PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

GENERIC PROPERTIES OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Introduction

In general, we have very little information concerning the structure of the set of solutions of nonlinear elliptic boundary value problems. If the operator is monotone, then usually the solution is unique or the set of solutions is convex. In the nonconvex case, the bifurcation theory gives valuable information concerning the branching of solutions. Apart from these important theories, very little is known and no general method seems available.

Our purpose in this article is to present a topological approach which gives generic results concerning the set of solutions of nonlinear elliptic boundary value problems. Our typical result is that, generically with respect to one of the parameters entering the problem, the set of solutions is *finite*; by parameter we mean for instance the coefficients of the differential operators, the boundary data, the open set under consideration.

Our main tool is a transversality theorem which is essentially due to Quinn [16]. This theorem is recalled in Section 1 where we also establish an abstract result concerning a nonlinear equation

$$A(u, \theta) = 0 \tag{0.1}$$

where θ is the parameter and u the unknown: under suitable assumptions, for "most values of θ", 0 is a regular value of the mapping $\{u \mapsto A(u, \theta)\}$, from which we infer that the number of u satisfying (0.1) is finite. Thus, for a given θ, either the set of solutions of (0.1) is finite or this property is true after an arbitrarily small perturbation of θ.

In Sections 2 through 4 some applications are given. In order to avoid too many technical details we have restricted ourselves to some typical cases, although it is our belief that the method applies to many other situations. In Section 2 we consider second order quasi-linear elliptic boundary value problems for which a priori estimates and the existence theory is available in Ladyženskaya–Uralceva [7]. We show that the number of solutions is finite for almost all values of the coefficients of the linear principal part of the operator. In Section 3 we consider a more specific problem:

$$\begin{cases} -\Delta u + g(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

and we establish that the set of solutions of (0.2) is finite for generic domains Ω.

In Section 4 we consider the non-homogeneous stationary Navier–Stokes equations. Our main result is that, for fixed viscosity, fixed second member and generic boundary data, the solution set is finite. (A similar result for generic second member was given by Foiaş–Temam [3] by use of different techniques.) Some consequences on the global structure of the solution set are derived.

These lectures are based on a joint work with R. Temam [17], [18].

Contents

1. An abstract result

We first recall the transversality theorem.

Let X, Y, Z be three real Banach spaces (1), $U \subset X$, $V \subset Y$ open subsets.

Let F be a mapping of class $\mathscr{C}^k$ ($k \geqslant 1$) from $U \times V$ into Z, such that

(1.1) for every $y \in V$, $F(\cdot, y)\colon x \mapsto F(x, y)$ is a Fredholm mapping from U into Z, of index l, $l < k$.

For the definition of nonlinear Fredholm mapping, the reader is referred to S. Smale [20]; (1.1) means that for every $x_0, y_0 \in U \times V$, the partial differential $F'_x(x_0, y_0)$ is a linear Fredholm mapping from X into Z and that its index (equal by definition to the index l of $F(\cdot, y_0)$) is less than k.

(1) Not necessarily separable.

We assume that

(1.2) z_0 is a regular value of F, which means that the total differential $F' \in \mathscr{L}(X \times Y, Z)$:

$$F'(x_0, y_0)\cdot(x, y) = F'_x(x_0, y_0)\cdot x + F'_y(x_0, y_0)\cdot y$$

is onto, at every point (x_0, y_0) such that $F(x_0, y_0) = z_0$.

Finally, we assume that F is proper in the following sense:

(1.3) The set of $x \in U$ such that $F(x, y) = z_0$ with y belonging to a compact set in Y is relatively compact in U.

We then have the following result (Quinn [16], Uhlenbeck [22], [23]) (a direct proof is given in the Appendix of [17]).

THEOREM 1.1. *Under assumptions* (1.1)–(1.3), *the set*

$$\mathcal{O} = \{y \in V,\ z_0 \text{ is a regular value of } F(\cdot, y)\} \tag{1.4}$$

is a dense open subset of V.

Now let us denote by X', Y', Z' the dual spaces of X, Y, Z. For $(x_0, y_0) \in U \times V$ the differential $F'_x(x_0, y_0)$, which is linear continuous from X into Z has an adjoint $F'_x(x_0, y_0)^*$, which is linear continuous from Z' into X'. Because of (1.1), its kernel has finite dimension.

From Theorem 1.1 we then derive the following result.

THEOREM 1.2. *Our assumptions are* (1.1), (1.3) *and for* z_0 *given we assume* (*instead of* (1.2)) *that for every* $(x_0, y_0) \in \bar{U} \times V$ *with* $F(x_0, y_0) = z_0$, *the following condition holds*:

(1.5) *If* $w \in \operatorname{Ker}\big(F'_x(x_0, y_0)\big)^*$ *and* $\langle F'_y(x_0, y_0)\cdot y, w\rangle = 0$ *for every* $y \in Y$, *then* $w = 0$.

Then (1.2) (*and the conclusions of Theorem* 1.1) *are valid. For every* y_0 *in* $\mathcal{O}$ (cf. (1.4)), *the set*

$$\{x \in U,\ F(x, y_0) = z_0\} \tag{1.6}$$

is the disjoint union of a finite number of compact and connected differentiable submanifolds of X *of class* $\mathscr{C}^k$ *and dimension* l.

Remark 1.1. If (1.3) is dropped and X, Y, Z are separable, Theorem 1.1 still holds, $\mathcal{O}$ being only a residual set.

If $l = 0$, (1.6) *is a finite set, its number of elements is constant on every connected component of* $\mathcal{O}$ *and, on such a component, every solution is a* $\mathscr{C}^k$*-function of* y.

Proof. (i) We first show that z_0 is a regular value of F. For $(x_0, y_0) \in F^{-1}(\{z_0\})$, and for any given $h \in Z$, we must find a pair $(x, y) \in X \times Y$ such that

$$F'_x(x_0, y_0)\cdot x + F'_y(x_0, y_0)\cdot y = h.$$

For simplicity we write $L = F'_x(x_0, y_0)$, and the equation becomes

$$Lx = h - F'_y(x_0, y_0)\cdot y. \tag{1.7}$$

Since L is a linear Fredholm operator, there exists x satisfying (1.7) if and only if there exists $y \in Y$ such that

$$\langle h - F'_y(x_0, y_0)\cdot y, w\rangle = 0$$

for every $w \in \operatorname{Ker} L^*$. Let $w_1, \ldots, w_N$ denote a basis of $\operatorname{Ker} L^*$, which has finite dimension.

We must find $y \in Y$ such that

$$\mathscr{L}_i(y) = \langle h, w_i\rangle, \quad i = 1, \ldots, N,$$

where $\mathscr{L}_i \in Y'$ is the linear continuous form on Y defined by

$$\mathscr{L}_i(y) = \langle F'_y(x_0, y_0)\cdot y, w_i\rangle, \quad i = 1, \ldots, N.$$

By a corollary to the Hahn–Banach theorem (which is explicited in L. Schwartz [19]) we are sure that such a y exists if the linear forms $\mathscr{L}_i$ are independent. This amounts to saying that if $\lambda_1, \ldots, \lambda_N \in \boldsymbol{R}$ and $\mathscr{L} = \sum_{i=1}^{N} \lambda_i \mathscr{L}_i$, then

$$\mathscr{L}(y) = 0, \ \forall y \in Y \Rightarrow \lambda_1 = \ldots = \lambda_N = 0. \tag{1.8}$$

Let $w = \sum_{i=1}^{N} \lambda_i w_i \in \operatorname{Ker} L^*$. Then $\mathscr{L}(y)$ is exactly

$$\langle F'_y(x_0, y_0)\cdot y, w\rangle.$$

and the condition is equivalent to the assumption (1.5).

(ii) The conclusions of Theorem 1.1 are valid. Let us assume that y_0 belongs to the set $\mathcal{O}$ in (1.4). Because of (1.3) and the continuity of F, the set (1.6) is compact. Since z_0 is a regular value of $F(\cdot, y_0)$, we immediately infer from the implicit function theorem that (1.6) is the union of disjoint connected differentiable manifolds of class $\mathscr{C}^k$ and dimension l. By the compactness of (1.6) the number of such manifolds is finite.

(iii) Finally, if $l = 0$, these manifolds are isolated points. Let us show that the number of solutions is constant on a connected component $\mathcal{O}'$ of $\mathcal{O}$. By the implicit function theorem the equation

$$F\big(\varphi_1(y), y\big) = z_0, \quad y \in \mathcal{O}',$$

admits a unique solution $x = \varphi_1(y)$ such that $\varphi_1(y_0) = x_1$, where x_1, y_0 are given, $x_1 \in U$, $y_0 \in \mathcal{O}'$, $F(x_1, y_0) = z_0$, and φ, like F, is of class $\mathscr{C}^k$. If $x_1, \ldots, x_N$ satisfy

$$F(x_i, y_0) = z_0, \quad i = 1, \ldots, N, \ y_0 \in \mathcal{O}',$$

then we get N different functions φ_i such that $\varphi_i(y_0) = x_i$, $1 \leqslant i \leqslant N$, and by the implicit function theorem,

$$\varphi_i(y) \neq \varphi_j(y), \quad \forall y \in \mathcal{O}', \ \forall i \neq j.$$

This proves that the number of points $x_i \in U$ such that $F(x_i, y_0) = z$ is independent of y_0 as long as y_0 remains in the same connected component $\mathcal{O}'$ of $\mathcal{O}$, and that each solution is a $\mathscr{C}^k$-function of y.

Remark 1.2. The determination of $F'_x(x_0, y_0)^*$ and the verification of (1.5) will be generally easier in Hilbert spaces. This leads us to the following remark.

We assume that $X \hookrightarrow \tilde{X}$, $Z \hookrightarrow \tilde{Z}$, where $\tilde{X}$, $\tilde{Z}$ are Hilbert spaces and the injections are continuous. We assume that $L = F'_x(x_0, y_0)$ admits a unique extension $\tilde{L}$ to a linear continuous operator from $\tilde{X}$ into $\tilde{Z}$ and that $(\tilde{L})^{-1}(Z) \subset X$. Then if $\tilde{L}^* \in \mathscr{L}(\tilde{Z}, \tilde{X})$ is the adjoint of $\tilde{L}$, we can replace (1.5) by

(1.5′) if $w \in \operatorname{Ker} \tilde{L}^*$ and $\langle F'_y(x_0, y_0)\cdot y, w\rangle = 0$ for every $y \in Y$, then $w = 0$.

Indeed, if (1.5′) is satisfied, we see, with the same proof as in Theorem 1.2, that for every $\tilde{h} \in \tilde{Z}$ there exist $\tilde{x} \in \tilde{X}$ and $y \in Y$ such that

$$\tilde{L}\tilde{x} = \tilde{h} - F'_y(x_0, y_0)\cdot y.$$

Now, since $\tilde{L}^{-1}(Z) \subset X$, if $\tilde{h} = h \in Z$, then $\tilde{x} \in X$ and the surjectivity of F' is again established.

2. Genericity with respect to the coefficients

Our first application of Theorem 1.2 concerns nonlinear elliptic equations of the second order. For simplicity we are not aiming at maximal generality and we only consider the following equations, which are studied in Ladyženskaja–Ural'ceva [7]: Chapter VI in [7] is devoted to the derivation of a priori estimates for these equations and the existence results are a particular case of Theorem 3.3 in the same chapter.

Let Ω be an open connected bounded set in $\boldsymbol{R}^n$, whose boundary Γ is of class $\mathscr{C}^{2,\alpha}$ for some α, $0 < \alpha < 1$, in the sense of [7]. Let a_{ij}, $1 \leqslant i$, $j \leqslant n$, denote n^2 real functions on Ω with

$$a_{ij} \in \mathscr{C}^{1,\alpha}(\overline{\Omega}), \quad 1 \leqslant i, j \leqslant n. \tag{2.1}$$

Let $g = g(x; u, p)$ denote a continuous function on $\bar{\Omega}\times \mathbf{R}\times \mathbf{R}^n$ such that

(2.2) $$g \in \mathscr{C}^a(K), \quad \text{for every compact set } K\subset \bar{\Omega}\times \mathbf{R}\times \mathbf{R}^n.$$

(2.3) $$\frac{\partial g}{\partial u}, \frac{\partial g}{\partial p_i}, \ 1\leqslant i\leqslant n, \text{ belong to } \mathscr{C}^a(K),$$
$$\forall K\subset \bar{\Omega}\times \mathbf{R}\times \mathbf{R}^n,\ K \text{ compact}.$$

The boundary value problem to be considered in this section is:

(2.4) $$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}+g(x, u, u_x) = 0 \quad \text{in } \Omega,$$

(2.5) $$u = 0 \quad \text{on } \Gamma,$$

where $u_{x_i} = \partial u/\partial x_i$ and $u_x = \operatorname{grad} u$.

The existence of solutions of (2.4)–(2.5) is established in [7] if a_{ij} and g satisfy the following conditions:

(2.6) There exist $b_1, b_2 \in \mathbf{R}$, $b_1 > 0$, $b_2 \geqslant 0$ such that
$$g(x, u, 0)u\leqslant -b_1u^2+b_2, \quad \forall x\in\bar{\Omega},\ \forall u \in \mathbf{R}.$$

(2.7) There exist $\mu_0, \mu_1 > 0$ such that
$$\mu_0\sum_{i=1}^{n}\xi_i^2\leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\leqslant \mu_1\sum_{i=1}^{n}\xi_i^2, \quad \forall x\in\Omega,\ \forall \xi\in \mathbf{R}^n.$$

(2.8) There exists $\mu_2 > 0$ such that for every $x\in\bar{\Omega}$, $u \in \mathbf{R}$,
$$|u|\leqslant M = \sqrt{b_2/b_1}, \quad \text{and} \quad p\in \mathbf{R}^n, \quad |g(x, u, p)|\leqslant \mu_2(1+|p|^2).$$

Then,

(2.9) Under the conditions (2.1)–(2.2), (2.6)–(2.8), there exists $u\in\mathscr{C}^{2,a}(\bar{\Omega})$, which satisfies (2.4)–(2.5).

Of course, no other information is available, as regards the set of solutions of (2.4)–(2.5).

We are going to apply Theorem 1.2. We set
$$X = \{u\in\mathscr{C}^{2,a}(\Omega),\ u = 0 \text{ on } \Gamma\},$$
$$Y = \mathscr{C}^{2,a}(\bar{\Omega})^{n^2}(^2),$$
$$Z = \mathscr{C}^{a}(\bar{\Omega});$$

(2) At a certain point in the proof of (1.5) below, $a_{ij}\in\mathscr{C}^{1,a}(\bar{\Omega})$ is not sufficient (cf. (2.1)).

these are obviously Banach spaces. We denote by u (or $u^0, v, \ldots$) and a (or $a^0, b, \ldots$) the elements of X and Y, $a = (a_{ij})_{1\leqslant i,j\leqslant n}$, $a_{ij}\in\mathscr{C}^{2,a}(\bar{\Omega})$. Then
$$U = X\setminus\{0\},$$
$$V = \Big\{a\in Y,\ \exists \mu_0 = \mu_0(a) > 0, \text{ such that } \sum a_{ij}(x)\xi_i\xi_j\geqslant \mu_0|\xi|^2,\ \forall x\in\Omega,\ \forall\xi\in\mathbf{R}^n\Big\}.$$

For $(u, a)\in X\times Y$ we now set

(2.10) $$F(u, a) = \sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}+g(x, u, u_x).$$

It follows easily from (2.1)–(2.2) and the properties of Hölder spaces that F maps $X\times Y$ into Z, that F is of class $\mathscr{C}^1$, and that its differential at some point (u^0, a^0) is given by

(2.11) $$F'(u^0, a^0)\cdot(u, a) = F'_u(u^0, a^0)\cdot u+F'_a(u^0, a^0)\cdot a,$$

(2.12) $$F'_u(u^0, a^0)\cdot u = \sum_{i,j=1}^{n} a^0_{ij}u_{x_ix_j}+\frac{\partial g}{\partial u}(x, u^0, u^0_x)\cdot u+ +\sum_{i=1}^{n}\frac{\partial g}{\partial p_i}(x, u^0, u^0_x)D_iu,$$

(2.13) $$F'_a(u^0, a^0)\cdot a = \sum_{i,j=1}^{n} a_{ij}u^0_{x_ix_j}, \quad \forall(u^0, a^0)\in X\times Y,\ \forall(u, a)\in X\times Y.$$

We are now going to verify (1.1)–(1.3) and (1.5).

Proof of (1.1). By Schauder's theory for linear elliptic equations (see for instance Agmon–Douglis–Nirenberg [2]), the mapping
$$u\mapsto\sum_{i,j=1}^{n} a^0_{ij}u_{x_ix_j}$$
is an isomorphism from X onto Z (a^0 given in V).

The injection of $\mathscr{C}^{1,a}(\bar{\Omega})$ into $\mathscr{C}^{a}(\bar{\Omega})$ is compact, and this easily implies that
$$u\mapsto\frac{\partial g}{\partial u}(x, u^0, u^0_x)\cdot u+\sum_{i=1}^{n}\frac{\partial g}{\partial p_i}(x, u^0, u^0_x)D_iu$$
is a linear compact operator from X into Z. Then the linear operator $u\mapsto F'_u(u^0, a^0)u$ given in (2.12) is a compact perturbation of an isomorphism. It is a Fredholm operator of index $l = 0$ from X into Z ($\forall u^0\in X$, $\forall a^0\in V$).

Proof of (1.3). It is sufficient to show that if

$$F(u^m, a^m) = z_0, \tag{2.14}$$

with $u^m \in X$, and a^m converging to some limit $a \in V$, then u^m contains a convergent subsequence. But (2.14) is equivalent to

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}^m u_{x_i x_j}^m + g(x, u^m, u_x^m) = 0 & \text{in } \Omega, \\ u^m = 0 & \text{on } \Gamma. \end{cases} \tag{2.15}$$

Since a^m converges to a in Y and $a \in V$, we may assume without loss of generality that each a^m satisfies (2.7) with some μ_0 and μ_1 independent of m.

The a priori estimates of [7] (cf. Chapter VI, Theorems 1.1 and 2.3) imply that for some $\beta > 0$, which only depends on Ω, M, μ_0, μ_1, μ_2 (cf. (2.7)–(2.9)), the $\mathscr{C}^\beta$-norms of u^m and ∇u^m in $\overline{\Omega}$ are bounded by some number independent on m (and depending only on M, μ_0, μ_1, μ_2, Ω). By lowering β, we may assume that $\beta < \alpha$, and therefore the sequence

$$g(x, u^m, u_x^m) \tag{2.16}$$

is bounded in $\mathscr{C}^\beta(\overline{\Omega})$ and relatively compact in $\mathscr{C}^{\beta_0}(\overline{\Omega})$, $0 < \beta_0 < \beta$.

Then Schauder's theory for linear elliptic equation (cf. [2]) and (2.15) show that u^m is a relatively compact sequence in $\mathscr{C}^{2,\beta}(\overline{\Omega})$.

We now reiterate the reasoning; (2.16) is a relatively compact sequence of $\mathscr{C}^\alpha(\overline{\Omega})$, from which we conclude that u^m is a relatively compact sequence of $\mathscr{C}^{2,\alpha}(\overline{\Omega})$.

Proof of (1.5). Instead we verify (1.5′) in the context of Remark 1.2.

We take $\tilde{Z} = L^2(\Omega)$, and for $\tilde{X}$ the space $H_0^1(\Omega) \cap H^2(\Omega)$.(3) Then $L = F_u'(u^0, a^0)$ (cf. (2.12)) is easily extended to a linear continuous operator $\tilde{L}$ from $H_0^1(\Omega) \cap H^2(\Omega)$ into $L^2(\Omega)$, and if $\tilde{L}u = h \in \mathscr{C}^\alpha(\overline{\Omega})$, and $u^0, a^0 \in U \times V$, we write

$$\begin{cases} u \in H_0^1(\Omega) \cap H^2(\Omega), \\ \sum_{i,j=1}^{n} a_{ij}^0 u_{x_i x_j} = h - \frac{\partial g}{\partial u}(x, u^0, u_x^0) \cdot u - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(x, u_x^0) D_i u \end{cases} \tag{2.17}$$

(3) $H^m(\Omega)$ is the Sobolev space of order m; $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of the space of smooth functions with a compact support in Ω, cf. Lions–Magenes [10], J. Nečas [14].

and by reiterated applications of the regularity results of [7] and [2] we obtain that $u \in X = \mathscr{C}^{2,\alpha}(\overline{\Omega})$.(4)

The adjoint of $\tilde{L}$ in $L^2(\Omega)$ is the ordinary adjoint. Saying that $w \in \operatorname{Ker}\tilde{L}^*$ amounts to saying that $w \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}^0 w) + \frac{\partial g}{\partial u}(x, u^0, u_x^0) w - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial g}{\partial p_i}(x, u^0, u_x^0) w\right) = 0 \quad \text{in } \Omega, \tag{2.18}$$

$$w = 0 \quad \text{on } \Gamma. \tag{2.19}$$

The regularity results of [7], [2] show that in fact $w \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$. (Here and below we need $a^0 \in \mathscr{C}^{2,\alpha}(\overline{\Omega})^{n^2}$ and not only $\mathscr{C}^{1,\alpha}(\overline{\Omega})^{n^2}$.)

According to the expression (2.13) of $F_a'(u^0, a^0)$, in order to establish (1.5′) we must prove the following:

If $w \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ satisfies (2.18)–(2.19) and for every $a \in \mathscr{C}^{2,\alpha}(\overline{\Omega})^{n^2}$

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} u_{x_i x_j}^0 w \, dx = 0,$$

then $w = 0$.

By taking $a_{ij} = \varphi \delta_{ij}$, $\varphi > 0$ arbitrary in $\mathscr{C}^{2,\alpha}(\overline{\Omega})$, we obtain

$$\Delta u^0 w = 0 \quad \text{in } \Omega.$$

Now, since $u^0 = 0$ on Γ and $u^0 \not\equiv 0$ ($u^0 \in U = X \setminus \{0\}$), Δu^0 cannot vanish identically in Ω. Then there exists an open set $\omega \subset \Omega$ such that $\Delta u^0(x) \neq 0$, $x \in \omega$, from which we conclude that $w(x) = 0$, $x \in \omega$. The fact $w = 0$ then follows from the uniqueness of solutions of the Cauchy problem for the equation (2.18) (cf. S. Agmon [1], Hörmander [6]). The uniqueness theorems apply since (2.18) can be rewritten in the form

$$\sum_{i,j=1}^{n} a_{ij}^0 w_{x_i x_j} + \sum_{i=1}^{n} d_i w_{x_i} + d_0 w = 0,$$

where d_i and d_0 are continuous on $\overline{\Omega}$, since $a_{ij}^0 \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$:

$$d_i = \sum_{j=1}^{n} (a_{ij}^0 + a_{ji}^0)_{x_j} - \left(\frac{\partial g}{\partial p_i}(x, u^0, u_x^0)\right)_{x_i},$$

$$d_0 = \sum_{i,j=1}^{n} a_{x_i x_j}^0 + \frac{\partial g}{\partial u}(x, u^0, u_x^0) - \sum_{i=1}^{n} D_i\left(\frac{\partial g}{\partial p_i}(x, u^0, u_x^0)\right).$$

(4) [7] implies that u and $\operatorname{grad} u$ are uniformly bounded in $\overline{\Omega}$ so that the right-hand side of (2.17) is uniformly bounded in $\overline{\Omega}$; then [2] shows that u is in the Sobolev space $W^{2,r}(\Omega)$, $\forall r$, $1 < r < \infty$, so that $u \in \mathscr{C}^{1,\beta}(\overline{\Omega})$, $\forall \beta < 1$. Then we find that the right-hand side of (2.17) is in $\mathscr{C}^\alpha(\overline{\Omega})$ and finally $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$.

Theorem 1.2 applies. We arrive at the main result of this section.

THEOREM 2.1. *We assume that Ω is an open bounded set of $\boldsymbol{R}^n$ and that its boundary Γ is of class $\mathscr{C}^{2,\alpha}$, for some α, $0<\alpha<1$.*

Let there be given a function g from $\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n$ into $\boldsymbol{R}$ which satisfies (2.2)–(2.3) *and* (2.8).

Then there exists a dense open subset $\mathcal{O}$ of $V\subset\mathscr{C}^{2,\alpha}(\overline{\Omega})^{n^2}$,

$$V=\Big\{a=(a_{ij})_{1\leqslant i,j\leqslant n},\ a_{ij}\in\mathscr{C}^{2,\alpha}(\overline{\Omega}),$$

$$\sum_{i,j=1}^{n}a_{ij}(x)\,\xi_i\,\xi_j\geqslant\mu_0\sum_{i=1}^{n}\xi_i^2,\ \forall x\in\overline{\Omega},\ \forall\xi\in\boldsymbol{R}^n,\ \mu_0=\mu_0(a)>0\Big\}$$

such that for every $a\in\mathcal{O}$, the elliptic boundary value problem (2.4)–(2.5) *possesses a finite number of solutions.*

The number of solutions is constant on every connected component of $\mathcal{O}$, and on such a component, every solution is a $\mathscr{C}^1$-function of the a_{ij}.

Remark 2.1. Theorem 2.1 can be extended in different fashions: perturbation of the coefficients in the class of *symmetric* operators, or perturbation of only one coefficient; cf. [17] for details.

3. Genericity with respect to the domain

We are interested in the nonlinear elliptic problem

$$-\Delta u+g(x,u)=0\quad\text{in }\Omega,\tag{3.1}$$

$$u=0\quad\text{on }\Gamma,\tag{3.2}$$

where Ω is a connected bounded open set in $\boldsymbol{R}^n$, whose boundary Γ is of class $\mathscr{C}^{2,\alpha}$ for some α, $0<\alpha<1$, and g is a given function subjected to the conditions:

$$g\text{ is a real continuous function on }\boldsymbol{R}^n\times\boldsymbol{R},\ g(x,0)=0,\tag{3.3}$$

$$g\text{ and }\frac{\partial g}{\partial u}\in\mathscr{C}^{\alpha}(K)\text{ for every compact set}\subset\boldsymbol{R}^n\times\boldsymbol{R}.\tag{3.4}$$

We want to obtain generic results concerning the number of solutions of (3.1), (3.2) when Ω varies. We will restrict ourselves to a local study, Ω varying in the neighbourhood of a given open set Ω_0.

We assume that Ω_0 is the image of the unit ball of $\boldsymbol{R}^n$ under a diffeomorphism of class $\mathscr{C}^{2,\alpha}$, its boundary being the image of the boundary of the unit ball. We consider open sets Ω which are the images of Ω_0 under diffeomorphisms having similar properties and close to I, the identity in $\mathrm{Diff}(\boldsymbol{R}^n)$.

More precisely, let Q be an open bounded set in $\boldsymbol{R}^n$, $Q\supset\overline{\Omega}_0$, and let $T=I+\theta\in\mathscr{C}^{2,\alpha}(\overline{\Omega})^n$ with $\theta\in\mathscr{C}_0^{2,\alpha}(Q)^n$, i.e. θ and its first and second order derivatives vanish on ∂Q. If the norm of θ in $\mathscr{C}^{2,\alpha}(\overline{Q})^n$ is sufficiently small:

$$\|\theta\|_{\mathscr{C}_0^{2,\alpha}(Q)^n}\leqslant c_n,\tag{3.5}$$

so that the Jacobian of T, $\det T'$, satisfies

$$\det T'=\det[(I+\theta)']\geqslant\varepsilon_0>0,\ \text{for some }\varepsilon_0>0,\tag{3.6}$$

then $T(Q)=Q$, $T=I+\theta$ is one-to-one in Q (i.e. T is *globally* one-to-one) and T^{-1} belongs to $\mathscr{C}^{2,\alpha}(Q)^n$, $T^{-1}-I$ belonging to $\mathscr{C}_0^{2,\alpha}(Q)^n$.(5) Furthermore, $T\Omega_0$ is an open set and $T\Gamma_0$ is exactly the boundary Γ of Ω. In the sequel we will only consider open sets Ω of this nature:

$$\begin{cases}\Omega=T\Omega_0,\quad \Gamma=T\Gamma_0,\\ T=I+\theta,\ \theta\in\mathscr{C}_0^{2,\alpha}(Q)^n\text{ satisfying (3.5), (3.6).}\end{cases}\tag{3.7}$$

We are going to apply Theorem 1.2 with

$$U=X\setminus\{0\},\qquad X=\{u\in\mathscr{C}^{2,\alpha}(\overline{\Omega}_0),\ u=0\text{ on }\Gamma_0\},$$
$$Y=\mathscr{C}^{2,\alpha}(Q)^n,$$
$$V=\{\theta\in\mathscr{C}^{2,\alpha}(Q),\ \theta\text{ satisfies (3.5), (3.6)}\},$$
$$Z=\mathscr{C}^{\alpha}(\overline{\Omega}_0)$$

and our result is the following:

THEOREM 3.1. *Let g denote a real function on $\boldsymbol{R}^n\times\boldsymbol{R}$ satisfying* (3.3), (3.4) *for some α, $0<\alpha<1$, and let Ω_0, Γ_0 and Q be given as above.*

Then there exists a dense open subset $\mathcal{O}$ of V, such that for every $\theta\in\mathcal{O}$ the elliptic boundary value problem (3.1), (3.2) *in $\Omega=(I+\theta)(\Omega_0)$ admits a finite number of solutions.*

The number of solutions is constant on every connected component of $\mathcal{O}$ and on such a component, every solution is a $\mathscr{C}^1$-function of Ω (i.e. of θ).

Proof. We have already fixed the spaces X, Y, Z, U, V occurring in Theorem 1.2. For $u\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$, $\theta\in V$, we set

$$\tilde{u}=u\circ T,\qquad T=I+\theta;\tag{3.8}$$

if $u\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$ and $u=0$ on Γ, then $\tilde{u}\in X$. We denote by x a generic point in Ω and by $\tilde{x}$ a generic point in Ω_0.

(5) Cf. also [5], [11], [12], [13] for boundary value problems in varying domains.

For θ fixed, $\theta \in V$ and $\Omega = (I+\theta)(\Omega_0)$, we consider the boundary value problem (3.1), (3.2). This problem is equivalent to

$$(3.8)\qquad \begin{cases} \tilde{u} \in X \text{ and} \\ -(\Delta_x u)(T\tilde{x}) + g(T\tilde{x}, u(T\tilde{x})) = 0, \ \tilde{x} \in \Omega_0. \end{cases}$$

We denote by $F(\tilde{u}, \theta)$ the left-hand side of (3.8):

$$(3.9)\qquad F(\tilde{u}, \theta) = -(\Delta_x u)(T\tilde{x}) + g(T\tilde{x}, u(T\tilde{x})).$$

We must first express $F(\tilde{u}, \theta)$ in terms of $\tilde{u}$ and θ in a more explicit manner. This can be done using some technical lemmas of Murat–Simon [13]. Lemma 4.1 of [13] gives

$$(3.10)\qquad F(v, \theta) = -\mathrm{div}_{\tilde{x}}\{(\det T')\cdot(T')^{-1}({}^tT')^{-1}\mathrm{grad}_{\tilde{x}} v\} + g(T\tilde{x}, v),$$
$$\forall v \in \mathscr{C}^{2,\alpha}(\overline{\Omega}_0), \ \theta \in V, \ T = I+\theta;$$

T' is the differential of T; $\mathrm{div}_{\tilde{x}}$ and $\mathrm{grad}_{\tilde{x}}$ indicate that these operators are related to the $\tilde{x}$-variable.(6) Lemmas 4.2, 4.3 and 4.5 of [13] show that F is differentiable with respect to θ and F'_θ is continuous. It is obvious, on the other hand, that F is differentiable with respect to v and that F'_v is continuous. Thus F is of class $\mathscr{C}^1$.

Instead of giving an explicit form of F'_v and F'_θ at any point (v^0, θ^0) of $U \times V$, we observe that by changing Ω_0 we can always assume that $\theta^0 = 0$, $T^0 = I$, so that we only need know the differential $F'_v(v^0, 0)$, $F'_\theta(v^0, 0)$.

As regards F'_v, we have

$$(3.11)\qquad F'_v(v^0, 0)\cdot v = -\Delta v + \frac{\partial g}{\partial u}(x, v^0)\cdot v.$$

Regarding F'_θ, we find, using [16],

$$(3.12)\qquad F'_\theta(v^0, 0)\cdot\zeta = \sum_{i=1}^{n}\big((\mathrm{div}\,\zeta)v^0_{x_i}\big)_{x_i} + \sum_{i=1}^{n}\big(({}^t[\zeta'] + [\zeta'])v^0_{x_i}\big)_{x_i} +$$
$$+ g(x, v^0)\,\mathrm{div}\,\zeta + \sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(x, v^0)\cdot\zeta_i + \sum_{i=1}^{n}\frac{\partial g}{\partial u}(x, v^0)v^0_{x_i}\cdot\zeta_i.$$

Proof of (1.1). It is clear that $F'_v(v^0, 0)$ is the sum of an isomorphism from X into Z (the operator $-\Delta$) and of a compact perturbation. Then $F'_v(v^0, 0)$ is a Fredholm mapping of index 0 from X into Z.

(6) At this point we are essentially working in a fixed domain Ω_0, with generic points $\tilde{x}$.

Proof of (1.3). We must show that if $u^m \in X$ satisfies

$$(3.13)\qquad F(u^m, \theta^m) = 0$$

with θ^m converging to some limit θ in V, then u^m contains a converging subsequence. The principle of the proof is the same as in Section 2; (3.13) is written in the form

$$(3.14)\qquad \sum_{i,j=1}^{n}(a^m_{ij}u^m_{\tilde{x}_j})_{\tilde{x}_i} = g(T\tilde{x}, u^m) \quad \text{in } \Omega_0,$$

$$(3.15)\qquad u^m = 0 \quad \text{on } \Gamma_0,$$

where

$$a^m_{ij} = [(\det T'_m)(T'_m)^{-1}\,{}^t(T'_m)^{-1}]_{i,j}.$$

Since θ^m converges to some limit θ in V, the a^m_{ij} belong to $\mathscr{C}^{1,\alpha}(\overline{\Omega}_0)$ and converge in this space to elements a_{ij} which satisfy (2.7) $(\theta \in V)$. Actually all the a^m_{ij} satisfy (2.7) with some μ_0, μ_1 independent of m. At this point the situation is exactly the same as in the proof of (1.3) in Section 2.

Proof of (1.5′). Let $u^0 \in X$ satisfy $F(u^0, 0) = 0$, i.e.,

$$(3.16)\qquad \begin{cases} -\Delta u^0 + g(x, u^0) = 0 & \text{in } \Omega_0, \\ u^0 = 0 & \text{on } \Gamma^0. \end{cases}$$

Let $L = F'_v(u^0, 0)$. We introduce, as in Remark 1.2 (and in Section 2),

$$\tilde{X} = H^1_0(\Omega_0) \cap H^2(\Omega_0), \qquad \tilde{Z} = L^2(\Omega_0);$$

L obviously admits a unique extension $\tilde{L} \in \mathscr{L}(\tilde{X}, \tilde{Z})$, and by the regularity results for elliptic operators, it is clear that $\tilde{L}^{-1}(Z) \subset X$, i.e., if $v \in H^1_0(\Omega_0) \cap$ $\cap H^2(\Omega_0)$ and $-\Delta v + \frac{\partial g}{\partial u}(x, u^0)v = h$ in Ω_0, $h \in \mathscr{C}^\alpha(\overline{\Omega}_0)$, then

$$v \in \mathscr{C}^{2,\alpha}(\overline{\Omega}_0).$$

$\tilde{L}$ is self-adjoint and $w \in \mathrm{Ker}\tilde{L}^*$ means $w \in L^2(\Omega_0)$ (and $w \in \mathscr{C}^{2,\alpha}(\overline{\Omega}_0)$ by regularity) and

$$(3.17)\qquad -\Delta w + \frac{\partial g}{\partial u}(x, u^0)w = 0 \quad \text{in } \Omega_0,$$

$$(3.18)\qquad w = 0 \quad \text{on } \Gamma_0.$$

Now we must show the following: if $w \in \mathscr{C}^{2,\alpha}(\Omega_0)$ satisfies (3.17)–(3.18) and

$$(3.19)\qquad \langle F'_\theta(u^0, 0)\cdot\zeta, w\rangle = 0$$

for every ζ in $\mathscr{C}^{2,\alpha}_0(Q)^n$, then $w = 0$.

Since u^0 satisfies (3.16), we can simplify the expression of $F'_\theta(u^0, 0)\cdot\zeta$. We have

$$\sum_{i=1}^{n}\big(\operatorname{div}(\zeta)u^0_{x_i}+({}^t[\zeta']+[\zeta'])u^0_{x_i}\big)_{x_i} = -\operatorname{div}[\zeta(\Delta u^0)]+\Delta(\zeta\cdot\operatorname{grad}u^0),$$

$$g(x, u^0)\operatorname{div}\zeta+\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(x, u^0)\cdot\zeta_i+\sum_{i=1}^{n}\frac{\partial g}{\partial u}(x, u^0)u^0_{x_i}\zeta_i = \operatorname{div}\big(g(x, u^0)\zeta\big),$$

whence

$$F'_\theta(u^0, 0)\zeta = \operatorname{div}\big(\zeta(-\Delta u^0+g(x, u^0))\big)+\Delta(\zeta\cdot\operatorname{grad}u^0), \tag{3.20}$$

and in view of (3.16)

$$F'_\theta(u^0, 0)\zeta = \Delta(\zeta\cdot\operatorname{grad}u^0). \tag{3.21}$$

The condition (3.19) is now written as

$$\int_{\Omega_0}\Delta(\zeta\cdot\operatorname{grad}u^0)\cdot w\,d\tilde{x} = 0 \quad \forall\zeta\in\mathscr{C}_0^{2,\alpha}(Q)^n.$$

Using Green's formula and (3.17), (3.18) we obtain

$$\int_{\Omega_0}(\zeta\cdot\operatorname{grad}u^0)\frac{\partial g}{\partial u}(x, u^0)w\,d\tilde{x}-\int_{\Gamma_0}(\zeta\cdot\operatorname{grad}u^0)\frac{\partial w}{\partial\nu}d\Gamma = 0, \quad \forall\zeta\in\mathscr{C}_0^{2,\alpha}(Q)^n. \tag{3.22}$$

It is easy to construct a sequence of $\zeta\in\mathscr{C}_0^{2,\alpha}(Q)^n$, which remains uniformly bounded in Ω_0, such that $\zeta = (\operatorname{grad}u^0)\left(\frac{\partial w}{\partial\nu}\right)$ on Γ_0 and such that the measure of the support of ζ tends to 0. The first integral in (3.22) tends to 0, while the second one remains constant and equal to

$$\int_{\Gamma_0}(\operatorname{grad}u^0)^2\left(\frac{\partial w}{\partial\nu}\right)^2 d\Gamma.$$

At the limit we find, using for instance a result of Protter ([15], p. 85), and (3.3):

$$\frac{\partial w}{\partial\nu} = 0 \quad \text{on some non empty open subset of } \Gamma_0. \tag{3.23}$$

Using again (cf. [1], [6]) the uniqueness of solution for the Cauchy problem associated to (3.17) (i.e. (3.17), (3.18), (3.23)), we obtain that $w = 0$.

The proof is complete.

4. Genericity with respect to the boundary data for the Navier–Stokes equations

We consider the stationary non homogeneous Navier–Stokes problem in a bounded open set Ω in $\boldsymbol{R}^n$, $n = 2, 3$, with regular boundary Γ, which is finitely connected: $\Gamma = \Gamma_1\cup\Gamma_2\cup\ldots\cup\Gamma_m$.

Given f and φ, find $u = (u_1, u_2, \ldots, u_n)$ and p such that

$$\begin{cases} -\nu\Delta u+\sum_{i=1}^{n}u_iD_iu+\operatorname{grad}p = f & \text{in } \Omega,\\ \operatorname{div}u = 0 & \text{in } \Omega,\\ u = \varphi & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Here $\nu > 0$ is the kinematic viscosity, u and p represent the velocity and the pressure of a viscous incompressible fluid filling Ω, the fluid being submitted to stationary volumic forces f, and Γ moving with the stationary velocity φ.

It is well known (Leray [9]) that (4.1) is equivalent to a functional equation involving only u; we shall denote by $\mathscr{S}(f, \nu, \varphi)$ the set of solutions (in a function space which will be explicited later) of this functional equation.

In [3], [4], C. Foiaş and R. Temam have obtained several generic results concerning $\mathscr{S}(f, \nu, \varphi)$; for instance, for fixed φ and ν, this set is finite for every f belonging to a dense open set of the function space to which f belongs. The aim of this section is to complete these results, in particular for the case where *f and ν are fixed*; this corresponds to many physical situations where f is 0 and the motion of the fluid is produced by the motion of the boundary.

4.1. We first recall some notations and well known facts on Navier–Stokes equations theory. For the details and omitted proofs, the reader is referred to the books [7], [21].

For $s\in\boldsymbol{R}$, we denote $\boldsymbol{H}^s(\Omega) = H^s(\Omega)^n$, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^n$. We set $\mathscr{D}(\Omega) = \{u\in C^\infty(\Omega)$ with a compact support in $\Omega\}$ and $\mathscr{V} = \{u\in\mathscr{D}(\Omega)^n;\ \operatorname{div}u = 0\}$; V = closure of $\mathscr{V}$ in $\boldsymbol{H}^1(\Omega)$, H = closure of $\mathscr{V}$ in $\boldsymbol{L}^2(\Omega)$, P = orthogonal projection in $\boldsymbol{L}^2(\Omega)$ on H.

Let $0 < \alpha < 1$ be fixed throughout the section. We denote by $\dot{C}^{K,\alpha}(\Gamma)$ the space of functions φ in $C^{k,\alpha}(\Gamma)$ such that

$$\int_{\Gamma_i} \varphi \cdot n \, d\Gamma = 0, \quad i = 1, \ldots, m, \tag{4.2}$$

where n is the unit outward normal to Γ.

Let A be the unbounded operator in H defined by $Au = -P\Delta u$. Then (Cattabriga–Yudovitch–Solonnikov Theorem), $D(A) = V \cap \boldsymbol{H}^2(\Omega)$, $|Au|$ being a norm on $D(A)$ which is equivalent to that induced by $\|\cdot\|_2 = \|\cdot\|_{\boldsymbol{H}^2}$. Moreover, if $f \in H \cap C^\alpha(\bar{\Omega})^n$, the solution u of $Au = f$ belongs to $C^{2,\alpha}(\bar{\Omega})^n$.

If $\varphi \in \dot{C}^{2,\alpha}(\Gamma)^n$, we denote by $u = \Lambda\varphi$ the unique element in $V \cap C^{2,\alpha}(\bar{\Omega})^n$ satisfying:

$$\begin{cases} -\Delta u + \operatorname{grad} p = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma. \end{cases} \tag{4.3}$$

For $u, v \in V$ we set $B(u, v) = P[(u \cdot \operatorname{grad})v]$. Then problem (4.1) is equivalent to the following one:

given $\varphi, f, \nu > 0$, find u such that

$$\nu A u + B(u + \Lambda\varphi, \, u + \Lambda\varphi) = f. \tag{4.4}$$

For given $\nu > 0, f, \varphi$, we shall denote by $S(f, \nu, \varphi)$ the set of functions u in V which satisfy (4.4).

The following lemma is classical:

LEMMA 4.1. *For $f \in H \cap C^\alpha(\bar{\Omega})^n$ and $\varphi \in \dot{C}^{2,\alpha}(\Gamma)^n$, $S(f, \nu, \varphi)$ is non-empty and is included in $C^{2,\alpha}(\bar{\Omega})^n \cap V$.*

Moreover, if f and φ remain in a bounded subset of $H \cap C^\alpha(\bar{\Omega})^n$ and $\dot{C}^{2,\alpha}(\Gamma)^n$ and if $\nu \geqslant \nu_0 > 0$, then u remains in a bounded subset of $C^{2,\alpha}(\bar{\Omega})^n \cap V$.

4.2. A first generic result

THEOREM 4.1. *Let $\nu > 0$ and $f \in H \cap C^\alpha(\bar{\Omega})^n$ be fixed. Then there exists a dense open set $\mathcal{O} = \mathcal{O}(f, \nu)$ in $\dot{C}^{2,\alpha}(\Gamma)^n$ such that, for every $\varphi \in \mathcal{O}$, $S(f, \nu, \varphi)$ is a finite set.*

Moreover, the number of elements of $S(f, \nu, \varphi)$ is odd and is locally constant on $\mathcal{O}$.

Proof. We will apply Theorem 1.2 with $U = X = V \cap C^{2,\alpha}(\bar{\Omega})^n$, $V = Y = \dot{C}^{2,\alpha}(\Gamma)^n$, $Z = C^\alpha(\bar{\Omega})^n \cap H$ and $F(u, \varphi) = \nu A u + B(u + \Lambda\varphi, u + \Lambda\varphi)$. F is clearly C^∞ (and even analytic!); moreover, $F(\cdot, \varphi)$ is Fredholm of index 0, its derivative being a compact perturbation of an isomorphism. The property (1.3) holds by Lemma 4.1 and classical arguments (cf. [18]).

The heart of the proof is to verify condition (1.5). It turns out to be equivalent to the following lemma (cf. [18] for omitted details):

LEMMA 4.2. *Let $a_0 \in C^{2,\alpha}(\bar{\Omega})^n \cap V$ and $w \in C^{2,\alpha}(\bar{\Omega})^n \cap V$ be a solution of*

$$-\nu \Delta w_i - D_j(a_{0_j} w_i) + (D_i a_{0_j}) w_j + D_i r = 0 \quad \text{in } \Omega. \tag{4.5}$$

Moreover, suppose that

$$\int_\Omega [(a_0 \cdot \nabla)\Lambda\varphi + (\Lambda\varphi \cdot \nabla)a_0] w \, dx = 0 \quad \forall \varphi \in \dot{C}^{2,\alpha}(\Gamma)^n. \tag{4.6}$$

Then $w \equiv 0$.

Proof of Lemma 4.2. We write (4.6) in the form (with summation in i, j, and setting $\Phi = \Lambda\varphi$)

$$\int_\Omega w_j a_{0_i} D_i \Phi_j \, dx + \int_\Omega w_j \Phi_i D_i a_{0_j} \, dx = 0. \tag{4.7}$$

Integrating the first term by parts, one gets:

$$\int_\Gamma a_{0_i} \varphi_j w_j n \, d\Gamma + \int_\Omega \Phi_j D_i(a_{0_i} w_j) \, dx - \int_\Omega \Phi_i (D_i a_{0_j}) w_j \, dx = 0. \tag{4.8}$$

But the first term in (4.8) is zero since w vanishes on Γ.

On the other hand, (4.5) implies

$$-\int_\Omega \nu \Delta w_j \Phi_j \, dx - \int_\Omega D_i(a_{0_i} w_j) \Phi_j \, dx + \int_\Omega (D_i a_{0_j}) w_j \Phi_i \, dx + \\ + \int_\Omega D_j r \Phi_j \, dx = 0. \tag{4.9}$$

Taking into account this equality in (4.8), we get

$$-\nu \int_\Omega \Delta w_j \Phi_j \, dx + \int_\Omega D_j r \Phi_j \, dx = 0. \tag{4.10}$$

And by Green's formula (using $w_{|\Gamma} = 0$), we have

$$-\int_\Omega D_j r \Phi_j \, dx = -\nu \int_\Omega w_j \Delta\Phi_j \, dx - \nu \int_\Gamma \frac{\partial w_j}{\partial n} \varphi_j \, d\Gamma. \tag{4.11}$$

But by definition, $\Delta\Phi$ is a gradient and $\int_\Omega w_j \Delta\Phi_j \, dx = 0$ since $w \in V$.

On the other hand, integrating by parts and using $\operatorname{div}\Phi = 0$, we get

$$\int_\Omega D_j r \Phi_j \, dx = \int_\Gamma r \varphi \cdot n \, d\Gamma$$

and (4.11) becomes

$$\int_\Gamma \left(r\varphi_j n^j - \nu \frac{\partial w_j}{\partial n}\varphi_j\right) d\Gamma = 0 \quad \forall \varphi \in \dot{C}^{2,\alpha}(\Gamma)^n, \tag{4.12}$$

where n^j is the jth component of n.

Let us first suppose that Γ is connected. We decompose $\varphi \in C^{2,\alpha}(\Gamma)^n$ into

$$\varphi = \varphi - \bar{\varphi} + \bar{\varphi} \quad \text{where} \quad \bar{\varphi} = \frac{n}{\int_\Gamma d\Gamma}\int_\Gamma \varphi \cdot n\, d\Gamma.$$

Then

$$\int_\Gamma \left(r\varphi\cdot n - \nu\frac{\partial w}{\partial n}\cdot\varphi\right) d\Gamma = \int_\Gamma \left[r(\varphi-\bar{\varphi})\cdot n - \nu\frac{\partial w}{\partial n}\cdot(\varphi-\bar{\varphi})\right] d\Gamma + \frac{1}{|\Gamma|}\left(\int_\Gamma \varphi\cdot n\, d\Gamma\right)\left(\int_\Gamma r\, d\Gamma\right) - \frac{\nu}{|\Gamma|}\left(\int_\Gamma \varphi\cdot n\, d\Gamma\right)\int_\Gamma \frac{\partial w}{\partial n}\cdot n\, d\Gamma, \ \forall \varphi \in C^{2,\alpha}(\Gamma)^n, \tag{4.13}$$

where $|\Gamma| = \int_\Gamma d\Gamma$.

The first integral on the right-hand side of (4.13) is zero since $\varphi - \bar{\varphi} \in \dot{C}^{2,\alpha}(\Gamma)^n$. Since r is defined up to a constant, we may choose r such that $\int_\Gamma r\, d\Gamma = 0$, and (4.13) reduces to

$$\int_\Gamma \left(r\varphi\cdot n - \nu\frac{\partial w}{\partial n}\cdot\varphi\right) d\Gamma = -c\int_\Gamma \varphi\cdot n\, d\Gamma, \quad \forall \varphi \in C^{2,\alpha}(\Gamma)^n, \tag{4.14}$$

with $c = \dfrac{\nu\int_\Gamma \frac{\partial w}{\partial n}\cdot n\, d\Gamma}{|\Gamma|}$.

Finally, (4.14) gives

$$\nu\frac{\partial w}{\partial n} - (r+c)n = 0 \quad \text{on } \Gamma. \tag{4.15}$$

In the general case where Γ is multi-connected, we proceed similarly to obtain

$$\nu\frac{\partial w}{\partial n} - (r+c_1)n = 0 \quad \text{on } \Gamma_1, \tag{4.16}$$

$$\nu\frac{\partial w}{\partial n} - \left(r - \frac{\int_{\Gamma_i} r\, d\Gamma_i}{|\Gamma_i|} + c_i\right) = 0 \quad \text{on } \Gamma_i, \quad i = 2, \ldots, m, \tag{4.17}$$

(r is now uniquely defined by the condition $\int_{\Gamma_1} r\, d\Gamma_1 = 0$).

To summarize, we arrive at the conclusion that w and r satisfy the system

$$\begin{cases} -\nu\Delta w_i - D_j(a_{0_j}w_i) + (D_i a_{0_j})w_j + D_i r = 0 & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w_{|\Gamma} = 0, \\ (4.16)+(4.17). \end{cases} \tag{4.18}$$

The proof of Lemma 4.2 is now reduced to showing that (4.18) implies $w \equiv 0$.

Taking the divergence of the first equation in (4.18) and using $\operatorname{div} w = \operatorname{div} a_0 = 0$ in Ω (since $a_0, w \in V$), we see that w, r satisfy:

$$\begin{cases} \left.\begin{aligned} &\Delta r - 2D_i a_{0_j}\cdot D_i w_j + \Delta a_{0_j}\cdot w_j = 0 \\ &\nu\Delta w_i + a_{0_j}D_j w_i + (D_i a_{0_j})w_j - D_i r = 0 \end{aligned}\right\} \text{ in } \Omega, \\ w_{|\Gamma} = 0, \\ (4.16)+(4.17). \end{cases} \tag{4.19}$$

But, using (4.19) one can show that the Cauchy data of r, w *are zero on* Γ_1, i.a. $w_{|\Gamma_1} = \dfrac{\partial w}{\partial n_{|\Gamma_1}} = r_{|\Gamma_1} = \dfrac{\partial r}{\partial n_{|\Gamma_1}} = 0$ (see [18] for details).

Hence, by the uniqueness theorem for the Cauchy problem associated to elliptic equations ([2], [6]), one obtains that w and r vanish identically in Ω.

This completes the proof of Lemma 4.2 and of the first part of Theorem 4.1.

The assertion concerning the oddness of the number of elements of $S(f, \nu, \varphi)$ follows from a classical degree theoretical argument and may be found in [18].

4.3. Genericity and analytic structure of the solution set. Using similar techniques one can get the following generic result on the global structure of the solution set (cf. [18] for a detailed proof):

THEOREM 4.2. *Let f be fixed in $C^\alpha(\bar{\Omega})^n \cap H$. Then there exists a residual set $\mathcal{O}(f)$ in $\dot{C}^{2,\alpha}(\Gamma)^n$ such that for $\varphi \in \mathcal{O}(f)$, $S = \bigcup_{\nu>0} S(f, \nu, \varphi)$ is a (not necessarily connected) 1-dimensional analytic submanifold of $V \cap C^{2,\alpha}(\bar{\Omega})^n \times]0, \infty[$.*

Remark 4.1. Theorem 4.2 shows that, for fixed f and generic φ, there is no bifurcation phenomenon with respect to the parameter ν. ■

The next theorem gives a more precise information about the analytic structure of $S(f, \nu, \varphi)$ for generic φ. A less precise result was given in [4] for generic f.

THEOREM 4.3. *Let* $f \in H \cap C^{\alpha}(\bar{\Omega})^n$ *and* $\varphi \in \mathcal{O}(f)$ (*cf.* *Theorem* 4.2). *There exists a subset* $E = E(f, \varphi)$ *of* $]0, \infty[$, $E \cap]\nu_0, +\infty[$ *being finite for each* $\nu_0 > 0$, *such that*

(i) *If* $\nu \in]0, \infty[\setminus E$, $S(f, \nu, \varphi)$ *is a finite set;*

(ii) *If* $\nu \in E$, $S(f, \nu, \varphi)$ *is the disjoint union of a finite number of connected* 1-*dimensional analytic manifolds, and of a finite number of points.*

Proof. Let $S = \bigcup_{\nu>0} S(f, \nu, \varphi)$ and $\Pi: S \to]0, \infty[$ be the natural projection. Let, further, $S_{\nu_0} = \{(u, \nu) \in S,\ \nu > \nu_0\}$. S_{ν_0} is clearly a one-dimensional analytic submanifold of $V \cap C^{2,\alpha}(\bar{\Omega})^n \times]0, \infty[$. Moreover, $\Pi_{|S_{\nu_0}}: S_{\nu_0} \to]\nu_0, \infty[$ is analytic and proper (this is a consequence of Lemma 4.1). By the analytic version of Sard's lemma, the set E_{ν_0} of singular values of $\Pi_{|S_{\nu_0}}$ is discrete (in fact, at most denumerable) and closed (since $\Pi_{|S_{\nu_0}}$ is proper). Moreover, E_{ν_0} is bounded, since $\Pi_{|S_{\nu}}$ is an isomorphism for ν large enough (cf. [4]). Therefore E_{ν_0} is a finite set. For $\nu \in E_{\nu_0}$, $S(f, \nu, \varphi)$ is a compact analytic set of dimension 1, and hence a finite disjoint union of connected analytic manifolds of dimension 0 and 1.

Remark 4.2. (i) Results similar to Theorem 4.1 for nonlinear elliptic operators of the type considered in Section 2 are derived in [17].

(ii) Results similar to Theorems 4.1 and 4.3 hold for time-periodic solutions of the 2-dimensional Navier–Stokes equations (cf. [18]).

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Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978