for $t \in [0, T]$. Since $u-w$ is a solution of (2.10) with $(u-w)(0,t) = 0$, the estimate (3.2) holds for $u-w$ and if follows that $|(u-w)(x,t)| < \varepsilon$ for $(x,t) \in [-1,1] \times [0,T]$. But by the definition of w this estimate is identical with (3.1).

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ON THE SPECTRAL FLOW AND THE INDEX THEOREM FOR FLAT VECTOR BUNDLES

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Let X be a smooth closed Riemannian manifold of dimension n and E a smooth Hermitian vector bundle over X. If $A: C^\infty(X, E) \to C^\infty(X, E)$ is a self-adjoint elliptic operator of order $m > 0$, then A has a discrete spectrum $\{\lambda_k\}_{k \in N}$ with $|\lambda_k| \to \infty$ as $k \to \infty$ ([6], part XII, Theorem 14). In [1] Atiyah, Patodi and Singer introduced the function

$$\eta(A;\, s) = \sum_{\lambda_k \neq 0} \operatorname{sign}(\lambda_k) \cdot |\lambda_k|^{-s},$$

holomorphic for $\operatorname{Re} s > n/2m$. More precisely, it is known ([1], [4]) that this function has a meromorphic extension onto the whole complex plane with only simple poles and that there is no pole at $s = 0$. This function has then been used to define an invariant $\operatorname{ind}(A;\, D_1, D_0)$ depending, for fixed flat connections D_1, D_0, only on the operator A.

In this paper we compute this invariant in a special case, where it has a clear interpretation. In particular, we obtain an expression for the index of the family of self-adjoint Fredholm operators over S^1 by an integer, which is a spectral invariant of the family, the so-called spectral flow. The main properties of this invariant, which were mentioned in [1], are also proved.

1. Extension of a pseudo-differential operator to an auxiliary flat vector bundle

DEFINITION 1.1. Let V be a smooth Hermitian vector bundle over a smooth manifold X. V is called *flat* if it has a trivialisation $\{U_i, \varphi_i\}$ with constant transition functions.

If this is the case, the exterior differential d acting locally on the sections of $U_i \times C^N \overset{\cong}{\to} V/U_i$ extends to a global connection D_V on the whole of V. The curvature Ω_V of this connection satisfies $\Omega_V = D_V^2 = 0$. Conversely, any connection with this property determines a flat structure in V.

Let E and F be vector bundles over X. If $A\colon C^\infty(X, E) \to C^\infty(X, F)$ is a differential operator, then A has a natural extension $A\otimes_{D_V} I\colon C^\infty(X, E\otimes V) \to C^\infty(X, F\otimes V)$. To define it, we write any $h \in C^\infty(X, V)$ as

$$h = \sum_{i\in I} \sum_{k=1}^{N} f_{ik} s_{ik},$$

where $f_{ik} \in C_0^\infty(U_i)$, $s_{ik} \in C^\infty(U_i, V\,|\,U_i)$, $s_{ik}(x) = \varphi_i^{-1}\big(x, (0, \ldots, \underset{k}{1}, \ldots, 0)\big)$, $N = \dim V$, and we put

$$(A\otimes_{D_V} I)(s\otimes h) = \sum_{i\in I} \sum_{k=1}^{N} A(f_{ik} s)\otimes s_{ik}.$$

THEOREM 1.2 ([6], part IV, §9). *Under the above conditions there exists a unique differential operator $A\otimes_{D_V} I$ with the properties*

(i) $\sigma(A\otimes_{D_V} I) = \sigma(A)\otimes I$,

(ii) $h \in C^\infty(X, V)$ *and* $D_V h = 0 \Rightarrow A\otimes_{D_V} I(s\otimes h) = As\otimes h$, $s \in C^\infty(X, E)$.

If A is merely pseudo-differential, the situation is a little more complicated, since A need not be local. We may, however, use a partition of unity $\{\psi_i\}$ subordinate to the covering $\{U_i\}$ and define

$$A\otimes_{D_V} I(s\otimes h) = \sum_{i} \sum_{k=1}^{N} \psi_i A(f_{ik} s)\otimes s_{ik}.$$

This depends, of course, on the partition of unity, but (i) and (ii) are still satisfied. Observe that, since the transition functions are constant unitary matrices, the operator $A\otimes_{D_V} I$ is self-adjoint if A is self-adjoint.

2. The invariant $\mathrm{ind}(A; D_1, D_0)$

Let $B\colon C^\infty(X, E) \to C^\infty(X, E)$ be a self-adjoint elliptic pseudo-differential operator of positive order. If we keep the bundle, the order and the principal symbol of B fixed, we obtain a convex subset J in the space of all such operators. Define

$$\xi(B;\ s) = \tfrac{1}{2}\big(\eta(B;\ s) + \dim\ker B\big).$$

It is proved in [1] (§§ 2 and 6) that the function

$$\mu(B) = \xi(B;\ 0) \bmod Z$$

is a smooth function on J with values in $\boldsymbol{R}/\boldsymbol{Z}$, so that $d\mu(B)$ is a well-defined real-valued 1-form on J. For all $B_0, B_1 \in J$ the integral

$$\int_{B_0}^{B_1} d\mu(B) = \int_0^1 d\mu(t)$$

does not depend on the choice of a smooth family joining B_0 with B_1. This integral will be denoted by $\mathrm{ind}(B_1, B_0)$. We thus have

$$\xi(B_1;\ 0) - \xi(B_0;\ 0) = k + \mathrm{ind}(B_1, B_0), \qquad k \in \boldsymbol{Z}.$$

We are interested in the value of $\mathrm{ind}(B_1, B_0)$. From now no the following special situation will be considered: A is a self-adjoint elliptic pseudo-differential operator of positive order; D_0 and D_1 are two flat connections on $X\times \boldsymbol{C}^N$. We then define

$$\mathrm{ind}(A;\ D_1, D_0) = \mathrm{ind}(A\otimes_{D_1} I, A\otimes_{D_0} I).$$

The following theorem was proved in [1], §§ 5 and 6:

THEOREM 2.1. $\mathrm{ind}(A;\ D_1, D_0)$ *is a real-valued homotopy invariant of the operator A, depending only on the element $\sigma_1(A) \in K^1(TX)$ determined by the principal symbol of A (for fixed flat connections D_1, D_0).*

Remark 2.2. A determines the family

$$\sigma(\{A_t\}) = \begin{cases} \cos t + i\sigma\sin t & \text{for} \quad t \in [0, \pi], \\ e^{it} & \text{for} \quad t \in [\pi, 2\pi], \end{cases}$$

where σ denotes the principal symbol of A. This family gives rise to an element in $K(S^1\times TX)$, trivial for $t = 0$, and thus to an element in $K^1(TX)$ (see [5], part II) denoted by $\sigma_1(A)$. It is proved in [1] that every class in $K^1(TX)$ may be obtained in this way.

Remark 2.3. Let $\alpha\colon \Pi_1(X) \to U(N)$ be a representation and $\tilde{X}$ the universal covering of X ($\tilde{X} \to X$ is a principal $\Pi_1(X)$-bundle). The bundle $V_\alpha = \tilde{X}\times_\alpha \boldsymbol{C}^N$ is flat. Assume that it is trivial and let $\varphi\colon V_\alpha \to X\times \boldsymbol{C}^N$ be a fixed unitary trivialisation. Then

$$\mathrm{ind}(\alpha, \varphi, A) = \mathrm{ind}(A;\ \varphi^* D_\alpha, d),$$

where the left-hand side is defined in [1], § 6; here D_α is the natural flat connection in V_α and $\varphi^* D_\alpha = \varphi D_\alpha \varphi^{-1}$.

3. The spectral flow

Let $\{B_t\}_{t\in I}$ be a smooth family of self-adjoint elliptic pseudo-differential operators of positive order acting on sections of a bundle F. If there exists a unitary automorphism γ of F such that $B_1 = \gamma B_0 \gamma^{-1}$ then $\xi(B_1;\ 0)$

$= \xi(B_0; 0)$, so that $k = -\operatorname{ind}(B_1, B_0)$.

The integer k equals the value at $t = 1$ of the following function j: $[0; 1] \to Z$:

(i) $j(0) = 0$;

(ii) at $t = t_0$ the value $j(t)$ increases by 1 for each eigenvalue $\lambda(t)$ of B_t changing sign from < 0 to $\geqslant 0$, and decreases by 1 for each eigenvalue changing sign from $\geqslant 0$ to < 0 (the eigenvalues are counted with their multiplicities).

The function j is well defined since we can approximate $\{B_t\}$ by an analytic or piece-wise linear family.

DEFINITION 3.1. The value $j(1)$ is called the *spectral flow of the family* $\{B_t\}_{t\in I}$.

The spectral flow may be defined not only for families of pseudo-differential operators of positive order as above. Let $\{A_t\}_{t\in I}$ be a family of self-adjoint Fredholm operators acting in separable Hilbert space H. For such operators zero is an eigenvalue of finite multiplicity, so in a neighbourhood of 0 we have only the discrete spectrum, hence we may still calculate the number of eigenvalues changing sign and $\operatorname{sf}\{A_t\}$ is well defined.

In fact, in this case there is an elementary proof that the eigenvalues near zero are continuous functions of the operator in the norm topology (cf. Theorem XIII.1, and Problem 2 in Chapter XIII of [7]).

LEMMA 3.2. *Let $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ be the families of self-adjoint Fredholm operators such that for each t $\|A_t - B_t\|$ is sufficiently small. Then* $\operatorname{sf}\{A_t\} = \operatorname{sf}\{B_t\}$.

Proof. It is obvious that in this case the eigenvalues of A change sign if and only if those of B do. This will be described in details in [8].

Assume, in particular, that there exists a unitary automorphism γ of H such that $A_1 = \gamma A_0 \gamma^{-1}$. The group of unitary automorphisms of H being contractible, there is a family $\{\gamma_t\}_{t\in I}$ of unitary transformations of H continuous in the norm topology such that $\gamma_0 = \gamma$ and $\gamma_1 = \mathrm{id}_H$. The family $\{A_t\}_{t\in I}$ yields then a family $\{\tilde{A}_s\}_{s\in I}$ of operators over S^1:

$$\tilde{A}_s = \begin{cases} A_{2s} & \text{for} \quad s \in [0, \frac{1}{2}], \\ \gamma_{2s-1} A_0 \gamma_{2s-1}^{-1} & \text{for} \quad s \in [\frac{1}{2}, 1]. \end{cases}$$

Observe that $\operatorname{sf}\{A_t\} = \operatorname{sf}\{\tilde{A}_s\}$, since all the operators $\tilde{A}_s$ for $s \in [\frac{1}{2}, 1]$ have the same eigenvalues, which in particular do not change sign in this interval. We may thus assume for our purposes that "the family over S^1" means a family $\{A_t\}_{t\in I}$ such that $A_1 = \gamma A_0 \gamma^{-1}$ for some γ as above. We then write $\{A_t\}_{t\in S^1}$.

THEOREM 3.3. *Let $\{A_t\}_{t\in S}$ be a family of self-adjoint Fredholm operators acting in a separable Hilbert space H. Then $\operatorname{sf}\{A_t\}$ is a homotopy invariant of the family.*

Proof. This is the consequence of Lemma 3.2.

4. The index theorem for families of self-adjoint operators

To compute value of $\operatorname{sf}\{A_t\}_{t\in S^1}$ we shall use the index theorem for families of self-adjoint operators.

Recall [2] (see also [8]) that the set $\hat{F}$ of all self-adjoint Fredholm operators in a separable Hilbert space H has 3 connected components $\hat{F}_+$, $\hat{F}_-$ and $\hat{F}_*$. The spaces $\hat{F}_\pm$ consist of operators that are essentially positive (respectively essentially negative), i.e. positive or negative modulo compact operators. Both these spaces are contractible. The space $\hat{F}_*$ is known to be the classifying space for the functor K^{-1}, which means that for any compact space X, $K^{-1}(X) \cong [X, \hat{F}_*]$, where the last symbol denotes the set of homotopy classes of continuous maps $X \to \hat{F}_*$.

Let now $\{A_p\}_{p\in P}$ be a continuous family of self-adjoint Fredholm operators over a compact space P. The element in $K^{-1}(P)$ determined by $\{A_p\}$ will be called the *analytic index* of $\{A_p\}$ and denoted by $\operatorname{index}\{A_p\}$.

If $\{A_p\}_{p\in P}$ is a family of self-adjoint elliptic pseudo-differential operators acting on sections of a Hermitian vector bundle E, we may express index $\{A_p\}$ in terms of topological invariants. In this case $\{A_p\}$ defines a family $\{A_{t,p}\}_{t\in S^1, p\in P}$ given by

$$A_{t,p} = \begin{cases} \cos t + iA_p \sin t & \text{for} \quad t \in [0, \pi], \\ e^{it} & \text{for} \quad t \in [\pi, 2\pi]. \end{cases}$$

The symbol $\sigma\{A_{t,p}\}$ gives an element in $K(S^1 \times P \times TX)$ trivial over $\{0\} \times P \times TX$, and thus an element $\sigma_1(A) \in K^1(TX \times P)$ called the *symbol of the family* $A = \{A_p\}_{p\in P}$. The topological index $\text{t-ind}\{\sigma(A_{t,p})\} \in K(S^1 \times P)$ is a homotopy invariant of $\{A_p\}$ and depends only on $\sigma_1(A)$. We may now formulate

THEOREM 4.1 (the index theorem for families of self-adjoint operators, [1], § 3). *Let $A = \{A_p\}_{p\in P}$ be a continuous family of self-adjoint elliptic pseudo-differential operators over a compact space P, acting on sections of a Hermitian vector bundle E. Then index A equals the value of the homomorphism* $\text{t-ind}\colon K^1(P \times TX) \to K^1(P)$ *on the symbol $\sigma_1(A)$ of this family.*

Assume now $P = S^1$. Then the map $\nu\colon K^1(S^1) \to Z$ given by $\nu(\sigma) = \operatorname{ch}(\sigma)[S^1]$ is an isomorphism. On the other hand, we have

LEMMA 4.2. sf *is an isomorphism* $\Pi_1(\hat{F}_*) \to Z$.

Proof. Let A_0 be the operator in $L^2(S^1)$ given by

$$A_0(e^{ikx}) = \begin{cases} \operatorname{sign}(k+1)e^{ikx} & \text{for} \quad k \geq 0, \\ \operatorname{sign} k e^{ikx} & \text{for} \quad k < 0 \end{cases}$$

and γ_k the automorphism of $S^1 \times C$ defined by

$$\gamma_k(x, z) = (x, e^{ikx}z),$$

where x is a real coordinate mod2π. Denote by $\{C_{m,t}\}$ any family joining A_0 with $\gamma_m A_0 \gamma_m^{-1}$. It is known that $\Pi_1(\hat{F}_*) = \Pi(\Omega F) \cong Z$ (the periodicity theorem) and that the classes of $\{C_{m,t}\}$ in $\Pi_1(\hat{F}_*)$ are different for different m. What is left is to observe that $\operatorname{sf}\{C_{m,t}\} = m$.

THEOREM 4.3. *Let* $A = \{A_t\}_{t\in S^1}$ *be a continuous family of self-adjoint elliptic pseudo-differential operators acting on sections of a Hermitian vector bundle* E. *Then*

$$\operatorname{sf} A = \nu(\operatorname{index} A).$$

Proof. Assume that $\operatorname{sf} A = m$. If A has the spectral decomposition $\{\lambda_k;\ \varphi_k\}_{k\in Z}$ and if $A_1 = \gamma A_0 \gamma^{-1}$ for a $\gamma \in \operatorname{aut} E$, then A_1 has spectral decomposition $\{\lambda_k;\ \gamma\varphi_k\}$. We may assume that A_0 is invertible. Let $|A_0| = (A_0^2)^{1/2}$. Then $\tilde{A}_0 = A_0|A_0|^{-1}$ has the spectral decomposition $\{\operatorname{sign}\lambda_k;\ \varphi_k\}$ and $\tilde{A}_1 = A_1|A_1|^{-1} = \gamma\tilde{A}_0\gamma^{-1}$ has the decomposition $\{\operatorname{sign}\lambda_k;\ \gamma\varphi_k\}$. The family joining $\tilde{A}_0$ with $\tilde{A}_1$ has, of course, the same symbol as the family A, since $|A_0|^{-1}$ is a positive operator of order $-p$ ($p = \operatorname{order} A$), which does not change the class of a symbol in K and a fortiori in K^1 ([6], part XII and [3], Theorem 2.2). Hence $\operatorname{index} A = \operatorname{index} \tilde{A}$ and moreover $\operatorname{sf} \tilde{A} = m$ by construction, so that index A is equal to the class of the family $\{C_{m,t}\}$ in $K^1(S^1)$ and $\operatorname{sf}\tilde{A} = \operatorname{sf}\{C_{m,t}\}$. The proof will be complete if we construct for any m a family B_m with spectral flow m and with $\nu(\operatorname{index}\{B_m\}) = m$; this is done in Section 6.

5. The class $b(D_1; D_0)$. A cohomological formula for $\operatorname{ind}(A; D_1, D_0)$

As follows from the construction in Section 1, a self-adjoint operator has many extensions to $X \times C^N$, which depend on the choice of the flat structure. The difference between two such extensions may be measured by the cohomological difference between two connections.

Let D_0 and D_1 be two connections on $X \times C^N$. Any family $\{D_t\}_{t\in I}$ of connections joining D_0 with D_1 defines a global connection $\tilde{D}$ on $I \times X \times C^N$ with the matrix $\omega(t, x) = \omega_t(x)$, where ω_t is the matrix of D_t. The form $\operatorname{ch}(\tilde{D}) - N$ determines an element in $H^{ev}(X \times I, X \times \{0, 1\}: R) = H^{odd}(X; R)$ that will be denoted by $\beta(D_1; D_0)$. The corresponding element in $K^{-1}(X) \otimes_Z R$ will be denoted by $b(D_1; D)$.

LEMMA 5.1. *Let* D *be a connection in* $X \times C^N$ *and* γ *a unitary automorphism of* $X \times C^N$. *Then*

$$\beta(\gamma^* D;\ D) = \operatorname{ch}[X \times C^N;\ \gamma],$$

where $[X \times C^N;\ \gamma]$ *is the element in* $K^{-1}(X)$ *determined by* γ.

Remark. The pair $(X \times C^N;\ \gamma)$ determines the following bundle W over $S^1 \times X$: $W = I \times X \times C^N/\sim$, where $(0, x, v) \sim (1, x, \gamma v)$. Then $[W] - N \in K(S^1 \times X)$ and gives an element in $K^{-1}(X)$ denoted by $[X \times C^N;\ \gamma]$.

Proof. In this case the connection $\tilde{D}$ defines a global connection on W. Thus

$$\beta(\gamma^* D;\ D) = \operatorname{ch}(\tilde{D}) - N = \operatorname{ch}(W) - N = \operatorname{ch}[X \times C^N;\ \gamma].$$

We now turn our attention back to the invariant $\operatorname{ind}(A;\ D_1, D_0)$ under the assumption of Theorem 2.1. Assume additionally that $D_1 = \gamma^* D_0 = \gamma D_0 \gamma^{-1}$, where γ is a unitary automorphism of $X \times C^N$.

LEMMA 5.2. *Let* D *be a flat connection in* $X \times C^N$ *and* γ *a unitary automorphism of* $X \times C^N$. *Then*

$$A \otimes_{\gamma^* D} I = (1 \otimes \gamma)(A \otimes_D I)(1 \otimes \gamma^{-1}).$$

Proof. Let h be a D-flat section of $X \times C^N$ over an open set $U \subset X$ (i.e. $Dh = 0$). Then γh is a $\gamma^* D$-flat section over U. If now $f \in C_0^\infty(U)$ and $s \in C^\infty(X, E)$, we obtain

$$A \otimes_{\gamma^* D} I\big(s \otimes f(\gamma h)\big) = \varphi A(fs) \otimes \gamma h = (1 \otimes \gamma)(A \otimes_D I)(1 \otimes \gamma^{-1})\big(s \otimes f(\gamma h)\big),$$

where $\varphi \in C_0^\infty(U)$, $\varphi|\operatorname{supp} f = 1$. By Theorem 1.2 the proof is complete.

THEOREM 5.3. *Let* $D_1 = \gamma^* D$ *with* D *and* γ *as above. Then the family* $\{A_t\}_{t\in I}$ *of operators joining* $A \otimes_D I$ *with* $A \otimes_{\gamma^* D} I$ *is a family over* S^1 *with symbol* $\sigma_1(A) \otimes [X \times C^N;\ \gamma]$.

Proof. In this case the family $\{A_t\}$ joining $A \otimes_D I$ with $A \otimes_{\gamma^* D} I$ defines a global operator on sections of $\pi^* E \otimes W$, where $\pi\colon S^1 \times X \to X$ is the projection and W is as above. The bundle $\pi^* E \otimes W$ may be obtained from $E \otimes C^N$ pulled back over $I \times X$ if we identify $(0, e \otimes v)$ with $(1, e \otimes \gamma v)$. Let now $e \in C^\infty(X, E)$, $v \in C^\infty(X, C^N)$, $dv = 0$. Then

$$A_0(0, e \otimes v) = (A \otimes_D I)(0, e \otimes v) = (0, Ae \otimes v) \sim (1, Ae \times \gamma v)$$
$$= (A \otimes_{\gamma^* D} I)(1, e \otimes \gamma v) = A_1(1, e \otimes \gamma v).$$

Hence it is clear that $\sigma\{A_t\} = \sigma(A)\otimes I$, where $\sigma(A)$ acts on π^*E lifted to $S^1\times(TX\setminus 0)$ and the identity acts on W lifted similarly. Thus, after the factorization,

$$\sigma_1\{A_t\} = \sigma_1(A)\otimes[X\times C^N;\ \gamma].$$

Note that this is an element in $K^0(TX)$ since $\sigma_1(A)\in K^1(TX)$; $[X\times C^N;\ \gamma]$ is viewed here as a lift to $K^{-1}(TX)$ of an element from $K^{-1}(X)$ and we have the pairing $K^{-1}(TX)\otimes K^1(TX)\to K^0(TX)$.

COROLLARY 5.4. *The following formulas hold:*

$$\begin{aligned}\operatorname{ind}(A;\ \gamma^*D, D) &= -t-\operatorname{ind}\big(\sigma_1(A)b(\gamma^*D;\ D)\big)\\ &= -\operatorname{ch}\big[t-\operatorname{ind}\big(\sigma_1(A)b(\gamma^*D;\ D)\big)\big][S^1]\\ &= -\operatorname{ch}\big(\sigma_1(A)\big)\beta(\gamma^*D;\ D)\mathcal{T}(X)[TX],\end{aligned}$$

where $\mathcal{T}(X)$ is the Todd class of the manifold X.

Proof.

$$\begin{aligned}\operatorname{ind}(A;\ \gamma^*D, D) &= -\operatorname{sf}\{A_t\} = -\nu(\operatorname{index}\{A_t\})\\ &= -\operatorname{ch}[t-\operatorname{ind}\sigma_1(A)b(\gamma^*D; D)]\\ &= -\operatorname{ch}\sigma_1(A)\beta(\gamma^*D; D)\mathcal{T}(X)[TX]\end{aligned}$$

(see [3], Theorem 5.1).

A more general formula has been proved in [1] in the case of $\dim X$ odd.

THEOREM 5.5 ([1], § 6). *Let X be an odd-dimensional (compact Riemannian) manifold. If D_1, D_0 are two flat connections in $X\times C^N$ and* $\operatorname{ind}_R = t-\operatorname{ind}\otimes I\colon K^0(TX)\otimes_Z R\to R$, *then*

$$\operatorname{ind}(A;\ D_1, D_0) = -\operatorname{ind}_R\sigma_1(A)b(D_1;\ D_0).$$

6. The case $X = S^1$; an example

Let $X = S^1$. Consider the operator $B_0\colon C^\infty(S^1, C)\to C^\infty(S^1, C)$,

$$B_0(f) = -i\frac{df}{dx}$$

(x is the real variable mod 2π). B_0 has the spectral decomposition $\{k, e^{ikx}\}_{k\in Z}$ so that $\eta(B_0;\ 0) = 0$ and $\xi(B_0;\ 0) = \frac{1}{2}$. In this case all connections are flat.

Consider a connection D_a in $S^1\times C$ with the form $\omega_a(x) = ia\,dx$, $a\in R$. Then $B_a = B_0\otimes_{D_a}I\colon C^\infty(S^1,\ C)\to C^\infty(S^1,\ C)$ is of the form

$$B_0(f) = -i\frac{df}{dx}+af.$$

B_a has the spectral decomposition $\{k+a, e^{ikx}\}_{k\in Z}$; thus $\eta(B_a;\ 0) = 1-2(a-[a])$, where $0\leqslant a-[a]\leqslant 1$ for $a>0$, $[a]\in Z$ and $0\leqslant [a]-a\leqslant 1$ for $a<0$. Thus

$$\xi(B_a;\ 0)-\xi(B_0;\ 0) = [a]-a;$$

in particular $\operatorname{ind}(B_0;\ D_a, d) = -a$.

Let us calculate the class $b(D_a;\ d)$. As a family of connections joining D_a with d we may take $\{D_t\}_{t\in I}$, $D_t = tD_a+(1-t)d$, which gives a connection in $I\times S^1\times C$ with matrix $\omega(t, x) = iat\,dx$ (in the trivial frame) and curvature $\Omega(t, x) = ia\,dt\wedge dx$. The corresponding Chern character equals

$$\operatorname{tr}\exp\left(\frac{i}{2\pi}\Omega\right) = 1+\frac{a}{2\pi}dx\wedge dt,$$

hence $\beta(D_a;\ d) = a\in H^1(S^1;\ R)\cong R$.

Let us compute the symbol $\sigma_1(B_0)$. It belongs to

$$K^1(TS^1) = K^1(DS^1/SS^1) = K^1(S^1\times S^1) = K^1(S^1)\oplus K^1(S^1) = Z\oplus Z.$$

The generators of $K^1(S^1\times S^1)$ are given by the following pairs:

$$[S^1\times S^1\times C;\ a(x, \xi)z = e^{ix}z],\qquad [S^1\times S^1\times C;\ a(x, \xi)z = e^{i\xi}z].$$

On the other hand,

$$\sigma(B_0)(x, \xi, t)(z) = \begin{cases}(\cos t-i\xi\sin t)z & \text{for}\quad t\in[0, \pi],\\ e^{it} & \text{for}\quad t\in[\pi, 2\pi].\end{cases}$$

We thus obtain a complex of bundles over $R^2_{(\xi,t)}$ determining the Bott class in $K(R^2) = \tilde{K}(S^2) = K^1(S^1)$. Hence

$$\sigma_1(B_0) = [S^1\times S^1\times C;\ a(x, \xi) = e^{i\xi}]\quad\text{and}\quad \operatorname{ch}\sigma_1(B_0) = -d\xi/2\pi.$$

By Theorem 5.5,

$$\begin{aligned}\operatorname{ind}(B_0;\ D_a, d) &= -\operatorname{ind}_R\sigma_1(B_0)b(D_a;\ d)\\ &= -\left[\frac{d\xi}{2\pi}\right]\left[\frac{a\,dx}{2\pi}\right][S^1\times S^1] = -\frac{a}{4\pi^2}\iint dx\wedge d\xi = -a.\end{aligned}$$

Let $a = k\in Z$. Then

$$B_k(x) = a_k(x)B_0(x)a_k^{-1}(x),$$

where $a_k(x)(x, z) = (x, e^{ikx}z)$. The family $\{B_t\}_{0\leqslant t\leqslant k}$ thus defines a family of differential operators over S^1. The spectral flow of this family equals

k and we compute index $\{B_t\}$ as above:

$$\operatorname{ch}(\operatorname{index}\{B_t\})[S^1] = \operatorname{ch}\sigma_1(B_0)\operatorname{ch}[S^1\times \boldsymbol{C};\ a_k]$$
$$= \left[-\frac{d\xi}{2\pi}\right]\left[\frac{kdx}{2\pi}\right][S^1\times S^1] = \frac{k}{4\pi^2}\int\int dx\wedge d\xi = k.$$

This completes the proof of the equality sf $= \nu$(index).

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