory is a cohomology theory especially shaped for the investigation of homotopy invariant aspects of manifolds, i.e. locally linear spaces, and is made up of stable equivalence classes of vector bundles instead of simplices in "orthodox" singular homology and cohomology. For a short account see [32]. The main theorem of K-theory is the Bott periodicity theorem

$$K(M) \cong K(M \times \boldsymbol{R}^2), \qquad M \text{ locally compact},$$

which can be interpreted as a parametrization of the Noether–Muskhelišvili index formula, identifying the space $K(\boldsymbol{R}^2)$ of "clutching functions"

$$f\colon S^1 \to \mathrm{GL}(k, \boldsymbol{C}), \qquad k \text{ large},$$

for bundles over S^2 with the space $K(\text{point}) \cong \boldsymbol{Z}$ of winding numbers.

There exists a global version of the Bott theorem for compact complex manifolds, namely that the tensor product for vector bundles which induces a ring structure on $K(M)$, via the projection, makes $K(TM)$ a free module over $K(M)$ with generator $[\sigma(\bar\partial)]$, where $\bar\partial$ is a certain invariantly defined "classical" differential operator on M, the Riemann–Roch operator. Here TM is the cotangent bundle. The isomorphism

$$K(M) \to K(TM),$$
$$[E] \mapsto [E]\cdot[\sigma(\bar\partial)]$$

can analytically be realized by constructing to every E a generalized Riemann–Roch operator $\bar\partial_E$ with "coefficients" in E. Modulo torsion, the same isomorphism holds for all compact oriented Riemannian manifolds of even dimension, if one substitutes the Hirzebruch signature operator D^+ for $\bar\partial$.

The three geometry-guided proofs elaborated by Atiyah, Bott, Patodi, and Singer are closely related to one of these Bott theorems respectively: The standard Bott periodicity states that from the point of index theory essentially there exists only one elliptic operator on a sphere, let us say of Wiener–Hopf type or equivalently (see § 7) of standard Riemann–Hilbert transmission type or whatever, and that all other elliptic operators on the sphere can be "deformed" into these standard operators, if not on the symbol level, so on the K-level. This yields the index formula in the Euclidean case for operators being the identity at ∞. The general formula for an arbitrary manifold M follows by embedding M in Euclidean space and applying again the Bott theorem (in the Thom form)

$$K(TM) \cong K(TN),$$

where N is an open tubular neighborhood for the M embedded and TN is considered as a complex vector bundle over TM.

That proof (from 1968) was shaped after Grothendieck's proof for the Riemann–Roch–Hirzebruch theorem, and is conceptionally the most elementary proof since it is based on its very heart on the notion of Wiener–Hopf operators only — and on the concept of pseudo-differential operators permitting the necessary deformations. However, the conceptional simplicity has to be paid by the rigidity of the embedding procedure which generally will destroy possibly present symmetries or other nice features of the original operator, if e.g. the embeddings are not complex algebraic.

The original proof (from 1963) is based on the global Bott theorem for complex manifolds which shows, roughly speaking, that the Riemann–Roch–Hirzebruch theorem is not just a special case of the index theorem but a typical one, containing the general case. Since every even-dimensional manifold is "cobordant" to formal sums of spheres and complex projective spaces, one has in a certain sense only to show the Riemann–Roch theorem in those standard cases, and then to combine the results. The performance of the calculations claims an arithmetic virtuosity, demonstrated earlier by Hirzebruch, which is not always sufficiently transparent, however.

The "heat equation proof" (1973) is conceptionally and technically the most advanced proof of the index theorem involving some ideas on parabolic equations and a great deal more differential geometry. It is based on the global Bott theorem in its generalized version, executing all the calculations with classical operators directly on the manifold, with every operation bearing its concrete geometric meaning, and expressible in curvature terms or in characteristic classes.

These three K-theory based proofs are compared in greater detail in [17]. They were supplemented by the work of Fedosov [30], [31], and recently by Hörmander [44] and also Rempel [57], who were able to obtain the index formula in the Euclidean case by straightforward calculations for operators with arbitrary coefficients. The calculations are in spirit closely related to the performance of cancellations of higher derivatives when having regard to the very concrete form of the operators involved. Thus they have removed the K-theory from the scene totally, since Hörmander had earlier in [43] shown a way how to procure the embedding procedure, investing more analysis instead of using K-theory. The use of K-theory in index theory is, however, as natural as is the concept of vector bundles and homotopy classes of matrix-valued functions for analysis at large (cf. the historical note [35] by Grauert and Schneider). Therefore the challenge of the Fedosov–Hörmander approach is found less in the puristic concept of avoiding K-theory than in the explicitness of the calculations which shed new light on the interaction of analysis and topology present in index theory. (Note that on the other hand, Atiyah has in [6] given an idea of proving the index theorem, by the use of much more refined topological means, avoiding pseudo-differential operators altogether. These deformation methods for polynomial symbols and related vector field problems of "hard" topology were elaborated by Lusztig but unfortunately not published.)

3. Elliptic operators of the Boutet de Monvel class

The above remarks about the index theory show that the corresponding methods have contributed not only results about topological questions on elliptic operators but also new aspects about the analytical properties. This concerns also elliptic boundary value problems, and the interest is not only confined to a proof of an analogue of the index theorem. In connection with the usual theory of elliptic boundary problems in sense of [1], [46] and the papers [18], [28], many open questions arise (a part of them of more technical nature). In order to discuss such problems, we recall some notations and definitions about a class of operators describing boundary value problems.

Let X be a compact C^∞-manifold with smooth boundary Y ($n = \dim X$). If E is a complex vector bundle over X, we denote by $C^\infty(X, E)$ the corresponding space of smooth sections. Analogous notations are used with respect to Y. By $\mathfrak{B}$ we denote the class of operators of the form

$$(1)\qquad \mathscr{A} = \begin{pmatrix} r^+A + r'B & K \\ r'T & Q \end{pmatrix} : \begin{matrix} C^\infty(X, E) \\ \oplus \\ C^\infty(Y, J) \end{matrix} \to \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, G) \end{matrix}$$

introduced by Boutet de Monvel [18], [19]. Here E, F and J, G are complex vector bundles over X and Y respectively. r^+A is a pseudo-differential operator over X satisfying the transmission property, $r'B$ a Green operator, $r'T$ a trace operator, K a potential operator and Q a pseudo-differential operator on Y. As usual the operators are described by homogeneous principal symbols $\sigma(A)(x, \xi)$, $\sigma(B)(x', \xi', \nu, \tau)$, $\sigma(T)(x', \xi', \nu)$, $\sigma(K)(x', \xi', \nu)$, $\sigma(Q)(x', \xi')$, and the order of $\sigma(A)$ has to be an integer a. The notation (ξ', ν) means a decomposition of a covector near Y into a tangent and a normal component ξ' and ν respectively (a Riemannian metric on X is fixed), and τ is an additional variable in $\boldsymbol{R}$.

If F_n denotes the Fourier transform on the x_n-axis $\boldsymbol{R}$ ($x = (x', x_n)$ near Y in local coordinates), we put $H^+ := F_n(\mathscr{S}(\overline{\boldsymbol{R}}_+))$, $H_0^- := F_n(\mathscr{S}(\overline{\boldsymbol{R}}_-))$. Here $\mathscr{S}(\overline{\boldsymbol{R}}_\pm)$ denotes the space of Schwartz functions on the half axis $\overline{\boldsymbol{R}}_\pm$, infinitely differentiable at the origin. Let H' be the space of polynomials in ν, $H^- := H_0^- \oplus H'$, $H := H^+ \oplus H^-$. By $\Pi^+: H \to H^+$ we denote the projection along H^- and by $\Pi': H \to \boldsymbol{C}$ the map with $\Pi'h := 0$ for $h \in H^-$ and $\Pi'h := \lim_{x_n \to 0} \varphi(x_n)$ for $h := F_n\varphi$, $\varphi \in \mathscr{S}(\overline{\boldsymbol{R}}_+)$.

If the given symbols $\sigma(A)$, $\sigma(B)$, $\sigma(T)$, $\sigma(K)$ belong to an operator in $\mathfrak{B}$, they are over Y elements of H_ν ($H_\nu^+ \oplus H_\tau^-$, H_ν^-, H_ν^+) for $\xi' \neq 0$ with respect to ν or (ν, τ). Let SY be the cosphere bundle over Y induced by the Riemannian metric and $p: SY \to Y$ the canonical projection. Then the bundle morphism

$$(2)\qquad \sigma_Y(\mathscr{A}) = \begin{pmatrix} \Pi^+\sigma(A) + \Pi'\sigma(B) & \sigma(K) \\ \Pi'\sigma(T) & \sigma(Q) \end{pmatrix} : \begin{matrix} p^*(H^+ \otimes E') \\ \oplus \\ p^*J \end{matrix} \to \begin{matrix} p^*(H^+ \otimes F') \\ \oplus \\ p^*G \end{matrix}$$

($E' := E|Y$, $F' := F|Y$) is called the *boundary symbol* of $\mathscr{A}$; for details see [18], [62], [58], [54], [59]. In connection with this terminology we call $\sigma(A)$ also *interior symbol* of $\mathscr{A}$ ($\sigma(A) =: \sigma_\Omega(\mathscr{A})$, $\Omega := X \setminus Y$).

The *composition* (if defined) of operators in $\mathfrak{B}$ belongs to $\mathfrak{B}$. The composition corresponds to compositions of interior and boundary symbols.

If

$$(3)\qquad a = \operatorname{ord}\sigma(A) \in \boldsymbol{Z}, \quad \gamma := \operatorname{ord}\sigma(T), \quad \lambda := \operatorname{ord}\sigma(K)$$

are the orders of homogeneity, we assume $a-1 = \operatorname{ord}\sigma(B)$, i.e.

$$\sigma(B)(x', t\xi', t\nu, t\tau) = t^{a-1}\sigma(B)(x', \xi', \nu, \tau), \quad t > 0.$$

Then $\mathscr{A}$ has a *continuous extension* as operator between Sobolev spaces

$$(4)\qquad \mathscr{A}: H^s(X, E) \oplus H^{t+\lambda+1/2}(Y, J) \to H^t(X, F) \oplus H^{s-\gamma-1/2}(Y, G)$$

($t := s - a$, s sufficiently large). In the definition of the scalar products in the Sobolev spaces we have fixed Hermitian metrics in the corresponding bundles.

An operator $\mathscr{A} \in \mathfrak{B}$ is called *elliptic* (or an elliptic boundary value problem) if both $\sigma_\Omega(A): \pi^*E \to \pi^*F$ ($\pi: TX\setminus 0 \to X$) and $\sigma_Y(\mathscr{A})$ are isomorphisms. The standard properties of elliptic operators in $\mathfrak{B}$ (elliptic regularity in Sobolev spaces, Fredholm property, index theorem) have been investigated in [18]. The index theorem for elliptic boundary value problems for differential operators was proved in [8]. The Fredholm property and elliptic regularity of more general operators of a matrix form similar to (1) (without the transmission property) was studied in [14], [73], [74], [75], [78] (cf. § 5).

Suppose that an elliptic symbol $\sigma(A): \pi^*E \to \pi^*F$ is given satisfying the transmission property in the above sense. Then

$$(5)\qquad \Pi^+\sigma(A): p^*(H^+ \otimes E') \to p^*(H^+ \otimes F')$$

is a family of Fredholm operators over SY. Thus we get an index element

$$(6)\qquad \operatorname{ind}\Pi^+\sigma(A) \in K(SY).$$

The condition

(7) $\operatorname{ind} \Pi^+\sigma(A) \in p^*K(Y) \quad (p\colon SY \to Y)$

is necessary and sufficient for the existence of an elliptic operator $\mathscr{A} \in \mathfrak{B}$ with $\sigma_\Omega(\mathscr{A}) = \sigma(A)$. For $\mathscr{A}$ in the form (1) and $\sigma(A) = \sigma_\Omega(A)$ can be easily proved

(8) $\operatorname{ind} \Pi^+\sigma(A) = [p^*G]-[p^*J].$

If V is a vector bundle over a space, we denote by 1_V the *identical bundle morphism*. In the case of a trivial bundle of fibre dimension N we write also 1_N. The set of symbols $\sigma\colon \pi^*E \to \pi^*F$ ($\pi\colon TX\setminus 0 \to X$) on X with the transmission property with respect to Y which are homogeneous of order m shall be denoted by $\mathfrak{A}^{(m)}$. By $\mathfrak{E}^{(m)}$ we denote the set of all elliptic symbols in $\mathfrak{A}^{(m)}$. By $\mathfrak{E}_1^{(0)}$ shall be denoted the set of all elliptic symbols σ in $\mathfrak{A}^{(0)}$ for which there exists a neighbourhood U of Y in X (depending on the symbol), so that σ is independent of the covector ξ over U (this means $\sigma|U$ is an isomorphism of bundles over TU including the zero section). For $\sigma \in \mathfrak{E}_1^{(0)}$ we have obviously $\operatorname{ind}\Pi^+\sigma = 0$. In [18] is proved the following

PROPOSITION 1. *Let $\sigma \in \mathfrak{E}^{(0)}$ and $\operatorname{ind}\Pi^+\sigma = 0$. Then there exists an integer $N \geqslant 0$ and a homotopy $\sigma\oplus 1_N \simeq \sigma_1$ in $\mathfrak{E}^{(0)}$ so that σ_1 belongs to $\mathfrak{E}_1^{(0)}$.*

This is a consequence of the next theorem. Let χ be a real non-negative C^∞-function on $\boldsymbol{R}_+ := \{t \geqslant 0\}$ with

$$\chi(t) := \begin{cases} 0 & \text{for } t < \varepsilon, \\ 1 & \text{for } t > 2\varepsilon \end{cases}$$

($\varepsilon > 0$ is fixed and sufficiently small). Put

$$\delta(\xi) := \chi(|\xi'|/|\xi|)|\xi'|.$$

Then the symbols $\delta(\xi)\pm iv$ (defined near Y) are elliptic and have the transmission property.

The symbol $z := (\delta(\xi)-iv)(\delta(\xi)+iv)^{-1}$ is also defined near Y and elliptic. Any vector bundle over Y can be lifted to some bundle over a tubular neighbourhood U ($\simeq Y\times[0,1)$) of Y. We use the same letters for objects over U defined as pull backs with respect to a projection $U \to Y$. Thus we have over U elliptic symbols $z^{-1}\cdot 1_{p*G}$, $z\cdot 1_{p*J}$ ($J, G \in \operatorname{Vect}(Y)$). One can prove (cf. [18])

$$\operatorname{ind}\Pi^+(z^{-1}\cdot 1_{p*G}) = [p^*G], \qquad \operatorname{ind}\Pi^+(z\cdot 1_{p*J}) = -[p^*J].$$

THEOREM 2. *Let $\sigma \in \mathfrak{E}^{(0)}$. Then the following properties are equivalent:*

(i) *there exist bundles $J, G \in \operatorname{Vect}(Y)$ with*

(9) $\operatorname{ind}\Pi^+\sigma = [p^*G]-[p^*J];$

(ii) *there exist bundles J_0, G_0, $L \in \operatorname{Vect}(Y)$ and in a neighbourhood U of Y a homotopy through symbols in $\mathfrak{E}^{(0)}$*

(10) $$\begin{pmatrix} \sigma & 0 \\ 0 & 1_{p*L} \end{pmatrix} \simeq a \begin{pmatrix} z^{-1}\cdot 1_{p*G_0} & 0 \\ 0 & z\cdot 1_{p*J_0} \end{pmatrix}$$

with $[p^*G]-[p^*J] = [p^*G_0]-[p^*J_0]$ and a bundle isomorphism $a\colon E'\oplus L \to F'\oplus L$.

(For a proof of Theorem 2 cf. [28], [63].)

COROLLARY 3. *Let $\sigma \in \mathfrak{E}^{(0)}$. Then condition* (7) *is equivalent to the existence of some integer $N \geqslant 0$, so that there exists near Y a homotopy $\sigma\oplus 1_N \simeq \sigma_1$ through elliptic symbols* (not necessarily in $\mathfrak{E}^{(0)}$), *so that σ_1 belongs to $\mathfrak{E}_1^{(0)}$.*

To any $\sigma \in \mathfrak{E}^{(0)}$ one can assign a *difference element* $d_2(\sigma) \in K(\boldsymbol{R}^2\times SY)$ (cf. [18], [62]). The above statements are then closely connected with the following

THEOREM 4. *If $\beta\colon K(SY) \to K(\boldsymbol{R}^2\times SY)$ is the Bott isomorphism, then*

(11) $\beta(\operatorname{ind}\Pi^+\sigma) = d_2(\sigma).$

A proof of this theorem is given in [59]. It is similar to the analytic proof of the Bott periodicity theorem (given in [4]). Theorem 4 is an essential part of the proof of the index theorem for boundary value problems given in [18]. Introducing an Abelian additive group $\operatorname{Ell}(X, Y)$ of equivalence classes of elliptic operators of the form (1) ([59], [62]) one can formulate the *index theorem* as follows:

THEOREM 5. *There exists a homomorphism*

(12) $d\colon \operatorname{Ell}(X, Y) \to K(T\Omega)$

($\Omega := X\setminus Y$), *so that* $\operatorname{ind}_a = \operatorname{ind}_t\circ d$. (Here $\operatorname{ind}_a\colon \operatorname{Ell}(X, Y) \to \boldsymbol{Z}$ and $\operatorname{ind}_t\colon K(T\Omega) \to \boldsymbol{Z}$ are the *analytical and topological index* respectively.)

4. Some problems about boundary problems

We can now formulate a question concerning an immediate generalization to the G-equivariant case. Suppose that an action of a Lie group G on X is given, inducing an action on Y. Then we can consider *G-invariant elliptic operators* of the form (1) ([38]). An analogue of the corresponding index theorem is not yet proved. Such a generalization seems to be not obvious since the corresponding versions of the Theorems 2 and 4 are not trivial ([60]).

Theorem 2 is also of independent interest. It means that near Y there exists a collection of typical symbols, so that the stable homotopy class of an arbitrary symbol in $\mathfrak{E}^{(0)}$ can be represented by such a simple symbol. This fact enables one also to prove the following

THEOREM 6 ([63]). *Suppose M is a closed compact manifold, $M = X^+ \cup X^-$, where $X^\pm$ are compact submanifolds with common boundary Y $(X^+ \cap X^- = Y)$. Let $A\colon C^\infty(M, E) \to C^\infty(M, F)$ be an elliptic pseudo-differential operator on M satisfying the transmission property with respect to Y. If $\mathscr{A}^\pm$ are elliptic operators on $X^\pm$ of the form analogous to* (1) *with operator part*

$$r^\pm A\colon C^\infty(X^\pm, E^\pm) \to C^\infty(X^\pm, F^\pm)$$

in the left upper corner ($E^+ := E|X^+$ etc.), *then there exists an elliptic pseudo-differential operator S on Y such that*

$$\operatorname{ind}\mathscr{A}^+ + \operatorname{ind}\mathscr{A}^- = \operatorname{ind}S + \operatorname{ind}\mathscr{A}. \tag{13}$$

In some sense this is a counterpart to the following so called *Agranovič–Dynin formula* ([2], [18], [62]).

THEOREM 7. *Let X be a compact smooth manifold with boundary Y and $\mathscr{A}_0, \mathscr{A} \in \mathfrak{B}$ elliptic with $\sigma_\Omega(\mathscr{A}_0) = \sigma_\Omega(\mathscr{A})$. Then there exists an elliptic pseudo-differential operator R on Y with*

$$\operatorname{ind}\mathscr{A}_0 - \operatorname{ind}\mathscr{A} = \operatorname{ind}R. \tag{14}$$

If $\mathscr{A}_0$ is fixed, the choice of an elliptic operator R on Y depending on $\mathscr{A}$ so that in case of ellipticity of $\mathscr{A}$ the formula (14) holds is called a *boundary reduction* of $\mathscr{A}$. A boundary reduction is possible also for certain non-elliptic operators in $\mathfrak{B}$. In such cases one obtains on Y operators R for which the ellipticity degenerates on a subset Z of Y ([65]).

Another question about elliptic operators in $\mathfrak{B}$ concerns *adjoints* with respect to Green formulas. If we have an elliptic operator

$$\mathscr{A}\colon C^\infty(X, E) \oplus C^\infty(Y, J) \to C^\infty(X, F) \oplus C^\infty(Y, G)$$

with $\sigma_\Omega(\mathscr{A}) = \sigma(\mathscr{A})$ and if $\sigma^*(\mathscr{A})$ is the adjoint of $\sigma(\mathscr{A})$ with respect to fixed Hermitian metrics in E and F respectively, we can ask for an elliptic operator

$$\mathscr{A}^*\colon C^\infty(X, F) \oplus C^\infty(Y, G_1) \to C^\infty(X, E) \oplus C^\infty(Y, J_1)$$

with $\sigma_\Omega(\mathscr{A}^*) = \sigma^*(\mathscr{A})$ and operators

$$\mathscr{R}\colon C^\infty(X, F) \oplus C^\infty(Y, G_1) \to C^\infty(X, F) \oplus C^\infty(Y, G),$$

$$\mathscr{S}\colon C^\infty(X, E) \oplus C^\infty(Y, J) \to C^\infty(X, E) \oplus C^\infty(Y, J_1)$$

in $\mathfrak{B}$, so that

$$\psi(\mathscr{A}f, \mathscr{R}g) = \omega(\mathscr{S}f, \mathscr{A}^*g) \tag{15}$$

for all

$$f \in C^\infty(X, E) \oplus C^\infty(Y, J), \quad g \in C^\infty(X, F) \oplus C^\infty(Y, G_1).$$

Here ψ and ω are Hermitian scalar products in the L^2-spaces $H^0(X, F) \oplus H^0(Y, G)$ and $H^0(X, E) \oplus H^0(Y, J_1)$ respectively.

Expressions of the form (15) are deduced in [64], [66]. The *Green formulas* for classical elliptic boundary value problems ([46]) are expressions of the type (15). One question is how to arrange the constructions in [64] for general elliptic operators in $\mathfrak{B}$, so that the classical Green formulas are reproduced in some sense. Green formulas for boundary problems are useful in the investigation of more precise *regularity properties* in Sobolev spaces with $p \neq 2$ and uniform estimates ([64]). If $W^{p,l}(X, E)$ denotes the space of distribution sections in E belonging locally to the Sobolev space $W^{p,l}$, the regularity of elliptic operators in $W^{p,l}$ $(1 < p < \infty)$ means that

$$\mathscr{A}\colon W^{p,s}(X, E) \oplus W^{p,t+\lambda+1/p'}(Y, J) \to W^{p,t}(X, F) \oplus W^{p,s-\gamma-1/p}(Y, G) \tag{16}$$

(with $t := s-\alpha$, $1/p' := 1-1/p$) is a Fredholm operator ($\alpha := \operatorname{ord}\sigma(A)$, $\gamma := \operatorname{ord}\sigma(T)$, $\lambda := \operatorname{ord}\sigma(K)$) ([59]). This implies the corresponding a priori estimate.

In the proof of the Fredholm property of (1) the *symbolic calculus* (specially for the boundary symbols) ([18], [58], [59]) is an improvement from a methodical point of view. It gives more insight and simplifies many things in the theory of elliptic boundary value problems. Under such a point of view it would be also interesting to have an analogue of the *Schauder estimates* for elliptic operators in $\mathfrak{B}$, which are not yet proved.

In connection with the usual theory of classical elliptic boundary value problems a lot of other things should be generalized to the class $\mathfrak{B}$, for instance *spectral properties* of the study of problems where boundary conditions have discontinuities ([28]), sometimes called *mixed boundary value problems*.

The general index formulas for elliptic operators on closed compact manifolds or for elliptic boundary value problems are sometimes rather "theoretical", i.e. it can be difficult to calculate the index explicitly. On the other hand there is a lot of papers in which special cases are investigated (e.g. in [2]). It is desirable to collect systematically the results and methods and to enlarge the classes of examples. One should compare formulas of Vekua, Calderon, Seeley, Fedosov and others and try to find interpretations with simple geometrical meaning.

5. Further Fredholm mappings connected with elliptic pseudo-differential operators

If X is a manifold with boundary Y as in § 3 and if a pseudo-differential operator

$$r^+A: C^\infty(X, E) \to C^\infty(X, F) \tag{17}$$

is given satisfying the transmission property and having an elliptic symbol $\sigma(A)$, it is quite obvious that (17) is in general no Fredholm mapping (this is true only for certain special cases). In order to connect with (17) a Fredholm mapping one defines (if possible) additional operators $r'T$, K, Q (trace operators, potential operators and operators on Y). This gives rise to the class of operators in $\mathfrak{B}$ and the concept of ellipticity. We mentioned a necessary and sufficient condition for the existence of an elliptic $\mathscr{A} \in \mathfrak{B}$ with $\sigma_\Omega(\mathscr{A}) = \sigma(A)$, namely (7). The investigation of Fredholm mappings connected with an elliptic symbol is an interesting general question and not limited to the above special situation. We shall discuss here some aspects of this problem.

Let $\mathscr{A} \in \mathfrak{B}$ be an operator of the form (1) with an elliptic interior symbol $\sigma_\Omega(\mathscr{A})$. Suppose that the boundary symbol is bijective only over $Y \setminus Z$, where Z is a smooth submanifold of Y of codimension 1. Then one can ask for the existence of an operator

$$\begin{pmatrix} r^+A + r'B & K & K_1 \\ r'T & Q & K_2 \\ T_1 & T_2 & Q \end{pmatrix}: \begin{matrix} C^\infty(X, E) \\ \oplus \\ C^\infty(Y, J) \\ \oplus \\ C^\infty(Z, J_1) \end{matrix} \to \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, G) \\ \oplus \\ C^\infty(Z, G_1) \end{matrix} \tag{18}$$

which is a Fredholm operator. Here J_1, G_1 are vector bundles over Z. The existence of a Fredholm operator (18) means that one can define *additional conditions*, namely trace and potential conditions with respect to Z so that the corresponding problem has a parametrix in the C^∞-spaces. In [65] is given a construction of Fredholm operators of the type (18) under certain conditions about the nature of degeneration of the Šapiro–Lopatinskij condition of $\mathscr{A}$. The method in the construction of (18) is a reduction to the boundary which yields an operator on Y with degenerating ellipticity over Z. One has then to investigate so called *interior boundary value problems* on Y ([69]). The question in general is as follows: Given a pseudo-differential operator $S: C^\infty(Y, J) \to C^\infty(Y, G)$ on a closed compact manifold Y which is elliptic on $Y \setminus Z$, where Z is a submanifold of Y of codimension k. How can one define a matrix of operators

$$\begin{pmatrix} S & K' \\ T' & Q' \end{pmatrix}: \begin{matrix} C^\infty(Y, J) \\ \oplus \\ C^\infty(Z, J') \end{matrix} \to \begin{matrix} C^\infty(Y, G) \\ \oplus \\ C^\infty(Z, G') \end{matrix}$$

which is Fredholm (J', G' suitable vector bundles over Z)? The answer gives a solution of a corresponding problem with degenerate Šapiro–Lopatinskij condition over Z ([65]).

Interior boundary problems (for $k = 1$) have been considered by many authors (e.g. [69], [27]). The case $k > 1$ is clear only in very special cases. Moreover the type of degenerations of ellipticity considered in the papers mentioned should be generalized. Such problems can also be studied in various distribution spaces. Extensions of the operators with respect to suitable norms preserving the Fredholm property are not so simple as in the elliptic theory. In the simplest cases one is led to *subelliptic operators* ([25], [26]). A classical special case for the situation of degenerate ellipticity of the boundary conditions is the *oblique derivative problem* ([13]).

Another problem of independent interest is how to connect with an operator (17) a Fredholm problem, if the condition (7) is not fulfilled. For certain special important operators this problem has been considered in [9]. One should study systematically the phenomenon $\operatorname{ind} \Pi^+\sigma \notin p^*K(Y)$ under this aspect, because many simple differential operators (such as the Cauchy–Riemann operator) are of this kind. The study of problems with *degenerating Šapiro–Lopatinskij conditions* could contribute a new aspect here. If boundary conditions can be formulated for the given elliptic operator such that the ellipticity degenerates on a submanifold Z of Y of the desired type and the corresponding interior boundary value problem can be treated, then a Fredholm problem can be found by adding new conditions over Z. At this point it should be useful to treat also microlocal versions of the constructions given in [65].

In the usual theory of elliptic boundary value problems it is a standard assumption that the boundary has codimension 1. But in the analysis the study of elliptic operators in *more general open sets* is of considerable interest. There exist many papers devoted to this question under various aspects ([67]).

From the point of view of general elliptic boundary value problems the class of bounded domains Ω is comparatively simple for which the components of $\partial\Omega$ are C^∞-manifolds of dimensions $n-k$ for $k = 1, \ldots, n$. Boundary problems for such domains are usually called *problems of Sobolev type*. In [70] boundary problems are studied for the polyharmonic operator Δ^m in a bounded domain in $\mathbf{R}^n$, where on the components of $\partial\Omega$ the normal derivatives of the solution are prescribed up to an order depending on the codimension k, the dimension n of Ω and m. General differential operators and general boundary conditions satisfying an analogue of the Šapiro–Lopatinskij condition are treated in [71] where also the existence of a parametrix is proved. Various K-theoretic aspects and the index of problems of Sobolev type for pseudo-differential operators are discussed in [72]. One has then operators of matrix form similarly as in the usual theory (cf. also [50], [51])

$$\begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} H^s(X, E) \\ \oplus \\ \bigtimes_{k=1}^{l} H^{s_k}(Z_k, J_k) \end{matrix} \to \begin{matrix} H^t(X, F) \\ \oplus \\ \bigtimes_{k=1}^{l} H^{t_k}(Z_k, G_k) \end{matrix}$$

where Z_k denotes the component of $\partial\Omega$ with dimension k. Here J_k and G_k are vector bundles over Z_k. The orders in the Sobolev spaces are fixed and depend on the orders of the operators, the dimension of X and of k. Unfortunately the last papers are partially unintelligible, and some propositions are not proved. Thus it seem to be still a problem to build up a systematic Fredholm theory for Sobolev type boundary problems including the topological aspects. Also further analytical questions are of interest, motivated by the theory of ordinary elliptic boundary problems discussed in § 3, such as the symbolic calculus (for boundary symbols), the algebra of operators and the type of the parametrix of a given problem. The operators in the case of Sobolev type problems are considered in Sobolev spaces with fixed orders of derivatives. The class of boundary problems for a given elliptic pseudo-differential operator depends on the choice of these orders (this shows a connection with the question of removability of singularities of solutions of elliptic equations). The corresponding effects should also be studied more systematically. Moreover regularity in L^p and in Hölder spaces can be studied and the regularity with respect to other uniform norms ([67]).

Next we return to the case of smooth compact manifolds X ($n = \dim X$) with boundary Y of dimension $n-1$. Given an arbitrary elliptic symbol (homogeneous of order a) $\sigma(A)\colon \pi^*E \to \pi^*F$ on X *without the transmission property*, one can consider corresponding operators

$$r^+A\colon H_0^s(X, E) \to H^{s-a}(X, F). \tag{19}$$

Here $H_0^s(X, E)$ denotes the subspace of those elements in $H^s(\tilde{X}, \tilde{E})$ with support in X ($\tilde{X}$ is an open neighbouring manifold of X of dimension n and $\tilde{E}$ a vector bundle over $\tilde{X}$ with $E = \tilde{E}|X$). Operators of this type and corresponding Fredholm problems

$$\mathscr{A} = \begin{pmatrix} r^+A & K \\ r'T & Q \end{pmatrix} : \begin{matrix} H_0^s(X, E) \\ \oplus \\ H^{t+\lambda+1/2}(Y, J) \end{matrix} \to \begin{matrix} H^t(X, F) \\ \oplus \\ H^{s-\gamma-1/2}(Y, G) \end{matrix} \tag{20}$$

($t = s-a$) have been studied in [28], [73], [74], [76]. The operators K and $r'T$ are defined by homogeneous symbols $\sigma(K)$ and $\sigma(T)$ respectively ($\lambda := \operatorname{ord}\sigma(K)$, $\gamma := \operatorname{ord}\sigma(T)$) and Q is a pseudo-differential operator on Y. The Fredholm property of $\mathscr{A}$ is ensured by an analogue of the Šapiro–Lopatinskij condition. The form of the operators K, r', T and Q depends

essentially on the choice of s. Important analytical properties of the operators of the form (20) are studied in [28] and in the mentioned papers of Višik and Èskin. On the other hand some interesting things are not complete. Here, questions analogous to those in § 4 can be raised. The properties of composition and the type of parametrices of operators (20) should be studied, and besides the symbolic calculus and continuity properties of the correspondence between the symbols and the operators in both directions.

One should prove also the index theorem analogous to the corresponding Theorem 5 in § 3 for operators (20). For this some properties of stable homotopy classes of elliptic operators of the type (20) are needed, specially properties about deformation of elliptic symbols similar to Theorem 2. A further question consist in an analogue of the necessary and sufficient condition (7) to $\sigma(A)$ for the existence of an elliptic boundary problem for (19). One can ask for theorems of Agranovič–Dynin type and of type of Theorem 6. Finally the investigation of elliptic operators r^+A for which elliptic boundary problems (20) do not exist and the construction of suitable other Fredholm theories can be a program for further studies.

6. Unique continuation

An operator A over a connected paracompact Riemannian manifold M operating between sections of vector bundles E and F over M has the *unique continuation property*, iff every $u \in \ker A$ vanishes automatically on the whole of M, if it vanishes on an open subset $\Omega \subset M$. Without the condition of global solvability, we may define A having the *local unique continuation property* at the point $x \in M$, iff there exists a neighbourhood M' of x in M such that $A|M'$ has the unique continuation property or more precisely: $Au|M' = 0$ implies $u|M' = 0$, if $u \in C^\infty(M, E)$ vanishes identically on an open subset $\Omega \subset M'$.

By definition the local unique continuation property for all points implies the global unique continuation property, but not vice versa. For differential operators of order m the unique continuation property (u.c.) can be reformulated via the Cauchy boundary value problem, e.g. demanding that all local solutions $Au|M' = 0$ vanish on M', if they fulfil in local coordinates the conditions

$$\left(\frac{\partial}{\partial x_n}\right)^j u = 0; \quad j = 0, \ldots, m-1,$$

along the hypersurface $x_n = 0$ (cf. also [21]).

Since Holmgren (1901) has obtained the u.c. for differential operators with analytic coefficients, it has been repeatedly conjectured that u.c. holds generally for elliptic equations with sufficiently smooth coefficients.

But Pliś showed with a series of counterexamples that all these conjectures were false. Nevertheless several sufficient and a few necessary conditions for u.c. were established, see e.g. [15], [21], [56].

Some of these conditions are formulated in terms of the dimensions of M and E and of the order of A; others gain their information from a careful analysis of the distribution and multiplicity of the complex zeros of the operator's symbol in different directions or from a priori inequalities for the operator. The mutual interrelation between these criteria, however, has not been systematically investigated as yet.

In the context of index theory we want to point out a problem posed by Schwartz [68] which is still unsolved, namely whether u.c. for an elliptic operator A implies u.c. for the adjoint operator A^*. Malgrange [47] has given some criteria under which u.c. holds simultaneously for A and A^*, and most of the criteria given in the literature for u.c. of A yield automatically u.c. for A^*. But this is not true for all criteria.

Why is a unified theory still missing which would connect the isolated bricks of our knowledge of u.c.? Some of the inherent difficulties are apparently due to the non-homotopy-invariance of that property which for example follows from the fact that we are able to approximate all coefficients by analytic ones. Moreover, one of the Cohen [23] counterexamples (but without C^∞-coefficients) having the form

$$Au := \Delta^3 u - a(x) D^5 u = 0$$

indicates that u.c. is definitely not dependent on the top symbol alone.

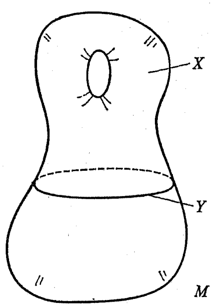

Fig. 2

So it might be interesting to reexamine the whole problem by more refined topological means, starting with one aspect, e.g. as in [16] with a theory of *relative indices*

$$\operatorname{ind}_X(A) := \dim \ker_X(A) - \dim \ker_X(A^*)$$

where X is a submanifold of M of codimension 0 with smooth boundary Y and

$$\ker_X(A) := \{u;\ u \in \ker A \text{ and } \operatorname{supp} u \subset \mathring{X}\},$$

$\ker_X(A^*)$ being defined similarly.

If u.c. holds, $\dim \ker_X(A)$ vanishes. However, one easily finds pseudo-differential operators with non-trivial $\ker_X(A)$ and with $\operatorname{ind}_X A \neq 0$, e.g. if A with $\operatorname{ind} A \neq 0$ has "support in X", i.e. being the identity outside $\mathring{X}$. Then one obtains $\operatorname{supp} u \subset X$ for all $u \in \ker X$ and hence $\ker_X A = \ker A$. Similarly one obtains $\ker_X A^* = \ker A^*$ and, in that case, $\operatorname{ind}_X A = \operatorname{ind} A$. In the same way, every boundary value problem over X with non-vanishing index leads to an operator A over M with $\operatorname{ind}_X A \neq 0$. Moreover, the above mentioned Pliś example being globally defined gives even a differential operator A with $\operatorname{ind}_X A \neq 0$ where X is the upper halfsphere on the sphere M.

Therefore the relative index $\operatorname{ind}_X(A)$ is not meaningless, even if — and perhaps just because — it reflects only one aspect of the unique continuation problem. Its study, however, would require analytical and topological methods beyond the realm of present index theory.

7. The transmission problem

We now consider two manifolds X^+ and X^- glued together along their common boundary Y, i.e. a closed oriented Riemannian manifold M with submanifolds X^+, X^-, Y such that

$$X^+ \cup X^- = M \quad \text{and} \quad X^+ \cap X^- = Y.$$

Instead of looking separately at boundary value problems on (X^+, Y) and (X^-, Y), it is quite natural to look at pairs of solutions u^+, u^- defined on X^+ and X^- with

$$Au^+ = 0 \text{ on } X^+ \quad \text{and} \quad Au^- = 0 \text{ on } X^-,$$

imposing m "transmission" or "coupling" conditions for u^+, u^- and some derivatives of u^+, u^- on Y. Here m is the order of the elliptic operator A on X. The ellipticity of the coupling conditions is then defined in every point of Y by the Lopatinskij ellipticity conditions for the locally folded up elliptic system $A \oplus A$.

One may also consider the jump case where P splits in P^+ and P^-, or a finite number of manifolds $X^0, X^1, \ldots, X^t$ glued together along their common boundary. Motivated by an elasticity problem of civil engineering, Picone [53] has generalized the problem to include situations where the boundaries Y^+ and Y^- of X^+ and X^- have only a submanifold Z of codimension 0 in common where some transmission conditions shall be fulfilled, whereas on $Y^+\setminus Z$ and $Y^-\setminus Z$ "traditional" boundary value problems are posed. Schechter [61] and other authors obtained conditions under which this problem which at the same time generalizes the intricate mixed boundary value problem (cf. § 4), has a finite and homotopy invariant index.

A closely related classical situation, with paracompact $M = C$ and $A = \partial/\partial\bar{z}$, is given by the already mentioned Riemann–Hilbert problem where one is asking for a piecewise holomorphic function Φ with $\Phi(\infty) = 0$ and the coupling condition

$$\Phi^+(t) = g(t)\Phi^-(t) + h(t), \qquad t \in Y,$$

if

$$\Phi^+(t) := \lim_{z\to t, z\in X^+} \Phi(t)$$

and similarly $\Phi^-(t) := \ldots$ and if $g \in C^0(Y)$, $g(t) \neq 0$ for $t \in Y$, and $h \in L^2(Y)$.

It was shown by Hilbert that the function-theoretical transmission problem can be translated into the singular integral equation

$$B\mu := a(t)\mu + \frac{b(t)}{\pi i}\int_Y \frac{\mu(\tau)}{\tau - t}\,d\tau = 0$$

where $\mu \in L^2(Y)$ and $g = (a+b)/(a-b)$ since every solution of the Riemann–Hilbert problem can be obtained by the Cauchy integral

$$\Phi(z) := \frac{1}{2\pi i}\int_Y \frac{\mu^2(\tau)}{t-\tau}\,d\tau$$

for $\mu \in \ker B$.

For several reasons the transmission problem belongs to the core of the index theory:

(i) *Historically* it was the Riemann–Hilbert problem which lead F. Noether in [49] to the discovery of non-vanishing indices for integral equations — against Hilbert's erroneous conjecture.

(ii) The *topological character* of the Noether–Muskhelišvili index formula is most apparent in that particular case since one has simply

$$\operatorname{ind} B = \deg(g).$$

So B is one more candidate of a "prototype" operator for building up the whole index theory, like the shift operator or the other standard operators which served as examples *and* patterns in the K-theoretical proofs of the general index formula. Bojarski [14] has actually constructed "abstract Riemann–Hilbert transmission problems" generating all Fredholm pairs

(in the sense of Kato) in a natural way. This opens the possibility for a purely analytical proof of the Atiyah–Singer index theorem for general manifolds by induction over series of transmission problems thus uniting the Fedosov–Hörmander approach for complicated operators on simple manifolds with the topological approach of emphasizing calculations for simple classical operators on more complicated manifolds. Since in all transmission problems there is an inherent unity of the topological and analytical aspects, they may very well play a mediating role in index theory.

(iii) We can interpret the transmission problems as being the analytical pendant to the topological problems of reconstructing a manifold out of its pieces. Therefore it is not surprising that one meets *challenging topological problems* along the way. This was discovered by Novikov and Wall [77] when calculating the Hirzebruch signature of "composed" manifolds from the signature of their components.

For a better understanding of the difficulties involved one could follow [16] and replace the general transmission problem with a problem which deals only with the closed manifolds Y^+, Y^- which in turn have a common submanifold Z of codimension 0 with boundary ∂Z. For pseudo-differential operators over Y^+, Y^- the ellipticity is easily explained even if the operators are amalgamated over Z, and the usual index properties are obtained; in particular, the index depends only on the symbol data η^+, η^-, ζ where η^+ and η^- and ζ are isomorphisms between the corresponding vector bundles over the covariant sphere bundles $S(Y^+\setminus\mathring{Z})$, $S(Y^-\setminus\mathring{Z})$, and $S(Z)$ with the coupling condition

$$\zeta|(SZ|\partial Z) = \eta^+|(SZ|\partial Z) \oplus \eta^-|(SZ|\partial Z).$$

In [16] a series of conditions was given when one can disamalgamate the symbol data by homotopies leading to two separate elliptic problems on Y^+ and Y^- with indices adding up to the index of the original problem. However, there exist topological obstructions against this deformation argument, e.g. if Z is a cell of even dimension. Therefore one has to make the necessary deformations not on the symbol level, but on the K-theory level.

Actually, it turns out that one has to switch to the dual "homological" K_*-theory and that Atiyah's theory [7] of "global elliptic operators" applies. In the meantime Kasparov [45] and independently Brown, Douglas and Fillmore [22] have succeeded in determining the exact equivalence relations on the global elliptic operators generating K_*. It might be worthwile at this stage to tackle the index problem for transmission problems again.

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