если g принадлежит линеалу сходимости оператора $\mathfrak{A}_2$ относительно $\{P_l, Q_l\}$.

Заметим, что проекционные методы приближённого решения систем сингулярных интегральных уравнений с оператором вида $\mathfrak{A}_2$ достаточно хорошо разработаны [10].

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AN ANALYTICAL INDEX FORMULA FOR ELLIPTIC PSEUDO-DIFFERENTIAL BOUNDARY PROBLEMS IN THE HALF-SPACE

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1. Introduction

In this note we give an analogue to a nice analytical index formula derived by B. V. Fedosov [2] (cf. also L. Hörmander [4]) for elliptic pseudo-differential operators (pdo) in $\boldsymbol{R}^n$, resp. on closed compact manifolds with trivial normal bundle. We consider a class of pd boundary problems introduced by L. Boutet de Monvel [1] (cf. also [7] or [5]). It is assumed that the reader is familiar with standard facts of the theory of pd boundary problems.

Our starting point is the so-called "coarse" index formula (Theorem 1) from [6]. Using homotopy arguments it is possible to reduce the number of derivatives involved in this formula in the expressions for densities on the half-space and on the boundary. The result is formulated in Theorem 2. A proof will be published elsewhere. Also the case of manifolds with a boundary will be treated in another paper.

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2. Preparations

Denote by $S^m(\boldsymbol{R}^n_+, \boldsymbol{R}^n)$ the space of pd symbols of order m having an extension to S^m-symbols in some open neighbourhood of $\boldsymbol{R}^n_+$. In the following we shall assume everywhere that the symbols considered are independent of x_n near $x_n = 0$ ((x', x_n) are the coordinates in $\boldsymbol{R}^n_+ = \{x_n \geqslant 0\}$). Remark that this is not an essential restriction, as regards the index, since it is always satisfied after a suitable homotopy (cf. [3]). Moreover, we assume the so-called transmission property of a pd symbol $\sigma_A \in S^m$ (cf. [1], [5] or [7]), which implies that the corresponding pdo A applied to extension-by-zero maps sends $C_0^\infty(\boldsymbol{R}_+^n, \boldsymbol{C}^k)$ into $C^\infty(\boldsymbol{R}_+^n, \boldsymbol{C}^k)$ (that means, functions smooth up to the boundary go into functions smooth up to the boundary). The space of S^m-symbols with transmission property is denoted by $\mathfrak{A}^m$. Denote by BS^m the space of boundary symbols of order m. This is the space of families of Wiener–Hopf operators

$$(1)\qquad \sigma_Y(\mathscr{A})(x', \xi') = \begin{pmatrix} \pi^+\sigma'_A + \pi'\sigma_B & \sigma_K \\ \pi'\sigma_T & \sigma_Q \end{pmatrix} : \begin{matrix} H^+\otimes \boldsymbol{C}^k \\ \oplus \\ \boldsymbol{C}^l \end{matrix} \to \begin{matrix} H^+\otimes \boldsymbol{C}^k \\ \oplus \\ \boldsymbol{C}^j \end{matrix}$$

with parameter space $T^*\boldsymbol{R}^{n-1}$ (for the exact definition cf. [5] or [7]).

A couple $(\sigma_A, \sigma_Y(\mathscr{A})) \in S^m \times \mathrm{BS}^m$ is called *compatible* if the function $\sigma'_A(x', \xi', \nu)$ in $\sigma_Y(\mathscr{A})$ equals to the restriction of σ_A to $x_n = 0$. A compatible couple $(\sigma_A, \sigma_Y(\mathscr{A})) \in \mathfrak{A}^m \times \mathrm{BS}^m$ is called a *symbol of order m in the half-space.* The space of all such symbols is denoted by $\mathscr{S}^m$. With the use of symbols in $\mathscr{S}^m$ one defines the class Op^m of operators of the form

$$\mathscr{A} = \begin{pmatrix} r+A+r'B & K \\ r'T & Q \end{pmatrix} : \begin{matrix} C_0^\infty(\overline{\boldsymbol{R}_+^n}, \boldsymbol{C}^k) \\ \oplus \\ C_0^\infty(\boldsymbol{R}^{n-1}, \boldsymbol{C}^l) \end{matrix} \to \begin{matrix} C^\infty(\overline{\boldsymbol{R}_+^n}, \boldsymbol{C}^k) \\ \oplus \\ C^\infty(\boldsymbol{R}^{n-1}, \boldsymbol{C}^j) \end{matrix}$$

To the operation of composition of (properly supported) operators in L^∞ there corresponds the composition rule in S^∞:

$$(2)\qquad \sigma_A^{(1)}\circ\sigma_A^{(2)} \sim \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha \sigma_A^{(1)}(x, \xi) D_x^\alpha \sigma_A^{(2)}(x, \xi)$$

and $S^\infty/S^{-\infty}$ is a graded algebra.

Similarly, $\mathrm{BS}^\infty/\mathrm{BS}^{-\infty}$ is turned into a graded algebra by the composition rule

$$\sigma_Y(\mathscr{A}_1)\circ\sigma_Y(\mathscr{A}_2) \sim \sum_{\alpha'} \frac{1}{\alpha'!}\,\partial_{\xi'}^{\alpha'}\sigma_Y(\mathscr{A}_1) D_{x'}^{\alpha'}\sigma_Y(\mathscr{A}_2)$$

where on the right one takes the composition in the sense of operators (1). The same holds for $\mathscr{S}^\infty/\mathscr{S}^{-\infty}$ if we set for $\sigma^{(i)} = (\sigma_A^{(i)}, \sigma_Y(\mathscr{A}_i))$, $i = 1, 2$,

$$\sigma^{(1)}\circ\sigma^{(2)} \sim (\sigma_A^{(1)}\circ\sigma_A^{(2)}, \sigma_Y(\mathscr{A}_1)\circ\sigma_Y(\mathscr{A}_2)).$$

Remark that x_n-independence of $\sigma_A^{(i)}$ near $x_n = 0$ implies that compatibility is preserved under composition.

Define for any $N \in \boldsymbol{N}$

$$\sigma_A^{(1)}\circ \ldots \circ\sigma_A^{(k)}|_N = \sum_{\Sigma\alpha_i<N} c_{\alpha_1,\ldots,\alpha_{k-1},\beta_2,\ldots,\beta_k}\,\partial_\xi^{\alpha_1}\sigma_A^{(1)}\partial_\xi^{\alpha_2}D_x^{\beta_2}\sigma_A^{(2)}\ldots D_x^{\beta_k}\sigma_A^{(k)},$$

where the constants $c_{\alpha_1,\ldots,\alpha_{k-1},\beta_2,\ldots,\beta_k}$ are obtained from (2). Similarly, define $\sigma_Y(\mathscr{A}_1)\circ \ldots \circ\sigma_Y(\mathscr{A}_k)|_N$ and $\sigma^{(1)}\circ \ldots \circ\sigma^{(k)}|_N$.

Recall the definition of the trace of a symbol $\sigma = (\sigma_A, \sigma_Y(\mathscr{A})) \in \mathscr{S}^m$ for $m < -n$, $l = j$, for which σ_A has compact x-support and $\sigma_Y(\mathscr{A})$ has compact x'-support. Set

$$(3)\qquad \operatorname{Tr}\sigma = \int_{x_n>0} \operatorname{tr}\sigma_A(x, \xi)\,d\xi\,dx + \int \{\operatorname{tr}(\pi'_\nu\sigma_B(x', \xi', \nu, \nu) + \sigma_Q(x', \xi'))\}\,d\xi'\,dx'$$

where tr denotes the matrix trace. The assumptions about σ yield absolute convergence of the integrals in (3). We shall use the notation

$$\operatorname{tr}'\sigma_Y(\mathscr{A}) := \operatorname{tr}(\pi'_\nu\sigma_B(x', \xi', \nu, \nu) + \sigma_Q(x', \xi')).$$

In what follows we shall assume the stabilization of the considered symbols near ∞, i.e., we assume that for $\sigma \in \mathscr{S}^m$, $\sigma = (\sigma_A, \sigma_Y(\mathscr{A}))$,

$$\sigma_A(x, \xi) = \sigma_A(\infty, \xi) \quad \text{for } |x| \geqslant c,$$
$$\sigma_Y(\mathscr{A})(x', \xi') = \sigma_Y(\mathscr{A})(\infty, \xi') \quad \text{for } |x'| \geqslant c$$

for a suitable constant c.

DEFINITION. Let $\sigma = (\sigma_A, \sigma_Y(\mathscr{A})) \in \mathscr{S}^m$. The symbol σ is called *elliptic* if there is an $R > 0$ such that:

(i) $\sigma_A(x, \xi)$ is for $|\xi| \geqslant R$ an invertible matrix and $\sigma_Y(\mathscr{A})(x', \xi')$ is for $|\xi'| \geqslant R$ an invertible boundary symbol (in the sense of (1));

(ii) $|\sigma_A^{-1}(x, \xi)| \leqslant c\langle\xi\rangle^{-m}$ for $|\xi| \geqslant R$ and $\sigma_Y(\mathscr{A})^{-1}(x', \xi') \in \mathrm{BS}^{-m}$ for $|\xi'| \geqslant R$.

An operator $\mathscr{A} \in \mathrm{Op}^m$ is called *elliptic* if its symbol is elliptic.

3. The index formula

Recall the main result of [6] applied to the half-space situation where symbols stabilize near ∞.

THEOREM 1. *Let $\mathscr{A} \in \mathrm{Op}^m$ be an elliptic operator with symbol σ, which stabilizes near ∞. Then*

$$\mathscr{A}: \begin{matrix} C_+^\infty(\overline{\boldsymbol{R}_+^n}, \boldsymbol{C}^k) \\ \oplus \\ C_+^\infty(\boldsymbol{R}^{n-1}, \boldsymbol{C}^l) \end{matrix} \to \begin{matrix} C_+^\infty(\overline{\boldsymbol{R}_+^n}, \boldsymbol{C}^k) \\ \oplus \\ C_+^\infty(\boldsymbol{R}^{n-1}, \boldsymbol{C}^j) \end{matrix}$$

(C_+^∞ *denotes the space of* C^∞*-functions, having together with all its derivatives, limits at* ∞), $\mathscr{A}$ *is a Fredholm operator and for any* $N > n$ *and* N' *sufficiently large we have*

$$\operatorname{ind}\mathscr{A} = \operatorname{Tr}\Big(\big(1-\sigma(\mathscr{R})\circ\sigma(\mathscr{A})\big)^N\big|_{N'} - \big(1-\sigma(\mathscr{A})\circ\sigma(\mathscr{R})\big)^N\big|_{N'}\Big), \tag{4}$$

where $\sigma(\mathscr{R})$ *is an arbitrary extension of* $\sigma(\mathscr{A})^{-1}$ *for small* ξ, *respectively* ξ'.

Applying arguments similar to those used in [2] or [4], one can derive from (4) an easier formula. Consider $\sigma_A(x, \xi)$, respectively $\sigma_Y(\mathscr{A})(x', \xi')$, as exterior forms of degree 0 with values in the $k\times k$ matrices, resp. in the boundary symbols on the half-line. Then $d\sigma_A$, resp. $d'\sigma_Y(\mathscr{A})$ (d' involves only derivations with respect to x' and ξ'), are 1-forms with values in the corresponding spaces. If a is a p-form with values in the $k\times k$ matrices then $\operatorname{tr} a$ is a p-form with values in C. Similarly, if β is a p-form with values in the boundary symbols on the half-line then $\operatorname{tr}'\beta$ is a usual p-form.

THEOREM 2. *Under the conditions of Theorem* 1,

$$\operatorname{ind}\mathscr{A} = -\frac{(n-1)!}{(2n-1)!(2\pi i)^n}\int\limits_{S_R^*\mathbf{R}_+^n}\operatorname{tr}\big[(\sigma_A^{-1}d\sigma_A)^{2n-1}\big] - \tag{5}$$
$$-\frac{(n-1)!}{i(2n-2)!(2\pi i)^{n-1}}\int\limits_{S_R^*\mathbf{R}^{n-1}}\operatorname{tr}\pi_\nu'\big[(\sigma_A^{-1}d'\sigma_A)^{2n-3}\sigma_A^{-1}\partial_\nu\sigma_A\big] -$$
$$-\frac{(n-2)!}{(2n-3)!(2\pi i)^{n-1}}\int\limits_{S_R^*\mathbf{R}^{n-1}}\operatorname{tr}'\big[\big(\sigma_Y(\mathscr{A})^{-1}d'\sigma_Y(\mathscr{A})\big)^{2n-3}\big]$$

where S_R^*X *denotes the cosphere bundle over* X *with radius* R.

Remark. For boundary problems for differential operators a similar formula (with a more involved second term) was given by B. V. Fedosov in [3], even in the situation of manifolds with trivial normal bundle. We intend to devote another paper to a formula like (5) for pd boundary problems on manifolds.

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