ADJOINTS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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The present paper was prepared for the Banach Semester "Partial Differential Equations". We describe here some results concerning elliptic boundary value problems for pseudo-differential operators in sense of [10], [9], [2], [7], namely a certain type of adjoints of such operators. We derive here Green formulas generalizing classical Green formulas ([14]) and obtain a Fredholm alternative. Specially we calculate the index element of the adjoint of a symbol as a Fredholm family over the cosphere bundle on the boundary of the given manifold. Adjoints of operators of the type discussed in [10], [9], [2], [7] have been considered also in [3], [11], [9], [8]. They are useful also for investigation of overdetermined and underdetermined boundary value problems and elliptic complexes on manifolds with boundary. First we recall some notations and definitions. We denote by $\mathcal{G}$ the class of all operators of the form

\[
\mathcal{G} = \left\{ \begin{array}{l}
\mathcal{A} = r^*A + r^*B = C^m(X, E_\alpha) \\
\mathcal{B} = \mathcal{C}^{m}(X, E_\beta) \\
\mathcal{C}^{m}(Y, J_\alpha) \\
\mathcal{D} = \mathcal{C}^{m}(Y, J_\beta)
\end{array} \right. 
\]

Here $X$ is a compact $C^\infty$-manifold with boundary $\partial X$, $n = \dim X$, $n - 1 = \dim Y$, $\Omega = X \setminus Y$ and $E_\beta$, $J_\beta$ are complex vector bundles over $X$, $Y$, $\beta = 0, 1$. By $r^*A$ we denote a pseudo-differential operator over $X$ with the transmission property ($r^*$ is, roughly speaking, the restriction to $X$ of a distribution over some neighbouring manifold of $X$), $r^*B$ is a Green operator, $\mathcal{C}$ a potential operator, $r^*C$ a trace operator, and $Q$ is a pseudo-differential operator over $Y$ ($r^*$ is the restriction to the boundary $Y$). We suppose that the operators are described by complete symbols $a(z, \xi)$, $b(z', \xi', z, \xi)$, $h(z', \xi', z', \xi')$, $t(z', \xi', z', \xi')$, $q(z', \xi')$. By $(z, \xi)$ are denoted local coordinates in $X \times X$ near $Y$ we consider local coordinates of the form $(z', \xi_n) = z$, $z' = (z_1, \ldots, z_{n-1})$ where $X$ corresponds to $z_n = 0$. Then we have

\[ \text{[233]} \]

$\mathcal{G} = \text{Banach Center t. X}$
\[ \xi = (t', \nu), \xi' = (\xi_1, \ldots, \xi_n), \nu \in \mathbb{R}^1. \] The variable \( \nu \in \mathbb{R}^1 \) plays an analogous role as \( \nu \). We assume that the complete symbols have asymptotic expansions with respect to homogeneous symbols of integer orders (the excision function for the definition of \( b, h, t, g \) depends only on \( \xi' \)). The homogeneous principal symbols are denoted by \( \sigma_{a}, \sigma_{b}, \sigma_{x}, \sigma_{p} \) and \( \sigma_{q} \) respectively and the corresponding degrees of homogeneity are denoted by

\[ a = \text{ord} \sigma_{a}, \quad a - 1 = \text{ord} \sigma_{b}, \quad \lambda = \text{ord} \sigma_{x}, \quad \gamma = \text{ord} \sigma_{p}, \quad \lambda + \gamma = \text{ord} \sigma_{q}. \]

(a \in \mathcal{Z}). We use here the same notations as in [6], [8], specially the spaces \( H, H^{+}, H^{-}, H^{0}, \) where \( H^{+} = H^{+} \otimes \mathcal{S}(\mathbb{R}^{n}), H^{-} = H^{-} \otimes \mathcal{S}(\mathbb{R}^{n}), H^{0} = H^{0} \otimes H^{0} \), space of polynomials in \( v \) and \( H = H^{+} \otimes H^{-} \otimes H^{0}, H^{+} = H^{+} \otimes H^{0}, H^{-} = H^{-} \otimes H^{0}, F_{\sigma_{a}}, \sigma_{b}, \sigma_{x}, \sigma_{p} \) the Fourier transform on the \( a_{0} \)-axis, \( \mathcal{S}(\mathbb{R}^{n}) \) the space of functions of the form \( \theta(a_{0})u(a_{0}) \) and \( (1 - \theta(a_{0}))u(a_{0}) \) respectively with \( u \in \mathcal{S}(\mathbb{R}^{n}), \theta(a_{0}) = 1 \) for \( a_{0} > 0, = 0 \) for \( a_{0} < 0 \). The type of the symbols as degree with respect to \( v \) is defined in [2]. We put \( \mathcal{E} = \mathcal{E}_{[v]}, \mathcal{E}^{-} = \mathcal{E}^{+} \otimes H^{+} \) if \( \mathcal{E} \) is a bundle over \( X \). It is supposed that on \( X \) some Riemannian metric is fixed. On \( X \) we consider the induced metric. By \( S^{X}Y \) we denote the unit cosphere bundle over \( Y \) and by \( p: S^{X}Y \rightarrow Y \) the canonical projection. Besides the interior symbol \( \sigma_{a}(\mathcal{A}) = \sigma_{a} \) of \( \mathcal{A} \in \mathcal{G} \) we use also the boundary symbol

\[ \sigma_{y}(\mathcal{A}) = \left( \begin{array}{cc} \Pi^{+} \sigma_{a} + \Pi^{+} \sigma_{b} - \sigma_{x} & \mathcal{P}^{+} \mathcal{E}^{+}_{Y} \to \mathcal{P}^{+} \mathcal{E}^{+}_{Y} \bigoplus \mathcal{P}^{\mathcal{J}} \mathcal{E}^{+} \bigoplus \mathcal{P}^{\mathcal{J}} \mathcal{E}^{+} \\ \Pi^{+} \sigma_{a} & \sigma_{x} \end{array} \right). \]

(3)

(11)], [6]. If \( \mathcal{A} \) is elliptic, we have for arbitrary \( \sigma_{a} \) a Fredholm family over \( S^{X}Y \)

\[ \Pi^{+} \sigma_{a} + \Pi^{+} \sigma_{b} - \sigma_{x} = \mathcal{P}^{+} \mathcal{E}^{+}_{Y}, \]

and

\[ \text{ind}_{\sigma_{y}}(\Pi^{+} \sigma_{a} + \Pi^{+} \sigma_{b} - \sigma_{x}) = [\mathcal{P}^{+} \mathcal{J}_{Y}]. \]

(5)

in \( \mathcal{P}^{+} \mathcal{K}(X) \subset \mathcal{K}(S^{X}Y) \), independently of \( \sigma_{y} \). Any \( \mathcal{A} \in \mathcal{G} \) with the orders (2) has a continuous extension as operator between corresponding Sobolev spaces

(6)

\[ \mathcal{A}: H^{0}(X, E_{0} \otimes H^{0}(X, J_{0}) \rightarrow H^{1}(X, E_{0} \otimes H^{1}(X, J_{0}) \]

with \( a_{0} \in \mathbb{R} \) sufficiently large and

\[ s_{a} = a_{0} - a, \quad s_{b} = a_{0} + \lambda + \frac{1}{2}, \quad s_{p} = s_{x} - \gamma - \frac{1}{2}. \]

(7)

In the following we suppose that \( s_{a}, s_{b} \) are nonnegative integers. We assume that in the bundles \( E_{j}, J_{j} \) are given fixed Hermitian metrics \( (j = 0, 1) \).

In \( E_{j} \) are then also induced Hermitian metrics. We have then Hermitian scalar products in the space \( H^{0}(X, E_{j} \otimes H^{j}(X, J_{j}) \), which we denote by \( \psi(\cdot, \cdot) \) for \( j = 0 \) and \( \psi(\cdot, \cdot) \) for \( j = 1 \). By \( \psi^{\prime} \) and \( \psi^{\prime} \) we denote corresponding scalar products in the fibres of \( p^{+}(E_{j}^{+} \otimes J_{j}) \) for \( j = 0 \) and \( j = 1 \) respectively. For

\[ G_{j} = J_{j} \oplus (\oplus J_{j}) \]

(8)

we denote by \( \varphi \) and \( \chi \) the Hermitian scalar product in \( H^{0}(X, E_{j} \otimes H^{j}(X, J_{j}) \) for \( j = 0 \) and \( j = 1 \) respectively. Consider an elliptic symbol

\[ \sigma_{a}(\mathcal{A}) = \mathcal{P}^{a} \mathcal{E}_{Y} \to \mathcal{P}^{a} \mathcal{E}_{Y} \]

(9)

(\pi): \mathcal{E}^{+} \otimes \mathcal{E}^{+} \subset \mathcal{E}^{+} \otimes \mathcal{E}^{+} \subset \mathcal{E}^{+} \otimes \mathcal{E}^{+} \]

and suppose \( \sigma_{a} = \sigma_{a}(\mathcal{A}) \) for some elliptic \( \mathcal{A} \in \mathcal{G} \) of the form (1). Our aim is to construct an elliptic operator

\[ \sigma_{y}(\mathcal{A}) \]

(10)

\[ \mathcal{J}^{\pi}: \mathcal{C}^{0}(X, E_{j} \otimes \mathcal{C}^{0}(Y, G_{j}) \to \mathcal{C}^{0}(X, E_{j} \otimes \mathcal{C}^{0}(Y, G_{j}) \]

with \( \sigma_{a}(\mathcal{A}) = \sigma_{a} \), where \( \sigma_{a} \) is the adjoint of \( \sigma_{a} \) with respect to the Hermitian metrics in \( E_{j} \) and \( E_{j} \) respectively. We have here the following fact proved in [1]: Suppose that we are given an operator \( \mathcal{A} \in \mathcal{G} \), where the operator part in the left upper corner has type 1 and the trace operator the type 0. If \( \mathcal{A} \) is a continuous operator

\[ \mathcal{A}: H^{0}(X, E_{j} \otimes H^{1}(X, J_{j}) \to H^{0}(X, E_{j} \otimes H^{1}(X, J_{j}) \]

then its adjoint \( \mathcal{A}^{*} \) in the sense of

\[ \psi(\mathcal{A}^{*}f, g) = \varphi(f, \mathcal{A}g) \]

belongs to \( \mathcal{G} \), and one has \( \sigma_{a}(\mathcal{A}) = \sigma_{a}(\mathcal{A})^{*} \) with respect to the Hermitian metrics and

\[ \psi(\sigma_{a}(\mathcal{A})h, \lambda) = \varphi(\lambda, \sigma_{a}(\mathcal{A})h) \]

(11)

for all \( h \in p^{+}(E_{j}^{+} \otimes J_{j}) \), \( \lambda \in p^{+}(E_{j}^{+} \otimes J_{j}) \). Ellipticity of \( \mathcal{A} \) implies ellipticity of \( \mathcal{A}^{*} \). In [2] is proved the existence of elliptic operators \( \mathcal{A}^{*} \in \mathcal{G} \) (\( j = 0, 1 \)), defining isomorphisms

\[ \mathcal{A}^{*} p_{j} \mathcal{A} = \left( \begin{array}{cc} (r^{+} A_{j})^{0} & 0 \\ 0 & (r^{-} A_{j})^{0} \end{array} \right) \to \left( \begin{array}{cc} H^{0}(X, E_{j}) & H^{1}(X, E_{j}) \\ 0 & H^{0}(X, E_{j}) \end{array} \right). \]

(12)

Near \( Y \) the interior symbol of \( A_{j}^{*} \) has the form

\[ \Gamma^{*}(t) = \delta(t) - (t - i)J_{\eta^{*}} \]

where \( \delta(t) = \chi(|t|/|t|) \) and \( \chi \in \mathcal{C}^{0}(\mathbb{R}) \), real, \( 0 < \chi < 1, \chi(t) = 0 \) for \( t < e, \chi(t) = 1 \) for \( t > 2e, e > 0 \) sufficiently small.
Applying (5) we obtain

Proposition 2. Let \( s_0, s_1 \) be integers. Suppose \( \sigma_{d \alpha} : \pi^* E_1 \to \pi^* E_1 \) is an elliptic symbol with \( \alpha = \text{ord} \sigma_d = s_0 - s_1 \) and \( \alpha \in \mathfrak{M} \) elliptic with \( \sigma_d(\alpha) = \sigma_d \).

Then

\[
\text{ind}_{\mathfrak{M}} \pi^* \sigma_d = -\text{ind}_{\mathfrak{M}} \pi^* \sigma_d + \text{sgn}(s_0 - s_1) \left( \mathcal{O} \begin{array}{c} E_1 \\ E_1 \end{array} \right) \]

\((\mathcal{F} = E_0 \simeq E_1)\).

An element \( \alpha \in \mathfrak{M} \) is called overdetermined (underdetermined) elliptic boundary value problem, if \( \sigma_d(\alpha) \) and \( \sigma_d(\alpha) \), respectively, are injective (surjective) mappings \([\mathfrak{M}, \mathfrak{M}] \). We obtain then the following

Remark 3. Suppose \( \alpha \in \mathfrak{M} \) is an overdetermined (underdetermined) elliptic boundary value problem, satisfying condition (14). Then \( \mathfrak{M} \) defined by formula (18) is an underdetermined (overdetermined) elliptic boundary value problem.

In [8] are also considered adjoints of elliptic complexes on manifolds with boundary. Using adjoints of complexes of operators acting in \( L^2 \)-spaces, one can construct \( C^0 \)-parametrices of elliptic complexes on manifolds with boundary.

In order to derive Green formulas for elliptic boundary value problems we make some assumptions about the orders. We suppose

\[
s_0 \leq s_1
\]

and consider operators \( \mathcal{A}, \mathcal{F} \in \mathfrak{M} \) which generate continuous mappings

\[
\begin{array}{c}
H^m(X, E_1) \rightarrow H^1(X, E_1) \\
\oplus
\end{array}
\]

with

\[
\sigma_d(\mathcal{A}) = (\pi^*(\mathcal{F}))^m I_{s_0 s_1}
\]

near \( Y \) (\( j = 0, 1 \)).

Thus \( \mathcal{A} \) is elliptic. One can suppose that the trace operators in \( \mathcal{A} \) have type 0. A proof of Lemma 1 is given in [8].

Next we define using (15)

\[
\mathfrak{M} = \mathcal{A} \mathcal{M} \mathfrak{M}^{-1}
\]

as a composition

\[
\begin{array}{c}
H^m(X, E_1) \\
\oplus
\end{array}
\]

where \( q_j \) are orders, depending on \( s_0, s_1, s_j \) (\( j = 0, 1 \)). We have then the

Theorem 4. Let \( \alpha \in \mathfrak{M} \) be elliptic, satisfying (14) and considered as a mapping (6). Then there exist operators \( \mathcal{A}, \mathcal{F} \in \mathfrak{M} \) as in (22), so that

\[
\psi(\mathcal{A}^\dagger f, \mathcal{F} g) = \langle \mathcal{A} f, \mathcal{F}^\dagger g \rangle
\]

for all \( f \in C^m(X, E_1) \oplus C^m(Y, E_1) \) and \( g \in C^m(X, E_2) \oplus C^m(Y, E_2) \).

In the case \( q_j = 0, r_j = 0 (j = 0, 1) \) one can set \( p_k = s_k, p_k = s_k \).

Proof. For simplicity we suppose \( s_1 = 0 \) and \( q_j = r_j = 0 (j = 0, 1) \). It is then obvious how to treat the general case. In (12) with the definition (15) of \( \mathcal{A} \) we set \( u = u_{\mathcal{A}^\dagger \mathcal{F}} f, v = \mathcal{A}^{-1} \).

Then we obtain

\[
\psi(\mathcal{A} u, v) = \psi(\mathcal{A}^\dagger \mathcal{F} f, \mathcal{A}^{-1} g) = \psi(\mathcal{A}^\dagger \mathcal{F} f, \mathcal{A}^{-1} g)
\]

and

\[
\psi(\mathcal{A} u, v) = \psi(\mathcal{A}^\dagger \mathcal{F} f, \mathcal{A}^{-1} g) = \psi(\mathcal{A}^\dagger \mathcal{F} f, \mathcal{A}^{-1} g)
\]
Thus (12) implies
\[ \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) = \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) \]
for all smooth \( f, g \). Since the operators in \( \mathcal{N}^{-1} \) have nonnegative orders and types, we obtain
\[ \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) = \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) \]
with \( \mathcal{N} = (\mathcal{N}^{-1})^{*} \mathcal{L}^m \). Moreover
\[ \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) = \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) \]
Since \( (\mathcal{N}^{-1})^{*} \mathcal{L}^m \) has a closure as continuous operator
\[ (\mathcal{N}^{-1})^{*} \mathcal{L}^m : H^1(X, B) \to H^1(Y, J) \to H^m(Y, B) \]
we can define its adjoint \( \mathcal{N}^{-1} = (\mathcal{N}^{-1})^{*} \mathcal{L}^m \) and obtain finally
\[ \psi(\mathcal{N}^{-1}f, \alpha^{-1}g) = \psi(\alpha^{-1}f, \alpha^{-1}g). \]
Thus Theorem 4 is proved.

Corollary 5. Under the above conditions the Fredholm alternative holds in the following form. The equation \( \alpha f = g \) for \( f \in C^\infty(X, B) \oplus C^\infty(Y, J) \) has a solution \( f \in C^\infty(X, B) \oplus C^\infty(Y, J) \) iff \( \psi(w, \alpha g) = 0 \) for all \( g \in \ker \alpha^* \).

Remark 6. From (22) follows
\[ \text{ind} \alpha^* = - \text{ind} \alpha. \]

References