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NEW IDEAS ON COMPLETE INTEGRABILITY

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The notion of complete integrability for nonlinear Hamiltonian systems is based on a theorem of Liouville. This result states that if a Hamiltonian system defined on $\boldsymbol{R}^{2N}$ has N independent first integrals in involution, then the system is completely integrable (i.e. can be integrated by quadrature). This notion has been extended to infinite dimensional Hamiltonian systems recently be Faddeev, Gardenar, Lax, Novikov (and others) who have shown that certain partial differential equations are integrable in this sense provided one lets $N \to \infty$. In particular, these authors study the Korteweg–de Vries equation

$$u_t = uu_x - u_{xxx}. \tag{1}$$

However, this notion of complete integrability seems limited to two-dimensional partial differential equations. Moreover, the methods developed in those studies, mentioned above, totally break down when a system "nearby" a given integrable system is examined. Finally, these methods do not seem to apply to nonlinear elliptic boundary value problems.

In this article we define a new type of complete integrability for nonlinear elliptic boundary value problem (in fact, for nonlinear continuous mappings between Banach spaces), and we show that this new notion does not suffer from the defects described above.

1. The nonlinear boundary value problem

For explicitness we shall study the following nonlinear elliptic Dirichlet problem:

$$\begin{cases} \Delta u + f(u) = g, \\ \quad u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ with boundary $\partial\Omega$ and $f(u)$ is a C^k

convex function of u $(k \geqslant 2)$, with the asymptotic behaviour

(3) $$0 < \lim_{t\to-\infty} f'(t) < \lambda_1 < \lim_{t\to\infty} f'(t) < \lambda_2$$

where λ_1 and λ_2 denote the lowest 2 eigenvalues of the Laplace operator on Ω relative to the null Dirichlet boundary conditions.

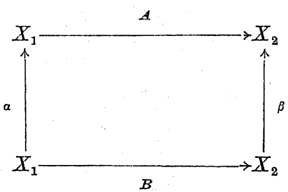
In [1] Ambrosetti and Prodi showed that for $g \in C^{0,\alpha}(\Omega)$ the number of solutions of (2) is either 0, 1 or 2. We now wish to show how the methods we used to study this problem in [2] can be pushed further. In fact, we shall define a notion of complete integrability and we show that (2) is completely integrable in this sense, independently of the domain Ω and the dimension N. Moreover, we show that the methods we use to establish the complete integrability of (2) are "stable" under perturbation in the sense that they yield the expected results on the perturbed problem.

2. The notion of complete integrability

We now define our notion of complete integrability for problem (2).

To this end, let A denote a given smooth mapping between two Banach spaces X_1, X_2. Then we say that A is *C^k-equivalent* to a mapping B if there are C^k-diffeomorphisms α and β such that the following diagram commutes:

(4)



$$A\alpha(x) = \beta B(x) \quad \text{for each } x \in X_1.$$

This just means that the mappings A and B differ by smooth coordinate changes. (Of course, we could extend this idea by assuming B to be a mapping between two other Banach spaces Y_1 and Y_2, provided the diffeomorphisms α and β are defined appropriately.)

Now, since we want to preserve the "Hamiltonian nature" of problem (2), we shall choose $X_1 = X_2$ to be separable Hilbert spaces. Then A can be regarded (relative to an orthonormal basis $(x_1, x_2, \ldots)$) as a coordinate mapping

(5) $$A(x_1, x_2, x_3, \ldots) = (A_1, A_2, A_3, \ldots).$$

The mapping A will be called *C^k-completely integrable* if there is a mapping B of the form

$$B(x_1, x_2, x_3, \ldots) = \big(g_1(x_1), g_2(x_2), g_3(x_3), \ldots\big)$$

such that A and B are C^k-equivalent $(k \geqslant 0)$.

To relate this notion to the boundary value problem (2), we define a mapping A between the Sobolev spaces $H = \mathring{W}_{1,2}(\Omega)$ as follows:

(6) $$(Au, \varphi)_H = \int_\Omega \{\nabla u \cdot \nabla\varphi - f(u)\varphi\}\, dx$$

for every $\varphi \in H$. It is easy to show the following

LEMMA 1. *A is a C^1-mapping of the Hilbert space $H = \mathring{W}_{1,2}(\Omega)$ into itself.*

We now state one of our main results on complete integrability.

THEOREM 2. *The mapping A is C^0-completely integrable in the sense that there are canonical homeomorphisms such that A is equivalent to the mapping $B\colon H \to H$ defined by $B(x_1, x_2, x_3, \ldots) = (x_1^2, x_2, x_3, \ldots)$.*

In this result we can choose the orthonormal basis $(x_1, x_2, \ldots)$ consisting of the normalized eigenfunctions of the Laplace operator relative to the null Dirichlet boundary conditions on $\partial\Omega$.

Simple applications of this result are:

COROLLARY 3. *The mapping A defined by* (6) *is a proper mapping.*

Proof. The mapping B of Theorem 2 is proper and the notion of being proper is preserved under C^0-equivalence.

COROLLARY 4. *The solutions of* (2) *can be found explicitly (provided they exist) in terms of the "canonical coordinate changes" and the eigenfunctions of the Laplace operator.*

COROLLARY 5. *All the singular points of the mapping A are "infinite dimensional"-fold in the sense of Whitney.*

Thus the mapping A defined by (6) is the "*simplest*" *nonlinear operator* that is associated with a nonlinear Dirichlet problem and exhibits bifurcation phenomena *independent* of the domain Ω and the dimension N.

Additional remark. If we suppose $f \in C^k$ $(k \geqslant 2)$, we may show that A is $C^{(k-2)}$-equivalent to B.

To this end we use our results coupled with Nirenberg [3].

3. Idea of the proof of Theorem 2

The proof divides into two distinct parts:

Part I. An analytical part consisting of 4 steps.

Step 1. Reduction to a finite-dimensional problem.

Step 2. Explicit cartesian representation for the singular points of $\varDelta$.

Step 3. Explicit cartesian representation for the singular values of A.

Step 4. Coerciveness estimates for the mapping A.

We sketch the main ideas of this part.

We write the mapping $A: H \to H$ in the form associated with the orthogonal decomposition $H = \operatorname{Ker}(\varDelta+\lambda_1)\oplus H_1$, i.e. we write an element $u \in H$ in the form $u = tu_1+\omega$ (where u_1 is a normalized eigenfunctions of $\varDelta$ on Ω associated with λ_1) and so $u_1 > 0$ in Ω, with $\omega \in H$. Then we show, for a fixed t, that the mapping A_1 defined by

$$\big(A_1(t, \omega), \varphi\big) = \int_\Omega \{\nabla\omega\cdot\nabla\varphi - f(tu_1+\omega)\varphi\}$$

(for $\varphi \in H_1$) is a global homeomorphism of H_1 into itself. This is achieved by using the Lax–Milgram theorem to prove that the inequality

$$\big(A_1'(t, \omega)\varphi, \varphi\big) \geqslant (\varepsilon/\lambda_2)\|\varphi\|_H^2$$

implies

$$\|[A_1'(t, \omega)]^{-1}\| \leqslant \lambda_2/\varepsilon \qquad \text{for a fixed } \varepsilon > 0.$$

The global result now follows from the Hadamard theorem [1].

Then we find out that the singular points and values can be determined by the coordinate representation

$$A(tu_1+\omega) = h(t)u_1+g_1 \tag{7}$$

or, more explicitly, writing $u(t) = tu_1+\omega(t)$,

$$\varDelta u(t)+f\big(u(t)\big) = h(t)u_1+g_1. \tag{8}$$

Let us examine what happens at a singular value; assume $h'(t) = 0$.

LEMMA. *At a singular value $t = t_0$ we have*

$$h''(t_0) = \int_\Omega f''\big(u(t)\big)[u'(t_0)]^3, \tag{9}$$

so that by our assumptions $h''(t_0) > 0$.

Sketch of proof. Consider (8) and differentiate twice with respect to t assuming $h'(t_0) = 0$:

$$\varDelta u'(t)+f'\big(u(t)\big)u'(t) = h'(t)u_1.$$

Since $h'(t_0) = 0$, $u'(t_0)$ is a nontrivial solution of (9) and by the asymptotic conditions (3) we may suppose $u'(t_0) > 0$ in Ω. (See [2].) Next,

$$\varDelta u''(t)+f'\big(u(t)\big)u''(t)+f''\big(u(t)\big)[u'(t)]^2 = h''(t_0)u_1. \tag{10}$$

Since $u''(t_0)$ is a nontrivial solution of (10) for this inhomogeneous equation, we have

$$(11) \qquad \begin{aligned} &\{f''\big(u(t_0)\big)[u'(t_0)]^2 - h''(t_0)u_1\} \perp \ker[\varDelta+f'(u)],\\ &\int_\Omega [f''\big(u(t_0)\big)[u'(t_0)]^2 - h''(t_0)u_1]\,u'(t_0) = 0. \end{aligned}$$

This relation proves (9). The result and the convexity of $f''(u)$ yields the lemma.

Another important fact is that $h(t) \to \infty$ as $t \to \infty$. This follows from the representation

$$h(t) = -\lambda_1 t + \int_\Omega f\big(tu_1+\omega(t)\big)u_1,$$

the asymptotic relation (3) and the fact that as $t \to \infty$, the contribution due to $\omega(f)$ in negligible via the a priori estimate

$$\|\omega'(t, g_1)\|_H \leqslant C \qquad (\text{independent of } t \text{ and } g_1). \tag{12}$$

These facts lead to the following picture of the mapping A.

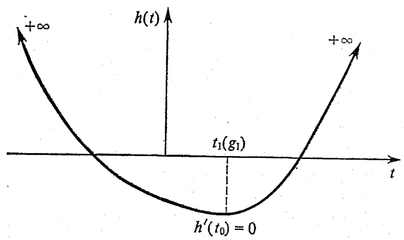


Fig. 1

From this picture we read off the cartesian representation of the singular points and singular values of A.

Part II. The second part of the proof is geometric; it consists in the construction of the diffeomorphisms α and β with use of the facts of Part I.

This part also consists of 4 steps.

Step 1. Layering of the mapping A in accordance with Step 1 of Part I by a diffeomorphism α_1.

Step 2. "Translation" of the singular points of the mapping A to those of B by a diffeomorphism α_2.

Step 3. "Translation" of the singular values of A to those of B by a diffeomorphism a_3.

Step 4. Construction of the final homeomorphism.

Indeed, after Step 3 we find

$$a_3 A a_1 a_2 = \big(a(t, \omega), \omega\big). \tag{13}$$

Using Step 4 of Part I, we represent the right-hand side of (13) as the composition $B\varrho$, where ϱ is a diffeomorphism $H \to H$. Thus

$$a_3 A a_1 a_2 = B\varrho, \tag{14}$$

which is the desired equation.

4. Stability of the methods introduced under a perturbation of A

Under a C^1-perturbation of A in the sense of the metric in H, our analytical results of Step 1 carry over to study perturbation problems.

Indeed, we prove

THEOREM. *Under a suitably restricted C^1-perturbation of A, the number of solutions of the perturbed problem is exactly the same as in* (2), *away from a neighbourhood of the singular values.*

Moreover, in this case, the solutions of (2) *are accurate approximations to the perturbed problem.*

The proof is based on a careful analysis of the steps in Part I above.

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INDEX THEORY AND ELLIPTIC BOUNDARY VALUE PROBLEMS REMARKS AND OPEN PROBLEMS

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1. Index theory for elliptic differential equations

Let M be an oriented Riemannian manifold of dimension n, and SM the *covariant sphere bundle* of differential forms of "length" 1. A linear differ-

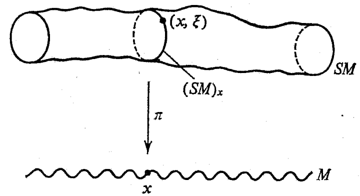

Fig. 1

ential operator of order m, operating between smooth sections of C^∞-vector bundles E and F over M of fibre dimension k can be written in local coordinates in the form

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha,$$

where $x \in M$, $D^\alpha = D^{\alpha_1 \cdots \alpha_n} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and a_α a matrix valued C^∞-function. A is called *elliptic* if

$$\sigma(A)(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x)\, \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n} \in \mathrm{GL}(k, C)$$

for all $(x, \xi) \in SM$.