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PARAMETER SHIFTING AND COMPLETE FAMILIES OF SOLUTIONS TO A SINGULAR PARABOLIC EQUATION

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1. Introduction

The singular parabolic equation

$$xu_{xx}+\lambda u_x = xu_t, \tag{1.1}$$

where λ is a real parameter, is known as the generalized heat equation (see e.g. [4], [5]). It is a special feature of this equation that it can be studied by using shifting procedures. These procedures allow to transfer a result known for a certain value λ_0 of the parameter (in many cases $\lambda_0 = 0$) to another value $\lambda \neq \lambda_0$. D. Colton [1] and the author [7] used methods of this type for the discussion of the Cauchy problem. But as it was shown in [3], these ideas can also be used for the construction of complete families of solutions to the generalized heat equations. However, the investigations in [3] also show that a deterioration of results can be a consequence of such shifts. So the result of [3] is only an approximation theorem of compact subdomains. In this paper, a combination of a parameter shifting with direct approximation steps leads to results free of that restriction. Moreover, the problem is studied for values of the parameter which cannot be covered by the method used in [3].

2. Lemmas

The lemmas fall into two groups. The first group includes an approximation result for the related equation

$$x^2u_{xx}+xu_x-n^2u = x^2u_t \qquad (n \in \boldsymbol{N}) \tag{2.1}$$

and the second group gives a base for the reduction of the original problem to the approximation result for equation (2.1).

LEMMA 1. (a) *Let J_n ($n \in \boldsymbol{N}$) be the Bessel function of the first kind of order n and $\mu > 0$ be a zero of J_n. Then*

$$u(x,t) = e^{-\mu^2 t} J_n(\mu x) \tag{2.2}$$

is a solution of (2.1) *for* $(x,t) \in \boldsymbol{R} \times (0,\infty)$ *and we have*

$$u(-1,t) = u(1,t) = 0.$$

(b) *Let u be the function in* (2.2). *Then u can be represented in the form*

$$u(x,t) = x^n \int_0^1 (1-y^2)^{n-1/2} v(xy,t)\,dy, \tag{2.3}$$

where v is a solution of the heat equation.

Proof. The first part can be proved by a simple calculation. The second part follows from the representation formula

$$J_\nu(z) = c_\nu z^\nu \int_0^1 (1-\zeta^2)^{\nu-1/2} \cos(z\zeta)\,d\zeta \qquad (\nu > 1/2,\ c_\nu \text{ some constants})$$

for Bessel functions (see [6]) and the fact that

$$v(x,t) = c_n \mu^n e^{-\mu^2 t} \cos(\mu x)$$

is a solution of the heat equation.

In the following we use the notation $P_{k,\nu}$ for the so-called generalized heat polynomials (see [4], [5]) defined by

$$P_{k,\nu}(x,t) = \sum_{j=0}^{k} 2^{2j} \binom{k}{j} \frac{\Gamma(\nu+k+1/2)}{\Gamma(\nu+k-j+1/2)} x^{2k-2j} t^j$$

for $k = 0,1,2,\ldots$, $\nu > 0$. It is easy to see that any function of the form

$$u(x,t) = x^n P_{k,n+1/2}(x,t)$$

is a solution of (2.1).

LEMMA 2. *Take the function*

$$\varphi(t) = \sum_{j=0}^{m} \beta_j t^j \qquad (\beta_j \in \boldsymbol{R}). \tag{2.4}$$

Then there exists a solution u of (2.1) *of the form*

$$u(x,t) = x^n \sum_{k=0}^{m} a_k P_{k,n+1/2}(x,t) \qquad (a_k \in \boldsymbol{R})$$

satisfying the conditions

$$u(1,t) = \varphi(t), \quad u(-1,t) = (-1)^n \varphi(t).$$

Proof. From equation (2.1) and from the form of u we see that the coefficients $a_0, \ldots, a_m$ have to satisfy the linear system

$$\sum_{k=j}^{m} a_k 2^{2j} \binom{k}{j} \frac{(n+k)!}{(n+k-j)!} = \beta_j \qquad (j = 0,1,\ldots,m).$$

They are uniquely determined by this system because the corresponding matrix is triangular with non-vanishing elements on the main diagonal.

LEMMA 3. *Let φ be as in* (2.4) *and let*

$$\psi(x) = x^n \sum_{i=0}^{r} \gamma_i x^{2i} \qquad (\gamma_i \in \boldsymbol{R},\ n \in \boldsymbol{N})$$

with $\varphi(0) = \psi(1)$. Then for given $T > 0$ and $\varepsilon > 0$ there exists a solution of (2.1) *of the form*

$$u(x,t) = x^n \sum_{k=0}^{s} a_k P_{k,n+1/2}(x,t) \qquad (s \in \boldsymbol{N},\ a_k \in \boldsymbol{R}) \tag{2.5}$$

such that

$$|\varphi(t) - u(1,t)| < \varepsilon \quad \text{for} \quad t \in [0,T] \tag{2.6}$$

and

$$|\psi(x) - u(x,0)| < \varepsilon \quad \text{for} \quad x \in [-1,1]. \tag{2.7}$$

Proof. Let u_1 be the solution of (2.1) constructed as in Lemma 2. Define $f(x) = \psi(x) - u_1(x,0)$. Then the function f is of the same form as ψ and $f(1) = 0 = f(-1)$. In virtue of results on Fourier–Bessel-series (see [6]) it follows that f can be approximated like this:

$$\Big| f(x) - \sum_{j=1}^{p} a_j J_n(\mu_j x) \Big| < \varepsilon/2 \quad \text{for} \quad x \in [-1,1]$$

($\mu_j > 0$ denoting the zeros of J_n, $a_j \in \boldsymbol{R}$). Now we define

$$u_2(x,t) = \sum_{j=1}^{p} a_j \exp(-\mu_j^2 t) J_n(\mu_j x).$$

This is a solution of (2.1) and in view of Lemma 1 we can write it in the form

$$u_2(x,t) = x^n \int_0^1 (1-y^2)^{n-1/2} v(xy,t)\,dy, \tag{2.8}$$

where v is the following solution of the heat equation:

$$v(x,t) = c_n \sum_{j=1}^{p} a_j \mu_j^n \exp(-\mu_j^2 t) \cos(\mu_j x).$$

From the results of Colton (see [2]) it follows that v can be approximated uniformly by a linear combination of heat polynomials h_{2k} of even order (see the remark in the bracket below). Therefore we can find real numbers b_k, $k = 0, 1, \ldots, q$, such that

$$\left|v(x,t)-\sum_{k=0}^{q} b_k h_{2k}(x,t)\right| < \varepsilon/2, \qquad (x,t)\in[-1,1]\times[0,t], \tag{2.9}$$

where h_{2k} are defined by

$$h_{2k}(x,t) = (2k)!\sum_{j=0}^{k}\frac{x^{2k-2j}t^j}{(2k-2j)!\,j!}.$$

(We can assume that all heat polynomials appearing in the linear combination have even order, because v is an even function with respect to x.)

The heat polynomials are connected with the generalized heat polynomials by the formula

$$P_{k,\nu}(x,t) = \varrho_{k,\nu}\int_0^1 (1-y^2)^{\nu-1} h_{2k}(xy,t)\,dy$$

(where $\varrho_{k,\nu}$ are some constants, see [1]).

From (2.8) and (2.9) it follows that

$$\left|u_2(x,t)-x^n\sum_{k=0}^{q}\varrho_{k,n+1/2}^{-1} b_k P_{k,n+1/2}(x,t)\right| < \varepsilon/2,$$
$$(x,t)\in[-1,1]\times[0,T].$$

Finally, the function

$$u_3(x,t) = u_1(x,t)+x^n\sum_{k=0}^{q}\varrho_{k,n+1/2}^{-1} b_k P_{k,n+1/2}(x,t)$$

is a solution of the form (2.5) of equation (2.1) and satisfies (2.6) and (2.7). In fact,

$$|\varphi(t)-u_3(1,t)| = \left|\sum_{k=0}^{q}\varrho_{k,n+1/2}^{-1} b_k P_{k,n+1/2}(1,t)\right| < \varepsilon/2,$$

$$|\psi(x)-u_3(x,0)| = \left|f(x)-x^n\sum_{k=0}^{q}\varrho_{k,n+1/2}^{-1} b_k P_{k,n+1/2}(x,0)\right|$$
$$\leqslant |f(x)-u_2(x,0)|+\varepsilon/2 < \varepsilon.$$

Remark. A result on the approximation of solutions of (2.1) by solutions of the form (2.5) can be proved by using Lemma 3 and a maximum principle argument.

LEMMA 4. *Let $u\in C^k\big((-1,1)\times(0,T)\big)$ $(k\geqslant 4)$ be a solution of* (1.1) *with $\lambda\neq 0$ and define $v(x,t) = \frac{1}{x}u_x(x,t)$. Then*

(a) $v\in C^{k-2}\big((-1,1)\times(0,T)\big)$,

(b) $xv_{xx}+(\lambda+2)v_x = xv_t$.

Proof. (a) follows from (1.1), namely we have:

$$\lambda v(x,t) = u_{xx}(x,t)-u_t(x,t).$$

Part (b) can be proved by showing that

$$x(u_{xx}-u_t)_{xx}+(\lambda+2)(u_{xx}-u_t)_x-x(u_{xx}-u_t)_t = 0.$$

But this follows from (1.1) by a simple calculation.

LEMMA 5. *Let $u\in C^{2n}\big((-1,1)\times(0,T)\big)$ be a solution of*

$$xu_{xx}-(2n-1)u_x = xu_t \qquad (n\in \boldsymbol{N}). \tag{2.10}$$

Then $D_t^n u(0,t) = 0$ for $t\in(0,T)$, where D_t^n denotes the partial n-th derivative with respect to t.

Proof. For $n = 1$ the assertion follows immediately: $u_x(0,t) = 0$ and therefore $\left(\frac{1}{x}u_x\right)_{x=0} = u_{xx}(0,t)$. From (2.10) it follows that $u_t(0,t) = 0$. For $n>1$ the assertion is proved by induction.

If $u\in C^{2n+2}\big((-1,1)\times(0,T)\big)$ is a solution of $xu_{xx}-(2n+1)u_x = xu$ then $-2n\left(\frac{1}{x}u_x\right)(0,t) = u_t(0,t)$. Define $v(x,t) = \frac{1}{x}u_x(x,t)$. Then we may assume $D_t^n v(0,t) = 0$ and it follows that $D_t^{n+1}u(0,t) = -2nD_t^n v(0,t) = 0$.

LEMMA 6. *Let $u\in C^{2n}\big((-1,1)\times(0,T)\big)$ be a solution of* (2.10). *Then there exists a polynomial solution u_1 of* (2.10) *such that $(u-u_1)(0,t) = 0$.*

Proof. Define

$$p_{k,n}(x,t) = \sum_{j=0}^{k}2^{2j}\binom{k}{j}(-1)^j\frac{(n-k-1+j)!}{(n-k-1)!}x^{2k-2j}t^j \tag{2.11}$$

for $n\in\boldsymbol{N}$, $k = 0,1,\ldots,n-1$. A simple calculation shows that these polynomials are solutions of (2.10). Since Lemma 5 shows that $u(0,t)$ is a polynomial of degree $n-1$, one can write $u(0,t) = \sum_{k=0}^{n-1}a_k t^k$.

We now define

$$(2.12)\qquad u_1(x,t) = \sum_{k=0}^{n-1} (-1)^k 2^{-2k} \frac{(n-k-1)!}{(n-1)!} \alpha_k p_{k,n}(x,t).$$

Since $p_{k,n}(0,t) = (-1)^k 2^{2k} \dfrac{(n-1)!}{(n-k-1)!} t^k$, we have $u(0,t) = u_1(0,t)$.

LEMMA 7. *Let $u \in C^{2n}\big((-1,1)\times(0,T)\big)$ be a solution of (2.10) with $u(0,t) = 0$. Then $D_x^j u(0,t) = 0$ for $j = 0,1,\ldots,2n-1$, $t\in(0,T)$, where D_x^j denotes the j-th partial derivative with respect to x.*

Proof. For $j = 0,1$ the assertion is trivial. For $n\geqslant 2$ and $j=2$ we can prove it by the following argument: Since $u_{xx}(0,t)-(2n-1)u_{xx}(0,t) = u_t(0,t)$, it follows from the assumptions of our lemma that $u_{xx}(0,t) = 0$. The rest again can be proved by induction.

Let $u\in C^{2n+2}\big((-1,1)\times(0,T)\big)$ be a solution of $xu_{xx}-(2n+1)u_x = xu_t$ with $u(0,t) = 0$. We define $v(x,t) = \dfrac{1}{x}u_x(x,t)$. Since $u_{xx}(0,t) = 0$, then $v(0,t) = 0$ and therefore we may assume $D_x^j v(0,t) = 0$ for $j = 0,1,\ldots,2n-1$, $t\in(0,T)$. But this implies $D_x^j(xv)(0,t) = 0$ for $j=0,1,\ldots,2n$ and $D_x^j u(0,t) = 0$ for $j = 0,1,\ldots,2n+1$.

3. The main result

The polynomials $p_{k,n}$ defined in (2.11) are solutions of (2.10), the same about the polynomials $q_{k,n}$ defined by $q_{k,n}(x,t) = x^{2n}P_{k,n+1/2}(x,t)$ for $k = 0,1,2,\ldots$ Since all polynomial solutions of (2.10) have to be even with respect to x (see [7]), the desired result on a complete family of solutions must concern approximation of solutions of (2.10) by linear combinations of the polynomials $p_{k,n}$. $(k = 0,1,\ldots,n-1)$ and $q_{k,n}$ $(k = 0,1,2,\ldots)$.

THEOREM 1. *Let $u\in C^{2n}\big((-1,1)\times(0,T)\big)\cap C([-1,1]\times[0,T])$ be a solution of (2.10) which is even with respect to x. Then for every $\varepsilon>0$ there exist $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\in\boldsymbol{R}$, $m\in\boldsymbol{N}$ and $\beta_0,\beta_1,\ldots,\beta_m\in\boldsymbol{R}$ such that*

$$\Big|u(x,t)-\sum_{k=0}^{n-1}\alpha_k p_{k,n}(x,t) - x^{2n}\sum_{k=0}^{m}\beta_k P_{k,n+1/2}(x,t)\Big| < \varepsilon$$

for $(x,t)\in[-1,1]\times[0,T]$.

Proof. Since the solution u_1 in Lemma 6 described by (2.12) is a linear combination of the polynomials $p_{k,n}$, it is sufficient to prove the theorem for solutions u with $u(0,t) = 0$. In fact, we shall show that in this case u can be approximated as follows:

$$(3.1)\qquad \Big|u(x,t)-x^{2n}\sum_{k=0}^{m}\beta_k P_{k,n+1/2}(x,t)\Big| < \varepsilon,\qquad (x,t)\in[-1,1]\times[0,T].$$

For solutions u with $u(0,t) = 0$ we can use the estimate

$$(3.2)\qquad |u(x,t)|\leqslant \max_{(x,t)\in B}|u(x,t)|\quad\text{for}\quad (x,t)\in[-1,1]\times[0,T]$$

with $B = ([-1,1]\times\{0\})\cup(\{-1\}\times(0,T])\cup(\{1\}\times(0,T])$.

There exists $\delta>0$ such that $|u(x,\delta)-u(x,0)|<\varepsilon/8$ for $x\in[-1,1]$. Since $D_x^j u(0,\delta) = 0$ for $j = 0,1,\ldots,2n-1$, we know that $x^{-2n}u(x,\delta)$ can be defined as a continuous function also at $x = 0$. Now, using the Weierstraß approximation theorem we approach this function by a polynomial f. We may assume that this polynomial is an even function. Therefore

$$|u(x,0)-x^{2n}f(x)|<\varepsilon/4\quad\text{for}\quad x\in[-1,1]$$

with $f(x) = \sum_{i=0}^{r}\gamma_i x^{2i}$.

The next step is to approximate $u(1,t)$ by a polynomial g. Suppose that

$$|u(1,t)-g(t)|<\varepsilon/8\quad\text{for}\quad t\in[0,T]$$

with $g(t) = \sum_{j=0}^{p} b_j t^j$. If we define $\varphi(t) = g(t)+f(1)-g(0)$, then we get

$$|u(1,t)-\varphi(t)| < |u(1,0)-g(0)|+|f(1)-u(1,0)|+\varepsilon/8<\varepsilon/2.$$

Write

$$\psi(x) = x^n f(x) = x^n\sum_{i=0}^{r}\gamma_i x^{2i}.$$

We get $\varphi(0) = f(1) = \psi(1)$ and using Lemma 3 we find a solution v of (2.1) of the form

$$v(x,t) = x^n\sum_{k=0}^{m}\beta_k P_{k,n+1/2}(x,t)$$

such that $|\varphi(t)-v(1,t)|<\varepsilon/2$ for $t\in[0,T]$ and $|\psi(x)-v(x,0)|<\varepsilon/2$ for $x\in[-1,1]$. Put $w(x,t) = x^n v(x,t)$. Then w is a solution of (2.10); moreover,

$$|u(x,0)-w(x,0)|\leqslant|u(x,0)-x^{2n}f(x)|+|x^n\psi(x)-x^n v(x,0)|<3\varepsilon/4$$

for $x\in[-1,1]$ and

$$|u(1,t)-w(1,t)|\leqslant|u(1,t)-\varphi(t)|+|\varphi(t)-v(1,t)|<\varepsilon$$

for $t \in [0, T]$. Since $u-w$ is a solution of (2.10) with $(u-w)(0, t) = 0$, the estimate (3.2) holds for $u-w$ and if follows that $|(u-w)(x, t)| < \varepsilon$ for $(x, t) \in [-1, 1] \times [0, T]$. But by the definition of w this estimate is identical with (3.1).

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ON THE SPECTRAL FLOW AND THE INDEX THEOREM FOR FLAT VECTOR BUNDLES

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Let X be a smooth closed Riemannian manifold of dimension n and E a smooth Hermitian vector bundle over X. If $A\colon C^\infty(X, E) \to C^\infty(X, E)$ is a self-adjoint elliptic operator of order $m > 0$, then A has a discrete spectrum $\{\lambda_k\}_{k\in N}$ with $|\lambda_k| \to \infty$ as $k \to \infty$ ([6], part XII, Theorem 14). In [1] Atiyah, Patodi and Singer introduced the function

$$\eta(A;\, s) = \sum_{\lambda_k \neq 0} \operatorname{sign}(\lambda_k) \cdot |\lambda_k|^{-s},$$

holomorphic for $\operatorname{Re} s > n/2m$. More precisely, it is known ([1], [4]) that this function has a meromorphic extension onto the whole complex plane with only simple poles and that there is no pole at $s = 0$. This function has then been used to define an invariant $\operatorname{ind}(A;\, D_1, D_0)$ depending, for fixed flat connections D_1, D_0, only on the operator A.

In this paper we compute this invariant in a special case, where it has a clear interpretation. In particular, we obtain an expression for the index of the family of self-adjoint Fredholm operators over S^1 by an integer, which is a spectral invariant of the family, the so-called spectral flow. The main properties of this invariant, which were mentioned in [1], are also proved.

1. Extension of a pseudo-differential operator to an auxiliary flat vector bundle

DEFINITION 1.1. Let V be a smooth Hermitian vector bundle over a smooth manifold X. V is called *flat* if it has a trivialisation $\{U_i, \varphi_i\}$ with constant transition functions.

If this is the case, the exterior differential d acting locally on the sections of $U_i \times C^N \overset{\cong}{\to} V/U_i$ extends to a global connection D_V on the