PARAMETRICES FOR A CLASS OF P. D. OPERATORS AND APPLICATIONS

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We consider a class of anisotropic P.D. operators defined in a region $\Omega \subset \mathbb{R}^n$, and degenerating on a symplectic submanifold of $T^*\Omega \setminus 0$, which is a generalization of certain well-known operator classes (see [6]). The results obtained are used in the construction of parametrices for degenerate operators.

1. Notation

Write $\mathbb{R}_n^\infty = \mathbb{R}_+ \times \mathbb{R}_+^{n-1}$ (1 \leq n < \infty) and let $(\zeta, \xi) = (\xi, \eta)$ be the points in $T^*\mathbb{R}^n$.

Let $M = (M_1, M_2)$ be a pair of positive integers and denote by \( \text{OPS}_M^N(\Omega) \) (\( M \in \mathbb{R}, \Omega \subset \mathbb{R}^n \)) the class of all P.D. operators of the form

\[
Pf(x) = (2\pi)^{-n} \int e^{i\xi \cdot \zeta} \phi(x, \zeta) d\zeta
\]

such that:

(i) the symbol $\phi(x, \zeta)$ belongs to $C^\infty(\Omega \times \mathbb{R}^n)$;

(ii) there exists a sequence $(p_{n-j})_{n-j \to 0}$ of $C^\infty$-functions on $\Omega \times (\mathbb{R}^n \setminus \{0\})$ with the following properties:

(a) $p_{n-j}(x, y; \lambda^{M_1} \xi, \lambda^{M_2} \eta) = \lambda^{-j} p_{n-j}(x, y, \xi, \eta), \forall j > 0$,

(b) $\phi \approx \sum p_{n-j}$, i.e.,

$$\left| \frac{\partial^{\alpha} \partial^{\beta} \phi}{\alpha! \beta!} \left[ p - \sum p_{n-j} \right] \right| = O((|\xi|^{M_1} + |\eta|^{M_2})^{n-N - |M_1| - |M_2|})$$

as $|\xi| + |\eta| \to \infty$, for all $N, \alpha, \beta, \gamma$, locally uniformly in $\Omega$.

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The function \( p_\alpha \) is called the principal symbol of \( P \), and we set
\[
\text{Char}(\mathcal{D}) = \{(x, \xi) \in T^*\Omega \setminus 0 \mid p_\alpha(x, \xi) = 0\}.
\]

Operators in \( \text{OPS}^m_\phi(\Omega) \) extend by continuity to operators from \( \mathcal{E}'(\Omega) \) to \( \mathcal{D}'(\Omega) \).

For every \( f \in \mathcal{D}'(\Omega) \) we define:
\[
\text{WF}_M(f) = \bigcap_{P \in \text{OPS}^m_\phi(\Omega)} \text{Char}(P).
\]

\( \text{WF}_M(f) \) is a closed subset of \( T^*\Omega \setminus 0 \), stable under the dilations \( (\tau, \eta) \mapsto (\tau^\alpha \xi, \tau^\beta \eta) \), \( \lambda > 0 \), and projects onto \( \text{supp}(f) \) under the natural mapping \( \pi: T^*\Omega \to \Omega \). It can be proved that for every \( f \in \text{OPS}^m_\phi(\Omega) \) we have
\[
\text{WF}_M(f) = \text{WF}_M(f) \cup \text{Char}(P), \quad \forall f \in \mathcal{D}'(\Omega).
\]

For a linear continuous operator \( A: \mathcal{C}_c^\infty(\Omega) \to \mathcal{D}'(\Omega) \) with distribution kernel \( K_A \in \mathcal{D}'(\Omega \times \Omega) \), we can define the wave front set \( \text{WF}_{M, k}(K_A) \in \mathcal{D}'(\Omega \times \Omega) \setminus 0 \) in the obvious way, and so we see that the first inclusion in (4) is a trivial consequence of the fact that for any \( f \in \text{OPS}^m_\phi(\Omega) \) we have
\[
\text{WF}_M(f) \subset \text{WF}_{M, k}(K_f).
\]

For any \( \omega \in \mathcal{D}'(\omega) \) we define \( H^s_{\text{loc}}(\omega) \) as the set of all distributions \( u \in \mathcal{D}'(\omega) \) such that:
\[
\int (1 + |\xi|^2 + |\eta|^2)^{-s} |\hat{u}(\xi, \eta)|^2 d\xi d\eta < \infty,
\]
for all \( \varphi \in \mathcal{C}_c^\infty(\omega) \).

We say that \( P \in \text{OPS}^m_\phi(\Omega) \) is hypoelliptic in \( \Omega \) with loss of \( r \) anisotropic derivatives, \( r \geq 0 \), if for any \( \omega \in \mathcal{D}'(\omega) \) and for all \( \varphi \in \mathcal{C}_c^\infty(\omega) \) the following implication holds:
\[
f \in \mathcal{D}'(\Omega), \quad Pf \in H^s_{\text{loc}}(\omega) \Rightarrow f \in H^{s-r}_{\text{loc}}(\omega).
\]

Operators with loss of \( \delta \)-derivatives are exactly those operators \( P \) for which \( \text{Char}(P) \) is empty. For most of the concepts introduced aboverefer, e.g. to R. Lascar [3].

2. A class of P. D. operators

Let
\[
\Sigma = \{(x, y; \xi, \eta) \in T^*\Omega \setminus 0 \mid x = \xi = 0\}
\]
and let \( k, l \) be positive rational numbers. By \( N^m_{k, l}(\Omega; \Sigma) \) we denote the class of all operators \( P \in \text{OPS}^m_\phi(\Omega) \) such that:
\[
(\text{i}) \quad \frac{\partial^k \partial^l p_\alpha}{\partial x^k \partial y^l} \mid_{x=y=0} = 0 \quad \text{if} \quad \{a, b, l, k \} < k
\]
and
\[
\sum_{\{a, b, l, k \} = k} \frac{1}{a!b!} \frac{\partial^k \partial^l p_\alpha}{\partial x^k \partial y^l} \mid_{x=y=0} r^p \neq 0 \quad \text{if} \quad (l, r) \neq (0, 0);
\]
\[
(\text{ii}) \quad \text{For} \quad j \leq kl/(1+l) \text{ one has:}
\]
\[
\frac{\partial^k \partial^l p_\alpha}{\partial x^k \partial y^l} \mid_{x=y=0} = 0 \quad \text{if} \quad \{a, b, l, k \} < k - \frac{j+1}{1-l}.
\]

Remark. When \( m = M = l = 1, k \in \mathbb{Z}_+ \), we obtain the classes considered by Sjöstrand [9] (written in a particular system of coordinates).

Various other classes, e.g. those considered by Menikoff [5], are included in \( N^m_{k, l}(\Omega) \).

Let us observe that \( \bigcup_{m \in \mathbb{Z}_+} N^m_{k, l}(\Omega) \) is an algebra with respect to composition and that it is closed under the involution \( P \mapsto P^* \).

Examples (Models of operators), \( m = 2 \):
\[
1) \quad P = a_1 \partial_x^2 + b \partial_x \partial_y + c \partial_y^2 \quad \text{if} \quad a, b, c \text{ are suitable functions}
\]
\[
2) \quad P = a_1 \partial_x^2 + b \partial_x \partial_y + c \partial_y^2 \quad \text{in} \quad C^\infty(\mathbb{R}^2).
\]

With any operator \( P \in N^m_{k, l}(\Omega; \Sigma) \) we associate a family of differential operators with polynomial coefficients in \( \mathbb{R}^2 \), depending on the parameter \( g \in \Sigma \). More precisely, for any \( g \in \Sigma \) we put:
\[
P^g; t, D_t \mid_{D_t} = \sum_{j=0}^{k+\frac{1}{l}} \sum_{\{a, b, l, k \} = k} \frac{1}{a!b!} (\partial^k \partial^l p_\alpha)(g) t^a D_x^b.
\]

It can be shown that \( P^g; t, D_t \mid_{D_t} \) has a finite index (ker \( P^g; t, D_t \mid_{D_t} \) has a finite index).

Main Theorem. Let \( P \in N^m_{k, l}(\Omega; \Sigma) \); then:

I. If \( P \) is hypoelliptic with loss of \( r \) derivatives in \( \Omega \), then \( r \vee k/(1+l) \).

II. If \( P \) is hypoelliptic with loss of \( kl/(1+l) \) derivatives, then
\[
\text{ker} P^g; t, D_t \mid_{D_t} = \{0\}, \quad \forall g \in \Sigma.
\]

III. Let \( \Sigma \) be satisfied (resp. let \( P^g; t, D_t \mid_{D_t} \) be, for any \( g \), a surjective map from \( \mathcal{D}'(\Omega) \) onto \( \mathcal{D}'(\Omega) \)); then there exists a linear operator
\[
R: H^s_{\text{loc}}(\Omega) \to H^{s-kl/(1+l)}_{\text{loc}}(\Omega),
\]
continuous for all \( t \), such that:
The isotropic principal symbol of \( P \) on \( \Gamma \) is \( \sigma^*(\mathcal{A}_\ell)(y, \eta) \), which vanishes exactly of order \( m_0 \) on the surface \( s = 0 \) and whose hamiltonian vector field is not collinear with the radial vector field. Let \( \mathcal{A} \) be a classical P.D. operator with principal symbol \( \sigma(\xi_1 + \eta_1)^{1/2} \); since \( \lambda_j \geq 2\lambda_j \) it is easy to see that we can find classical P.D. operators \( \mathcal{A}_\ell \) of order \( m_0 - m_1 \) such that \( P = \sum_{\ell} \mathcal{A}_\ell \lambda^\ell \) (Levi’s condition).

We can then apply a result of Chazarain [1] to conclude that not only the equality \( WF(f) = WF(Pf) \) fails to hold (though sing \( supp(f) = \text{singsupp}(Pf) \)), but, moreover, that there is a propagation of singularities in the fibers \( T^*_\mathbb{R}^n \cap \Gamma \). A typical example is given by the Kaniou operator \( iD_\xi - \xi D_\eta^\perp \).

### 3. Applications

We sketch only some of them.

**A. Operators with symplectic characteristic manifold.** Let \( M \) be a manifold and let \( P \in \text{OPS}^\infty(M) \) be a classical P.D. operator in \( M \) with symbol \( p \sim p_n + \ldots \). Let \( \Sigma \), \( \Sigma \) be two conic sub-manifolds of \( T^*M \times 0 \), of codimension \( n \), and such that

(i) \( \Sigma \) regular involutive (regular = radial vector field \( \mathcal{F}(\Sigma) \));

(ii) \( \Sigma_\perp \) is involutive;

(iii) \( \Sigma \cap \Sigma_\perp \neq \emptyset \) with transversal intersection, and such that \( \sigma \in \mathcal{S}(\Sigma) \) is not singular, \( \forall \not\in \Sigma \).

We impose on \( P \) the following hypothesis:

\[
\sum_{\ell} \sigma(\xi) \xi^{m - \ell} \psi = 0, \quad \forall \not\in \Sigma
\]

for some \( k < 1 \) or \( r < 1 \). In the cases \( k = 1 \) or \( r = 1 \) which imply \( s = 1 \), we obtain a parametrix (left or right) for a class of sub-elliptic operators. These results are well known, only the proof seems to be simple.

**Example.**

\[
D_\xi + i\varphi D_\eta, \quad D_\xi + i\varphi \psi D_\eta (r \text{ even}), \quad D_\xi + i\varphi \psi D_\eta
\]

When \( k \) and \( r \) are greater than 1, some hypotheses on the lower order terms in the symbol of \( P \) are needed to ensure e.g. hypoeellipticity. Suppose \( k = 2 \) and \( r = 2 \); and assume the hypothesis:

\[
| p_n(\xi, \eta) | \leq |\xi| \{ |D_\xi (\xi, \eta)| + \delta_\lambda (\xi, \eta) \}^{t_\lambda / r} + \delta_\lambda (\xi, \eta) t_\lambda = \max(0, \eta)
\]

where \( t_\lambda = \max(0, \eta) \). Note that conditions (11) and (12) are invariant under general homogeneous canonical transformations. Then for every \( p \in \Sigma \) we construct in \( N_\lambda = T^*M / T(M) \) a differential operator \( P_\lambda \) such that \( P \) is microlocally hypoeelliptic with loss of \( 2\sigma/(\sigma + 2) \) derivatives iff \( P_\lambda \) is injective for every \( \not\in \).

\[\mathcal{I} = \{ (x, y; \xi, \eta) \in T^*\mathbb{R}^n \times 0 : \xi \neq 0 \}.\]
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EXAMPLE. $D^2_x + \omega^2D^2_y + \lambda x^{-1}D_y$.

The proofs of all the above results consist in reducing by a suitable canonical transform, to the case of

$$\Sigma = \{(x, y; \xi, \eta) = T^x (F^x \times F^y) \cap \{x = \xi = 0\}$$

so that, microlocally, $P$ belongs to some $N^2_{\Omega}(\mathbb{R}^4; \Sigma)$ and recognizing that the hypotheses imposed on $P$ allow to apply the main theorem.

B. Operators with involutive characteristic manifold. We make the same assumptions on $M$, $\Sigma_1$, $\Sigma_2$ as before. Suppose that $P \in \text{OPS}^\infty(M)$ and suppose that:

$$p_m(\zeta, \xi) = \langle \gamma^m \delta_3(\zeta, \xi), \xi \rangle^k, \quad k \in \mathbb{N}.$$  

For well-known reasons we consider only the case $k \geq 2$ and define:

$$p_0(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^{n} \frac{\partial p_m(\xi, \xi)}{\partial \xi_j}, \quad k \in \mathbb{N},$$  

which is called the sub-principal symbol of $P$.

For every $q \in \Sigma_1$, let $X$ be a smooth vector field on $T^x M$ which is transversal to $\Sigma_1$ at $q$ and define:

$$I_p(q; X) = \frac{1}{h!} (X^p)_{p_m}(q) + p'(q).$$  

Then if $I_p(q; X) \neq 0$ for every $q$ and $X$, we get that $P$ is microlocally hypoelliptic with loss of 1 derivative.

EXAMPLE. $D^2_x + \lambda x^{-1}D_y$.

Condition $I_p(q; X) \neq 0$ implies that $p'(q) \neq 0$. In the case where $p'(x)$ is identically zero (and $k = 2$), various results are known concerning propagation of singularities for the solutions of $Pu = f$ (see [7]).

We consider here the case of $k = 2$ and suppose that $p'(x)$ vanishes exactly of order $r$ on $\Sigma_1 \cap \Sigma_2$. We make the following hypotheses:

(i) $p_m$ takes values in a closed convex cone $\Delta \subset C$ (with opening $\leq \gamma$);

(ii) $p'_m$ takes values in a closed convex cone $\Delta' \subset C$;

(iii) $\Delta \cap (-\Delta') = \{0\}$.

Under the above hypotheses $P$ is microlocally hypoelliptic with loss of $(2r+3)/2$ derivatives (a two-sided parametrix can be constructed).

EXAMPLE. $D^2_x + i\omega^2D_y$.

Lascar [4] has obtained similar results by a different technique. We can also consider the case of $k \geq 2$, $r = 1$ (under some hypotheses on the range of $p_m$ and $p'_m$) and obtain a two-sided microlocal parametrix.

REFERENCES
