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ADJOINTS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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The present paper was prepared for the Banach Semester "Partial Differential Equations". We describe here some results concerning elliptic boundary value problems for pseudo-differential operators in sense of [10], [5], [2], [7], namely a certain type of adjoints of such operators. We derive here Green formulas generalizing classical Green formulas ([4]) and obtain a Fredholm alternative. Specially we calculate the index element of the adjoint of a symbol as a Fredholm family over the cosphere bundle on the boundary of the given manifold. Adjoints of operators of the type discussed in [10], [5], [2], [7] have been considered also in [3], [1], [9], [8]. They are useful also for investigation of overdetermined and underdetermined boundary value problems and elliptic complexes on manifolds with boundary. First we recall some notations and definitions. We denote by $\mathfrak{G}$ the class of all operators of the form

$$\mathscr{A} = \begin{pmatrix} r^+A+r'B & K \\ r'T & Q \end{pmatrix} : \begin{matrix} C^\infty(X, E_0) \\ \oplus \\ C^\infty(Y, J_0) \end{matrix} \to \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(Y, J_1) \end{matrix}. \tag{1}$$

Here X is a compact C^∞-manifold with boundary Y ($n = \dim X$, $n-1 = \dim Y$, $\Omega = X \setminus Y$) and E_j (J_j) are complex vector bundles over X (Y), $j = 0, 1$. By r^+A we denote a pseudo-differential operator over X with the transmission property (r^+ is, roughly speaking, the restriction to X of a distribution over some neighbouring manifold of X), $r'B$ is a Green operator, K a potential operator, $r'T$ a trace operator, and Q is a pseudo-differential operator over Y (r' is the restriction to the boundary Y). We suppose that the operators are described by complete symbols $a(x, \xi)$, $b(x', \xi', \nu, \tau)$, $k(x', \xi', \nu)$, $t(x', \xi', \nu)$, $q(x', \xi')$. By (x, ξ) are denoted local coordinates in T^*X. Near Y we consider local coordinates of the form $(x', x_n) = x$, $x' = (x_1, \ldots, x_{n-1})$ where X corresponds to $x_n = 0$. Then we have

$\xi = (\xi', \nu)$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $\nu \in \boldsymbol{R}^1$. The variable $\tau \in \boldsymbol{R}^1$ plays an analogous role as ν. We assume that the complete symbols have asymptotic expansions with respect to homogeneous symbols of integer orders (the exicision function for the definition of b, k, t, q depends only on ξ'). The homogeneous principal symbols are denoted by σ_A, σ_B, σ_K, σ_T and σ_Q respectively and the corresponding degrees of homogeneity are denoted by

$$a = \operatorname{ord}\sigma_A, \quad a-1 = \operatorname{ord}\sigma_B, \quad \lambda = \operatorname{ord}\sigma_K, \tag{2}$$
$$\gamma = \operatorname{ord}\sigma_T, \quad 1-a+\lambda+\gamma = \operatorname{ord}\sigma_Q,$$

($a \in \boldsymbol{Z}$). We use here the same notations as in [6], [8], specially the spaces H, H^+, H_0^-, H', where $H^+ = F_{x_n}(\mathscr{S}(\overline{\boldsymbol{R}_+}))$, $H_0^- = F_{x_n}(\mathscr{S}(\overline{\boldsymbol{R}_-}))$, H' = space of polynomials in ν and $H = H^+ \oplus H_0^- \oplus H'$, $H^- = H_0^- \oplus H'$, F_{x_n} the Fourier transform on the x_n-axis, $\mathscr{S}(\overline{\boldsymbol{R}_\pm})$ the space of functions of the form $\theta(x_n)u(x_n)$ and $(1-\theta(x_n))u(x_n)$ respectively with $u \in \mathscr{S}(\boldsymbol{R})$, $\theta(x_n) = 1$ for $x_n \geq 0$, $= 0$ for $x_n < 0$. The type of the symbols as degree with respect to ν is defined in [2]. We put $E' = E|_Y$, $E^+ = E' \otimes H^+$ if E is a bundle over X. It is supposed that on X some Riemannian metric is fixed. On Y we consider the induced metric. By S^*Y we denote the unit cosphere bundle over Y and by $p\colon S^*Y \to Y$ the canonical projection. Besides the interior symbol $\sigma_\Omega(\mathscr{A}) = \sigma_A$ of $\mathscr{A} \in \mathfrak{G}$ we use also the boundary symbol

$$\sigma_Y(\mathscr{A}) = \begin{pmatrix} \Pi^+\sigma_A + \Pi'\sigma_B & \cdot\sigma_K \\ \Pi'\sigma_T & \sigma_Q \end{pmatrix} : \begin{matrix} p^*E_0^+ \\ \oplus \\ p^*J_0 \end{matrix} \to \begin{matrix} p^*E_1^+ \\ \oplus \\ p^*J_1 \end{matrix}. \tag{3}$$

([1], [6]). If $\mathscr{A}$ is elliptic, we have for arbitrary σ_B a Fredholm family over S^*Y

$$\Pi^+\sigma_A + \Pi'\sigma_B\colon p^*E_0^+ \to p^*E_1^+, \tag{4}$$

and

$$\operatorname{ind}_{S^*Y}(\Pi^+\sigma_A + \Pi'\sigma_B) = [p^*J_1] - [p^*J_0] \tag{5}$$

in $p^*K(Y) \subset K(S^*Y)$, independently of σ_B. Any $\mathscr{A} \in \mathfrak{G}$ with the orders (2) has a continuous extension as operator between corresponding Sobolev spaces

$$\mathscr{A}\colon H^{s_0}(X, E_0) \oplus H^{t_0}(Y, J_0) \to H^{s_1}(X, E_1) \oplus H^{t_1}(Y, J_1) \tag{6}$$

with $s_0 \in \boldsymbol{R}$ sufficiently large and

$$s_1 = s_0 - a, \quad t_0 = s_1 + \lambda + \tfrac{1}{2}, \quad t_1 = s_0 - \gamma - \tfrac{1}{2}. \tag{7}$$

In the following we suppose that s_0, s_1 are nonnegative integers. We assume that in the bundles E_j, J_j are given fixed Hermitian metrics ($j = 0, 1$).

In E_j are then also induced Hermitian metrics. We have then Hermitian scalar products in the space $H^0(X, E_j) \oplus H^0(Y, J_j)$, which we denote by $\varphi(\cdot,\cdot)$ for $j = 0$ and $\psi(\cdot,\cdot)$ for $j = 1$. By φ' and ψ' we denote corresponding scalar products in the fibres of $p^*(E_j^+ \oplus J_j)$ for $j = 0$ and $j = 1$ respectively. For

$$G_j = J_j \oplus \Big(\bigoplus_1^{s_j} E_j'\Big) \tag{8}$$

we denote by ω and χ the Hermitian scalar product in $H^0(X, E_j) \oplus H^0(Y, G_j)$ for $j = 0$ and $j = 1$ respectively. Consider an elliptic symbol

$$\sigma_A\colon \pi^*E_0 \to \pi^*E_1 \tag{9}$$

($\pi\colon T^*X \setminus 0 \to X$) and suppose $\sigma_A = \sigma_\Omega(\mathscr{A})$ for some elliptic $\mathscr{A} \in \mathfrak{G}$ of the form (1). Our aim is to construct an elliptic operator

$$\mathscr{A}^*\colon C^\infty(X, E_1) \oplus C^\infty(Y, G_1) \to C^\infty(X, E_0) \oplus C^\infty(Y, G_0) \tag{10}$$

with $\sigma_\Omega(\mathscr{A}^*) = \sigma_A^*$, where σ_A^* is the adjoint of σ_A with respect to the Hermitian metrics in E_0 and E_1 respectively. We use here the following fact proved in [1]. Suppose that we are given an operator $\mathscr{B} \in \mathfrak{G}$, where the operator part in the left upper corner has type 1 and the trace operator the type 0. If $\mathscr{B}$ is a continuous operator

$$\mathscr{B}\colon H^0(X, E_0) \oplus H^0(Y, J_0) \to H^0(X, E_1) \oplus H^0(Y, J_1) \tag{11}$$

then its adjoint $\mathscr{B}^*$ in the sense of

$$\psi(\mathscr{B}f, g) = \varphi(f, \mathscr{B}^*g) \tag{12}$$

belongs to $\mathfrak{G}$, and one has $\sigma_\Omega(\mathscr{B}^*) = (\sigma_\Omega(\mathscr{B}))^*$ with respect to the Hermitian metrics and

$$\psi'(\sigma_y(\mathscr{B})h, l) = \varphi'(h, \sigma_y(\mathscr{B}^*)l)$$

for all $h \in p^*(E_0^+ \oplus J_0)_{(x',\xi')}$, $l \in p^*(E_1^+ \oplus J_1)_{(x',\xi')}$. Ellipticity of $\mathscr{B}$ implies ellipticity of $\mathscr{B}^*$. In [2] is proved the existence of elliptic operators $\mathscr{L}^{s_j,t_j} \in \mathfrak{G}$ ($j = 0, 1$), defining isomorphisms

$$\mathscr{L}^{s_j,t_j} = \begin{pmatrix} (r^+\Lambda_{E_j}^-)^{s_j} & 0 \\ 0 & R_j \end{pmatrix} : \begin{matrix} H^{s_j}(X, E_j) \\ \oplus \\ H^{t_j}(Y, J_j) \end{matrix} \to \begin{matrix} H^0(X, E_j) \\ \oplus \\ H^0(Y, J_j) \end{matrix}. \tag{13}$$

Near Y the interior symbol of $\Lambda_{E_j}^-$ has the form

$$l^-(\xi) = (\delta(\xi) - i\nu)\cdot 1_{\pi^*E_j}$$

where $\delta(\xi) = \chi(|\xi'|/|\xi|)|\xi'|$ and $\chi \in C^\infty(\boldsymbol{R})$ real, $0 \leq \chi \leq 1$, $\chi(t) = 0$ for $t < \varepsilon$, $\chi(t) = 1$ for $t > 2\varepsilon$, $\varepsilon > 0$ sufficiently small.

From now on we suppose

(14) $$\operatorname{ord}_\nu \sigma_T^i \leqslant s_0 - 1$$

(if $t \sim \sum_{i=0}^{\infty} \sigma_T^i$, $\sigma_T^0 = \sigma_T$), that means the type of $r'T$ is not greater than s_0.

Then the operator

(15) $$\mathscr{B} = \mathscr{L}^{s_1,t_1}\mathscr{A}(\mathscr{L}^{s_0,t_0})^{-1}$$

satisfies the condition for the existence of $\mathscr{B}^*$ mentioned above. We have near Y

$$\sigma_\Omega(\mathscr{B}) = \big(l^-(\xi)\big)^{s_1} 1_{\pi^*E_1}\sigma_\Omega(\mathscr{A})\big(l^-(\xi)\big)^{-s_0} 1_{\pi^*E_0}.$$

From this follows near Y

$$\sigma_\Omega(\mathscr{B}^*) = \big(l^+(\xi)\big)^{-s_0}\cdot 1_{\pi^*E_0}\sigma_\Omega^*(\mathscr{A})\big(l^+(\xi)\big)^{s_1} 1_{\pi^*E_1},$$

$l^+(\xi) = \delta(\xi) + iv$.

We also use the following

LEMMA 1. *For arbitrary $s_j, q_j \in \mathbf{Z}$, $r_j = \frac{1}{2}r_j'$, $r_j' \in \mathbf{Z}$, $q_j, r_j \geqslant 0$ there exist isomorphisms $\mathscr{N}_j \in \mathfrak{G}$*

(16) $$\mathscr{N}_j\colon \begin{matrix} H^{q_j+s_j}(X, E_j) \\ \oplus \\ H^{q_j+s_j}(Y, J_j)\end{matrix} \to \begin{matrix} H^{q_j}(X, E_j) \\ \oplus \\ H^{r_j}(Y, G_j)\end{matrix}$$

with

(17) $$\sigma_\Omega(\mathscr{N}_j) = \big(l^+(\xi)\big)^{s_j} 1_{\pi^*E_j}$$

near Y $(j = 0, 1)$.

Thus $\mathscr{N}_j$ is elliptic. One can suppose that the trace operators in $\mathscr{N}_j$ have type 0. A proof of Lemma 1 is given in [8].

Next we define using (15)

(18) $$\mathscr{A}^* = \mathscr{N}_0\mathscr{B}^*\mathscr{N}_1^{-1}$$

as a composition

$$\begin{matrix} H^{s_0}(X, E_1) \\ \oplus \\ H^{s_0+r_1-q_1}(Y, G_1)\end{matrix} \xrightarrow{\mathscr{N}_1^{-1}} \begin{matrix} H^{s_0+s_1}(X, E_1) \\ \oplus \\ H^{s_0+s_1}(Y, J_1)\end{matrix} \xrightarrow{\mathscr{B}^*} \begin{matrix} H^{s_0+s_1}(X, E_0) \\ \oplus \\ H^{s_0+s_1}(Y, J_0)\end{matrix} \xrightarrow{\mathscr{N}_0} \begin{matrix} H^{s_1}(X, E_0) \\ \oplus \\ H^{s_1+r_0-q_0}(Y, G_0)\end{matrix}.$$

Then we have obvious by

$$\sigma_\Omega(\mathscr{A}^*) = \sigma_\Omega^*(\mathscr{A}).$$

Ellipticity of $\mathscr{A}$ implies ellipticity of $\mathscr{A}^*$.

Applying (5) we obtain

PROPOSITION 2. *Let s_0, s_1 be integers. Suppose $\sigma_A\colon \pi^*E_0 \to \pi^*E_1$ is an elliptic symbol with $a = \operatorname{ord}\sigma_A = s_0 - s_1$ and $\mathscr{A} \in \mathfrak{G}$ elliptic with $\sigma_\Omega(\mathscr{A}) = \sigma_A$. Then*

(19) $$\operatorname{ind}_{S^*Y}\Pi^+\sigma_A^* = -\operatorname{ind}_{S^*Y}\Pi^+\sigma_A + \operatorname{sgn}(s_0 - s_1)\Big[\bigoplus_1^{|s_0-s_1|} E'\Big]$$

$(E' = E_0' \cong E_1')$.

An element $\mathscr{A} \in \mathfrak{G}$ is called *overdetermined* (underdetermined) *elliptic boundary value problem*, if $\sigma_\Omega(\mathscr{A})$ and $\sigma_Y(\mathscr{A})$ are injective (surjective) mappings ([9], [8]). We obtain then the following

Remark 3. Suppose $\mathscr{A} \in \mathfrak{G}$ is an overdetermined (underdetermined) elliptic boundary value problem, satisfying condition (14). Then $\mathscr{A}^*$ defined by formula (18) is an underdetermined (overdetermined) elliptic boundary value problem.

In [8] are also considered adjoints of elliptic complexes on manifolds with boundary. Using adjoints of complexes of operators acting in L^2-spaces, one can construct C^∞-parametrices of elliptic complexes on manifolds with boundary.

In order to derive Green formulas for elliptic boundary value problems we make some assumptions about the orders. We suppose

(20) $$s_1 \leqslant s_0$$

and consider operators $\mathscr{R}, \mathscr{S} \in \mathfrak{G}$ which generate continuous mappings

(21) $$\mathscr{R}\colon \begin{matrix} H^{s_0}(X, E_1) \\ \oplus \\ H^{p_0}(Y, G_1)\end{matrix} \to \begin{matrix} H^{s_1}(X, E_1) \\ \oplus \\ H^{t_1}(Y, J_1)\end{matrix}, \qquad \mathscr{S}\colon \begin{matrix} H^{s_0}(X, E_0) \\ \oplus \\ H^{t_0}(Y, J_0)\end{matrix} \to \begin{matrix} H^{s_1}(X, E_0) \\ \oplus \\ H^{p_1}(Y, G_0)\end{matrix}$$

where p_j are orders, depending on s_j, q_j, r_j $(j = 0, 1)$. We have then the

THEOREM 4. *Let $\mathscr{A} \in \mathfrak{G}$ be elliptic, satisfying* (14) *and considered as a mapping* (6). *Then there exist operators $\mathscr{R}, \mathscr{S} \in \mathfrak{G}$ as in* (21), *so that*

(22) $$\psi(\mathscr{A}f, \mathscr{R}g) = \omega(\mathscr{S}f, \mathscr{A}^*g)$$

for all $f \in C^\infty(X, E_0) \oplus C^\infty(Y, J_0)$, $g \in C^\infty(X, E_1) \oplus C^\infty(Y, G_1)$. In the case $q_j = 0$, $r_j = 0$ $(j = 0, 1)$ one can set $p_0 = s_1$, $p_1 = s_0$.

Proof. For simplicity we suppose $t_1 = 0$ and $q_j = r_j = 0$ $(j = 0, 1)$. It is then obvious how to treat the general case. In (12) with the definition (15) of $\mathscr{B}$ we set $u = \mathscr{L}^{s_0,t_0}f$, $v = \mathscr{N}_1^{-1}g$. Then we obtain

$$\psi(\mathscr{B}u, v) = \psi(\mathscr{B}\mathscr{L}^{s_0,t_0}f, \mathscr{N}_1^{-1}g) = \psi(\mathscr{L}^{s_1,t_1}\mathscr{A}f, \mathscr{N}_1^{-1}g)$$

and

$$\varphi(u, \mathscr{B}^*v) = \varphi(\mathscr{L}^{s_0,t_0}f, \mathscr{N}_0^{-1}\mathscr{N}_0\mathscr{B}^*\mathscr{N}_1^{-1}g) = \varphi(\mathscr{L}^{s_0,t_0}f, \mathscr{N}_0^{-1}\mathscr{A}^*g).$$

Thus (12) implies

$$\psi(\mathscr{L}^{s_1,t_1}\mathscr{A}f, \mathscr{N}_1^{-1}g) = \varphi(\mathscr{L}^{s_0,t_0}f, \mathscr{N}_0^{-1}\mathscr{A}^*g)$$

for all smooth f, g. Since the operators in $\mathscr{N}_j^{-1}$ have nonnegative orders and types, we obtain

$$\varphi(\mathscr{L}^{s_0,t_0}f, \mathscr{N}_0^{-1}\mathscr{A}^*g) = \omega(\mathscr{S}f, \mathscr{A}^*g)$$

with $\mathscr{S} = (\mathscr{N}_0^{-1})^*\mathscr{L}^{s_0,t_0}$. Moreover

$$\psi(\mathscr{L}^{s_1,t_1}\mathscr{A}f, \mathscr{N}_1^{-1}g) = \chi\big((\mathscr{N}_1^{-1})^*\mathscr{L}^{s_1,t_1}\mathscr{A}f, g\big).$$

Since $(\mathscr{N}_1^{-1})^*\mathscr{L}^{s_1,t_1}$ has a closure as continuous operator

$$(\mathscr{N}_1^{-1})^*\mathscr{L}^{s_1,t_1}\colon H^0(X, E_1)\oplus H^0(Y, J_1) \to H^{s_1}(X, E_1)\oplus H^{s_1}(Y, G_1)$$

we can define its adjoint $\mathscr{R} = \big((\mathscr{N}_1^{-1})^*\mathscr{L}^{s_1,t_1}\big)^*$ and obtain finally

$$\psi(\mathscr{L}^{s_1,t_1}\mathscr{A}f, \mathscr{N}_1^{-1}g) = \psi(\mathscr{A}f, \mathscr{R}g).$$

Thus Theorem 4 is proved.

COROLLARY 5. *Under the above conditions the Fredholm alternative holds in the following form. The equation $\mathscr{A}f = w$ for $w \in C^\infty(X, E_1)\oplus C^\infty(Y, J_1)$ has a solution $f \in C^\infty(X, E_0)\oplus C^\infty(Y, J_0)$ iff $\psi(w, \mathscr{R}g) = 0$ for all $g \in \ker\mathscr{A}^*$.*

Remark 6. From (22) follows

$$\operatorname{ind}\mathscr{A}^* = -\operatorname{ind}\mathscr{A}.$$

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