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NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS IN THE ORLICZ–SOBOLEV SPACES

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This paper deals with the existence and approximation of the solutions of the initial-boundary value problems for nonlinear parabolic equations of the form

$$(\mathrm{E}_1) \qquad \frac{\partial u}{\partial t}+\sum_{|i|\leqslant m}(-1)^{|i|}D^iA_i(x,u,\ldots,\nabla^m u)=f(x,t)$$

in $(x,t)\in\Omega\times(0,T)$, where $\Omega\subset\boldsymbol{R}^n$ is a bounded domain with Lipschitzian boundary $\partial\Omega$ and $T<\infty$. These problems have been extensively studied by many authors in the case where the coefficients A_i have polynomial growth in u and its derivatives.

It is our goal here to extend the existence results to the cases where the coefficients do not necessarily satisfy this condition. When the coefficients A_i are rapidly (or slowly) increasing, then it seems to be appropriate to formulate the problem of existence in Banach spaces of the Orlicz–Sobolev type, which are not reflexive, in general. In such a case the corresponding operator of monotone type (i.e. the corresponding elliptic operator of (E_1)) is not bounded nor everywhere defined and, generally, not coercive. Nonlinear elliptic boundary value problems with operators of the type just described have been studied by J. P. Gossez in [1]. Applying Rothe's method (recently developed in [2]–[5]) and the results of [1], we obtain the existence results for the corresponding nonlinear parabolic initial-boundary value problems.

As an example for a rapidly increasing coefficient A_i stands, e.g.,

$$A_i(x,\xi)=\xi_i\exp(\xi_i^2) \quad \text{or} \quad A_i(x,\xi)=\xi_i\exp\Big(\sum_{|i|\leqslant m}a_i\xi_i^2\Big)$$

where $a_i\geqslant 0$ for $|i|\leqslant m$. As an example for a slowly increasing coefficient A_i stands, e.g., $A_i(x,\xi)=\dfrac{\xi_i}{|\xi_i|}\ln(|\xi_i|+1)$.

In Section 1 we present an abstract result and in Section 2 we present applications to the initial-boundary value problems with equations of type (E_1).

Notations and definitions. Let Y and Z be real Banach spaces in duality with respect to a continuous pairing $\langle\cdot,\cdot\rangle$ and let Y_0, Z_0 be subspaces of Y and Z, respectively.

DEFINITION 1 (see [1]). The system $(Y, Y_0; Z, Z_0)$ is a *complementary system* if, by means of $\langle\cdot,\cdot\rangle$, Y_0^* can be identified (i.e. is linearly homeomorph c) to Z and Z_0^* to Y.

Let H be a real Hilbert space with its scalar product (,) and the norm $\|\cdot\|$. We identify H^* with H.

We assume that Z and H are continuously imbedded into a linear locally convex space V. Moreover, we assume that

$$Y_0 \cap H \text{ is dense in } Y_0 \text{ and } H,$$

$$\text{(1.1)}\qquad Z_0 \cap H \text{ is dense in } Z_0 \text{ and } H,$$

$$\|y\|_H = 0 \text{ implies } \|y\|_Y = 0 \text{ for } y \in Y\cap H,$$

where $Y\cap H$ is the Banach space with the norm $\|y\|_{Y\cap H} = \|y\|_Y + \|y\|_H$. By $\sigma(Z, Y_0)$ we denote the weak topology in Z generated by Y_0 $(Z = Y_0^*)$. Similarly $\sigma(Y, Z_0)$ is the weak topology in Y generated by Z_0 $(Y = Z_0^*)$.

Let A be a mapping of $D(A) \subset Y$ into Z with $Y_0 \subset D(A)$.

DEFINITION 2. The operator A is *of type* $(\overline{\mathrm{M}})$ with respect to the complementary system $(Y, Y_0; Z, Z_0)$ if:

(a) $\langle Au - Av, u - v\rangle \geqslant 0$ for all $u, v \in D(A)$;

(b) A is a continuous map from finite-demensional subsets of Y_0 into Z in $\sigma(Z, Y_0)$ topology;

(c) There exists $\varepsilon > 0$ such that A is a bounded map from $B_\varepsilon(0, Y_0)$ into Z ($B_\varepsilon(0, Y_0)$ is the ball with radius ε in Y_0 centered at 0);

(d) For any net $\{y_i, z_i\}$ such that $Ay_i = z_i$, $y_i \in D(A)$, y_i bounded, the conditions $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $z_i \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup\langle z_i, y_i\rangle \leqslant \langle z, y\rangle$ imply $y \in D(A)$ and $Ay = z$.

In [1] a pseudomonotone operator with respect to $(Y, Y_0; Z, Z_0)$ has been defined. Our operator A of type $(\overline{\mathrm{M}})$ with respect to $(Y, Y_0; Z, Z_0)$ is also pseudomonotone.

Let $\|\cdot\|_Y$ be an (equivalent) norm on Y. Denote by $\|\cdot\|_{Y_0}$ the restriction of $\|\cdot\|_Y$ to Y_0, by $\|\cdot\|_Z$ the norm on Z dual to $\|\cdot\|_{Y_0}$ and by $\|\cdot\|_{Z_0}$ the restriction of $\|\cdot\|_Z$ to Z_0. If $\|\cdot\|_Y$ is dual to $\|\cdot\|_{Z_0}$ and $\langle y, z\rangle \leqslant \|y\|_Y\|z\|_Z$ holds for all $y \in Y$, $z \in Z$, then the norm $\|\cdot\|_Y$ is said to be *admissible* (see [1]).

Let C with or without indices stand for positive constans.

Section 1

Let us consider the abstract equation

$$\text{(E}_2\text{)}\qquad \frac{du(t)}{dt} + Au(t) = f(t), \qquad u(0) = u_0,$$

where A is a mapping from $D(A) \subset Y$ into Z which is of type $(\overline{\mathrm{M}})$ with respect to the complementary system $(Y, Y_0; Z, Z_0)$. Let f be an abstract function $\langle 0, T\rangle \to H$. We assume the following coercivity assumption on A:

$$(\mathrm{K}_0)\qquad \langle Au, u\rangle \|u\|_Y^{-1} \to \infty \quad \text{for } \|u\|_Y \to \infty.$$

In many cases this assumption cannot be verified. Assumption (K_0) may be replaced by the following two assumptions:

$$(\mathrm{K}_1)\qquad \langle Au, u\rangle \to \infty \text{ for } \|u\|_Y \to \infty, u \in D(A);$$

(K_2) for all $f \in Z_0 + H$ there exist a (norm) neighbourhood U_f in $Z + H$ and a number $C(f, \lambda)$ such that $\|u\|_{Y\cap H} \leqslant C(f, \lambda)$ for all $u \in D(A)\cap H$ satisfying $Au + \lambda u \in U_f$, where $\lambda > 0$ is a fixed parameter.

The abstract function f: $\langle 0, T\rangle \to H$ is said to be *of bounded variation* if

$$\sup_{\{t_i\}_{i=1}^n} \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| = \bigvee_{\langle 0,T\rangle} (f; H) < \infty,$$

the supremum being taken over finite partitions of the interval $\langle 0, T\rangle$.

Our main result is

THEOREM 1. *Let A be of type $(\overline{\mathrm{M}})$ with respect to the complementary system $(Y, Y_0; Z, Z_0)$ and let (1.1) be satisfied. We assume $u_0 \in D(A)\cap H$, $Au_0 \in H$, $f \in C(\langle 0, T\rangle, H)$ with $\bigvee_{\langle 0,T\rangle}(f; H) < \infty$. Let one of the following assumptions* I *or* II *be satisfied:*

I. (K_0) *is fulfilled;*

II. Y *admits an admissible norm and* (K_1), (K_2) *hold.*

Then there exists a unique $u \in L_\infty(\langle 0, T\rangle, Y\cap H)$ with the following properties

(i) $u(t)$: $\langle 0, T\rangle \to H$ *is Lipschitz continuous and* $u(0) = u_0$;

(ii) *the strong derivative du/dt exists for a.e. $t \in (0, T)$ and we have* $du/dt \in L_\infty(\langle 0, T\rangle, H)$;

(iii) *the equality*

$$\left(\frac{du(t)}{dt}, v\right)+\langle Au(t), v\rangle = (f(t), v)$$

holds for all $v \in Y \cap H$ *and for a.e.* $t \in (0, T)$.

Before proving Theorem 1 let us sketch the idea of the proof. We apply Rothe's method to (E_2) in the following way. We replace (E_2) by its time discretization and we solve the operator equation (successively for $i = 1, \ldots, n$)

$$(1.2)\qquad \frac{u-u_{i-1}}{h}+Au = f_i,$$

where $h = T/n$, $t_i = ih$, $f_i = f(t_i)$, u_0 is from (E_2) and n is a positive integer. By means of u_i $(i = 1, \ldots, n)$ we construct Rothe's function

$$(1.3)\qquad u_n(t) = u_{i-1}+\frac{t-t_{i-1}}{h}(u_i-u_{i-1}) \quad \text{for } t_{i-1}\leqslant t\leqslant t_i,$$

$i = 1, \ldots, n$, and then we prove certain a priori estimates for $u_n(t)$.

Finally, we prove that the $u(t) = \lim u_n(t)$ is a solution to our problem.

LEMMA 1. *For each* $i = 1, \ldots, n$ *there exists a unique* $u_i \in D(A)\cap H$ *such that the equality*

$$\left(\frac{u_i-u_{i-1}}{h}, v\right)+\langle Au_i, v\rangle = (f_i, v)$$

holds for all $v \in Y\cap H$.

Proof. Let us consider the system $(Y\cap H, Y_0\cap H;\ Z+H, Z_0+H)$ with respect to the pairing $[f, v] = \langle f_1, v\rangle+(f_2, v)$ for $v \in Y\cap H$ and $f_1+f_2 = f\in Z+H$. Owing to our assumptions on Z and H we can construct the Banach space $Z+H$ with the norm

$$\|f\|_{Z+H} = \inf_{\substack{f_1\in Z, f_2\in H\\ f_1+f_2=f}} \max(\|f_1\|_Z, \|f_2\|).$$

Moreover, we have

$$(1.4)\qquad [f, v]\leqslant \inf_{\substack{f_1\in Z, f_2\in H\\ f_1+f_2=f}} (\|f_1\|_Z\|v\|_Y+\|f_2\|\,\|v\|)\leqslant \|v\|_{Y\cap H}\|f\|_{Z+H},$$

which proves that the pairing is continuous. In view of (1.1) we have (see [6])

$$(1.5)\qquad \begin{aligned}(Y_0\cap H)^* &= Y_0^*+H^* = Z+H,\\ (Z_0+H)^* &= Z_0^*\cap H^* = Y\cap H\end{aligned}$$

(in the sense of sets and norms) after identifying Z_0^* with Y, Y_0^* with Z and H^* with H. Thus $(Y\cap H, Y_0\cap H;\ Z+H, Z_0+H)$ is a complementary system with respect to the continuous pairing $[\,,\,]$. From (1.4) and (1.5) we conclude that the norm $\|\cdot\|_{Y\cap H}$ is admissible in $Y\cap H$ if the norm $\|\cdot\|_Y$ is admissible in Y.

Let $\lambda > 0$ be a fixed parameter. We define a mapping A_λ from $D(A)\cap H$ into $Z+H$ by

$$[A_\lambda u, v] = \langle Au, v\rangle+(\lambda u, v) \quad \text{for all } v\in Y_0\cap H.$$

We prove easily that A_λ is of type $(\overline{\mathrm{M}})$ with respect to the complementary system $(Y\cap H, Y_0\cap H;\ Z+H, Z_0+H)$.

We claim that A_λ has property (d). Let $\{y_i, z_i\}$ be a net such that $A_\lambda y_i = z_i$, $y_i\in D(A)\cap H$, y_i bounded in $Y\cap H$, $y_i\to y$ for $\sigma(Y\cap H, Z_0+H)$, $z_i\to z\in Z+H$ for $\sigma(Z+H, Y_0\cap H)$ and $\limsup[A_\lambda y_i, y_i]\leqslant [A_\lambda y, z]$. Hence we obtain $\langle Ay_i, y_i\rangle\leqslant C(\lambda)$. From the property (a) of A we have

$$\langle Ay_i, v\rangle\leqslant \langle Ay_i, y_i\rangle+\langle Av, v\rangle-\langle Av, y_i\rangle,$$

from which we conclude

$$\|Ay_i\|_Z = \sup_{\substack{\|v\|_Y\leqslant 1\\ v\in Y_0}} |\langle Ay_i, v\rangle|\leqslant C,$$

because of the property (c) of A. Thus $\{Ay_i\}$ is bicompact in Z in $\sigma(Z, Y_0)$ topology. There exist a subnet (which we denote also by $\{Ay_i\}$) and an element $z_1\in Z$ such that $Ay_i\to z_1$ for $\sigma(Z, Y_0)$. Since $y_i\to y$ for $\sigma(H, H)$, we have

$$Ay_i+\lambda y_i\to z_1+\lambda y \quad \text{for } \sigma(Z+H, Y_0\cap H).$$

Thus $z = z_1+\lambda y$ and

$$\liminf(\lambda y_i, y_i)+\limsup\langle Ay_i, y_i\rangle\leqslant \limsup[A_\lambda y_i, y_i]\leqslant \langle z_1, y\rangle+(\lambda y, y).$$

Since $\|y\|^2\leqslant \liminf\|y_i\|^2$, we have $\limsup\langle Ay_i, y_i\rangle\leqslant\langle z_1, y\rangle$. Hence the property (d) of A implies $y\in D(A)$ and $Ay = z_1$, from which we obtain $y\in D(A)\cap H$ and $A_\lambda y = z$, proving the claim.

Now let us take $\lambda = 1/h$ and consider the equation

$$A_\lambda u = f_i+\frac{u_{i-1}}{h}\in H\subset Z_0+H.$$

Using the existence results of [1] (Theorem 3.1, Theorem 3.10 or Corollary 3.7) we conclude that there exists a unique $u_i\in D(A)\cap H$ such that

$$(1.6)\qquad \left(\frac{u_i-u_{i-1}}{h}, v\right)+\langle Au_i, v\rangle = (f_i, v)$$

holds for all $v \in Y_0 \cap H$. From (1.6) we conclude that the functional $\langle Au_i, v\rangle$ is continuous in v in the norm of the space H. Thus (1.6) holds for all $v \in Y \cap H \subset H$ and the proof is complete.

LEMMA 2. *There exists C such that*

$$\left\|\frac{u_i-u_{i-1}}{h}\right\| \leqslant C, \qquad \|u_i\|_{Y\cap H} \leqslant C$$

holds for all n, $i = 1, \ldots, n$.

Proof. Subtracting (1.6) for $i = j$ and $i = j-1$ and taking $v = (u_i - u_{i-1})/h$ we obtain

$$\left\|\frac{u_i-u_{i-1}}{h}\right\| \leqslant \left\|\frac{u_{i-1}-u_{i-2}}{h}\right\| + \|f_i - f_{i-1}\|;$$

the property (a) of A has been used here. From this recurrent inequality we get

$$\left\|\frac{u_i-u_{i-1}}{h}\right\| \leqslant \sum_{j=1}^{i}\|f_j - f_{j-1}\| + \left\|\frac{u_1-u_0}{h}\right\|.$$

Analogously from (1.6) we deduce the inequality

$$\left\|\frac{u_1-u_0}{h}\right\|^2 \leqslant \left|\left\langle Au_0, \frac{u_1-u_0}{h}\right\rangle\right| + \|f_1\|\left\|\frac{u_1-u_0}{h}\right\|.$$

Since $Au_0 \in H$, we estimate

$$\left|\left\langle Au_0, \frac{u_1-u_0}{h}\right\rangle\right| \leqslant \|Au_0\|\left\|\frac{u_1-u_0}{h}\right\|$$

and hence

$$\left\|\frac{u_i-u_{i-1}}{h}\right\| \leqslant \|Au_0\| + \max_{\langle 0,T\rangle}\|f(t)\| + \bigvee_{\langle 0,T\rangle}(f;\ H).$$

From this inequality and from the triangle inequality we conclude $\|u_i\| \leqslant C$ for all n, $i = 1, \ldots, n$. Then, from (1.6) we obtain $|\langle Au_i, u_i\rangle| \leqslant C$, which implies $\|u_i\|_Y \leqslant C$ for all n, $i = 1, \ldots, n$ (because of the assumption I or II) and the proof of Lemma 2 is complete.

Now we define Rothe's function $u_n(t)$ by means of (1.3) and the step function $\bar{u}_n(t)$: $\bar{u}_n(t) = u_i$ for $t_{i-1} < t \leqslant t_i$, $i = 1, \ldots, n$, $\bar{u}_n(0) = u_0$. Analogously we define $\bar{f}_n(t)$ by means of $f_i = f(t_i)$. On account of Lemma 2 we have

(1.7) $$\|u_n(t) - \bar{u}_n(t)\| \leqslant C/n \quad \text{for all } n \text{ and } t \in \langle 0, T\rangle.$$

LEMMA 3. *There exists $u(t) \in L_\infty(\langle 0, T\rangle, Y \cap H)$ with the following properties:*

(i) *$u_n(t) \to u(t)$ in the norm of the space $C(\langle 0, T\rangle, H)$;*

(ii) *$u(t)$ is Lipschitz continuous from $\langle 0, T\rangle$ into H;*

(iii) *the strong derivative $du(t)/dt$ exists for a.e. $t \in (0, T)$ and we have $du/dt \in L_\infty(\langle 0, T\rangle, H)$.*

Proof. The identity (1.6) can be rewritten in the form

(1.8) $$\left(\frac{d^- u_n(\tau)}{d\tau}, v\right) + \langle A\bar{u}_n(\tau), v\rangle = (\bar{f}_n(\tau), v)$$

for all $v \in Y \cap H$ and $\tau \in (0, T)$, where $\dfrac{d^- u_n(\tau)}{d\tau} = \dfrac{u_i - u_{i-1}}{h}$ for $t_{i-1} < \tau \leqslant t_i$, $i = 1, \ldots, n$. Subtracting (1.8) for $n = r$ and $n = s$ and putting $v = \bar{u}_r(\tau) - \bar{u}_s(\tau)$ we obtain

$$\left(\frac{d^-\big(u_r(\tau)-u_s(\tau)\big)}{d\tau}, u_r(\tau)-u_s(\tau)\right)$$
$$\leqslant \left(\frac{d^-\big(u_r(\tau)-u_s(\tau)\big)}{d\tau}, u_r(\tau)-u_s(\tau)-\big(\bar{u}_r(\tau)-\bar{u}_s(\tau)\big)\right) +$$
$$+\|\bar{u}_r(\tau)-\bar{u}_s(\tau)\|\,\|\bar{f}_r(\tau)-\bar{f}_s(\tau)\|;$$

again the property (a) of A has been used. Integrating this inequality over $(0, t)$ and using the estimates of Lemma 2 and (1.6) we obtain

$$\frac{1}{2}\|u_r(t)-u_s(t)\|^2 \leqslant C_1\left(\frac{1}{r}+\frac{1}{s}\right) + C_2\int_0^t \|\bar{f}_r(\tau)-\bar{f}_s(\tau)\|\,d\tau$$

for all positive integers r, s and $t \in (0, T)$. Hence we conclude that there exists $u \in C(\langle 0, T\rangle, H)$ such that $u_r(t) \to u(t)$ in H uniformly in $t \in \langle 0, T\rangle$ and the proof of assertion (i) is complete.

By Lemma 2 we have

$$\|u_n(t) - u_n(t')\| \leqslant C|t-t'|.$$

Hence and from assertion (i) we deduce

(1.9) $$\|u(t) - u(t')\| \leqslant C|t-t'| \quad \text{for all } t, t' \in \langle 0, T\rangle$$

and assertion (ii) is proved. From (1.9), in virtue of the result of Y. Komura (see [7]), follows assertion (iii). Now, we prove that $u \in L_\infty(\langle 0, T\rangle, Y \cap H)$. Owing to Lemma 2 we have the estimate

(1.10) $$\|u_n(t)\|_{Y\cap H} + \|\bar{u}_n(t)\|_{Y\cap H} \leqslant C$$

for all n and $t \in (0, T)$. Bounded sets in $Y \cap H$ are compact in $\sigma(Y \cap H$,

Z_0+H) topology, since $Y\cap H = (Z_0+H)^*$. Hence there exist $w_t \in Y\cap H$ and a subsequence $\{u_{n_k}(t)\}$ (t is fixed) such that $u_{n_k}(t) \to w_t$ in $\sigma(Y\cap H, Z_0+H)$ topology. On the other hand, $u_{n_k}(t) \to w_t$ in $\sigma(Y\cap H, H)$ topology, which is weaker than $\sigma(Y\cap H, Z_0+H)$ topology. According to Lemma 3 and (1.1) we have $w_t = u(t)$ and, moreover, the original sequence $u_n(t)$ converges to $u(t)$ in $\sigma(Y\cap H, Z_0+H)$ topology. Hence the conclusion of the proof follows from (1.10).

LEMMA 4. *Let $u(t)$ be as in Lemma 3. Then $u(t) \in D(A)\cap H$ and $A\bar{u}_n(t) \to Au(t)$ in $\sigma(Z, Y_0)$ topology for all $t \in (0, T)$.*

Proof. Let t be fixed. Just as in Lemma 3 we have $u_n(t) \to u(t)$ in $\sigma(Y\cap H, Z_0+H)$ topology, since (1.10) and (1.7) are fulfilled. From (1.8) and Lemma 2 we infer

(1.11) $$|\langle A\bar{u}_n(t), v\rangle| \leqslant C\|v\|$$

for all n and $v \in Y\cap H$. Hence and from Lemma 2 we obtain

(1.12) $$|\langle A\bar{u}_n(t), \bar{u}_n(t)\rangle| \leqslant C$$

for all n. Owing to the property (a) of A we estimate

$$\langle A\bar{u}_n(t), v\rangle \leqslant \langle A\bar{u}_n(t), \bar{u}_n(t)\rangle + \langle Av, v\rangle - \langle Av, \bar{u}_n(t)\rangle$$

for all $v \in D(A)$. Hence, using property (c) of A together with the estimates in Lemma 2 and (1.12), we obtain

$$\|A\bar{u}_n(t)\|_Z = \sup_{\substack{v\in Y_0 \\ \|v\|_Y\leqslant 1}} |\langle A\bar{u}_n(t), v\rangle| \leqslant C < \infty.$$

Thus there exist $w_t \in Z$ and a subsequence $\{A\bar{u}_{n_k}(t)\}$ (which we denote by $\{A\bar{u}_n(t)\}$) such that $A\bar{u}_n(t) \to w_t$ in $\sigma(Z, Y_0)$ topology. Passing to the limit in (1.11) we find out that the functional $\langle w_t, v\rangle$ is continuous in v in the norm of the space H. Owing to (1.11) and Lemma 3 we have

$$\langle A\bar{u}_n(t), u(t)\rangle - \langle A\bar{u}_n(t), \bar{u}_n(t)\rangle \to 0 \quad \text{for } n \to \infty.$$

From these facts and from $\langle A\bar{u}_n(t), v\rangle \to \langle w_t, v\rangle$ for $v \in Y_0\cap H$ we conclude that

$$\langle A\bar{u}_n(t), \bar{u}_n(t)\rangle \to \langle w_t, u(t)\rangle,$$

since $Y_0\cap H$ is a dense set in $H \supset Y\cap H$. Thus, from the property (d) of A, we obtain $u(t) \in D(A)\cap H$ and $Au(t) = w_t$ which ends the proof of Lemma 4.

Proof of Theorem 1. Integrating (1.8) over $(0, t)$ we obtain

(1.13) $$\big(u_n(t), v\big) - (u_0, v) + \int_0^t \langle A\bar{u}_n(\tau), v\rangle d\tau = \int_0^t \big(\bar{f}_n(\tau), v\big) d\tau$$

for all $v \in Y_0\cap H$. In view of Lemma 4 we have

$$\langle Au_n(\tau), v\rangle \to \langle Au(\tau), v\rangle \quad \text{for all } \tau \in (0, T) \text{ and } v \in Y_0\cap H.$$

From the estimate (1.11) we get

$$|\langle Au(t), v\rangle| \leqslant C \quad \text{for all } t \in (0, T)$$

and we see that $\langle Au(t), v\rangle$ is measurable, since $\langle A\bar{u}_n(t), v\rangle$ is a step function in t. Taking limit in (1.13) we obtain

$$\big(u(t), v\big) - (u_0, v) + \int_0^t \langle Au(\tau), v\rangle d\tau = \int_0^t \big(f(\tau), v\big) d\tau$$

for all $v \in Y_0\cap H$. Hence we deduce

(1.14) $$\left(\frac{du(t)}{dt}, v\right) + \langle Au(t), v\rangle = \big(f(t), v\big)$$

for all $v \in Y_0\cap H$ and a.e. $t \in (0, T)$. The functional $\langle Au(t), v\rangle$ is continuous in the norm of the space H (see (1.11) and Lemma 4). Thus (1.14) holds for all $v \in Y\cap H \subset H$. The uniqueness of $u(t)$ can be proved by the following standard argument. If $u_1(t)$, $u_2(t)$ are two solutions of (E_2) then $u(t) = u_1(t) - u_2(t)$ satisfies the inequality

$$\left(\frac{du(t)}{dt}, u(t)\right) \leqslant 0,$$

because of the property (a) of A and (1.14). Hence

$$\|u(t)\| = 0 \quad \text{for all } t \in (0, T)$$

and the proof is complete.

Remark 1. If $f\colon \langle 0, T\rangle \to H$ is Lipschitz continuous, i.e., $\|f(t) - f(t')\| \leqslant C|t-t'|$ holds for all $t, t' \in \langle 0, T\rangle$, then the estimate

$$\|u_n(t) - u(t)\|^2 \leqslant C/n$$

takes place. This fact follows from the proof of Lemma 3 and the estimate $\|\bar{f}_r(t) - f(t)\| \leqslant C/r$.

Section 2

Let us consider an equation of type (E_1), with rapidly (or slowly) increasing coefficients A_i (in their variables). In such a case it proves advisable to consider A_i ($|i| \leqslant m$) as a mapping from a space of Orlicz–Sobolev type (generating a complementary system) into an Orlicz space. Then Theorem 1 can be applied.

We now sketch the fundamental concepts of the theory of Orlicz spaces (for details see [8]). A real valued function $G(t)$ is said to be an *N-function* if it satisfies: $G(t) > 0$ for $t > 0$, $\frac{G(t)}{t} \to \infty$ for $t \to \infty$, $\frac{G(t)}{t} \to 0$ for $t \to 0$, $G(t)$ is convex and even for $t \in \mathbf{R}$. Let us denote by $\mathscr{L}_G(\Omega)$ the set $\{u \in L_1(\Omega);\ \int G(u(x))\,dx < \infty\}$ and by $L_G \equiv L_G(\Omega)$ the linear hull of $\mathscr{L}_G(\Omega)$. The set L_G is a Banach space (Orlicz space) with respect to the (Luxemburg) norm

$$\|u\|_{(G)} = \inf\Big\{r > 0;\ \int_\Omega G\Big(\frac{u}{r}\Big)dx \leqslant 1\Big\}.$$

The closure in L_G of the set of all bounded measurable functions in Ω is denoted by E_G. The function $G(t)$ is said to *satisfy the Δ_2-condition* if there exists a $k > 0$ such that $G(2t) \leqslant kG(t)$ holds for $t \geqslant t_1$ for some $t_1 > 0$. The inclusions $E_G \subset \mathscr{L}_G \subset L_G$ take place and the equalities $E_G = \mathscr{L}_G$, $\mathscr{L}_G = L_G$ hold if and only if $G(t)$ satisfies the Δ_2-condition. The dual space of E_G can be identified by means of $\int_\Omega uv\,dx$ with the Orlicz space $L_{\bar G} \equiv L_{\bar G}(\Omega)$, where $\bar G(t)$ is the N-function (conjugate to $G(t)$) defined by

$$\bar G(t) = \sup_{s \in R}\big(ts - G(s)\big).$$

Young's inequality $ts \leqslant G(t) + \bar G(s)$ takes place and we have $\bar{\bar G}(t) = G(t)$. The norm

$$\|u\|_G = \sup_{v \in L_{\bar G}}\Big\{\int_\Omega uv\,dx;\ \|v\|_{(\bar G)} \leqslant 1\Big\}$$

(Orlicz's norm) is equivalent to the norm $\|\cdot\|_{(G)}$ and the following Hölder's inequality

$$\int_\Omega uv\,dx \leqslant \|u\|_G \|v\|_{(\bar G)} \quad \text{for all } u \in L_G,\ v \in L_{\bar G}$$

is valid. Clearly, the system $(L_G, E_G;\ L_{\bar G}, E_{\bar G})$ is a complementary system with respect to the scalar product $\int_\Omega uv\,dx$.

Let $W^m L_G \equiv W^m L_G(\Omega) = \{u \in L_1(\Omega);\ D^i u \in L_G \text{ for all } |i| \leqslant m\}$, ($i = (i_1, \ldots, i_N)$ is a multiindex with $|i| = i_1 + \ldots + i_N$ and $D^i u$ is the distributional derivative of u) with the norm

$$\|u\|_{m,G} = \Big(\sum_{|i| \leqslant m} \|D^i u\|_G^2\Big)^{1/2}.$$

The space $W^m L_G$ can be canonically imbedded into the product $\prod_{|i| \leqslant m} L_G \equiv \Pi L_G$. Let $W_0^m L_G(\Omega)$ be the $\sigma(\Pi L_G, \Pi E_{\bar G})$ closure of $C_0^\infty(\Omega)$ in $W^m L_G$.

Let $W_0^m E_G$ be the intersection $W_0^m L_G \cap \Pi E_G$. We define $W_0^m E_G$ as the norm closure of $C_0^\infty(\Omega)$ in $W^m L_G$. In [1] the following density results are proved:

$C^\infty(\Omega)$ is $\sigma(\Pi L_G, \Pi L_{\bar G})$ dense in $W^m L_G$,

$C_0^\infty(\Omega)$ is $\sigma(\Pi L_G, \Pi L_{\bar G})$ dense in $W_0^m L_G$,

$C_0^\infty(\Omega)$ is (norm) dense in $W^m E_G$,

$W_0^m E_G$ is the intersection of $W_0^m L_G$ with ΠE_G.

Let us write $Y = W_0^m L_G$ and $Y_0 = W_0^m E_G$. Then the dual space to Y_0 is $Z = W^{-m} L_{\bar G}$, where

$$W^{-m} L_{\bar G} = \Big\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i| \leqslant m} (-1)^{|i|} D^i f_i \text{ with } f_i \in L_{\bar G}(\Omega)\Big\}$$

($\mathscr{D}'(\Omega)$ is the space of distributions) and $Z_0 = W^{-m} E_G$ (the dual of Z_0 being $W_0^m L_G$) is

$$W^{-m} E_{\bar G} = \Big\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i| \leqslant m} (-1)^{|i|} D^i f_i \text{ with } f_i \in E_{\bar G}(\Omega)\Big\}.$$

The quadruple $(Y, Y_0;\ Z, Z_0)$ is a complementary system (see [1]) with respect to the continuous pairing

$$\langle f, u\rangle = \sum_{|i| \leqslant m} \int_\Omega D^i u f_i\,dx \quad \text{for } u \in Y,\ f \in Z.$$

Let $H = L_2(\Omega)$. We can verify easily that $Y_0 \cap H$ is dense in Y_0 (because of the density results) and that $\|u\|_H = 0$ implies $\|u\|_{Y \cap H} = 0$ for $u \in Y \cap H$. The spaces $W^{-m} L_{\bar G}$ and H are continuously imbedded into the linear locally convex space $\mathscr{D}'(\Omega)$. We also find out without trouble that the set of functionals

$$\Big\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i| \leqslant m} (-1)^{|i|} D^i f_i,\ f_i \in C_0^\infty(\Omega)\Big\}$$

is dense in $W^{-m} E_{\bar G}$ and in $L_2(\Omega)$, again because of the density results. Thus the assumption (1.1) is satisfied.

Let q be the number of all multiindices j with $|j| \leqslant m$. By $\xi = (\xi_i;\ |i| \leqslant m)$ we denote a real vector in $\mathbf{R}^q$ and by $\xi(u)$ we denote the vector function $\{\xi(u) = D^i u,\ |i| \leqslant m\}$. Let M be the class of continuous functions $g(u)$ in $\mathbf{R}$ satisfying: $g(u) \to \infty$ for $u \to \infty$, $g(u)$ is odd and $ug(u)$ is convex for $u \geqslant u_1 > 0$, where u_1 is sufficiently big. It is well-known (see [8]) that for each $g(u) \in M$ there exists an N-function $G(u)$ (not uniquely determined) such that $G(u) = ug(u)$ for all $u \geqslant u_1$. All these N-functions are equivalent and generate the same Orlicz space L_G.

The coefficients $A_i(x, \xi)$ are supposed to satisfy the following conditions:

(2.1) $A_i(x, \xi)$ $(|i| \leqslant m)$ are real valued functions defined on $\xi \in \boldsymbol{R}^q$, which are measurable in x for fixed ξ and continuous in ξ for fixed x;

(2.2) There exist $g(u) \in M$, $a(x) \in E_{\bar{G}}(\Omega)$ and C_1, C_2 such that

$$|A_i(x, \xi)| \leqslant a(x) + C_1 \sum_{|j| \leqslant m} |g(C_2 \xi_j)|;$$

(2.3) $$\sum_{|i| \leqslant m} \big(A_i(x, \xi) - A_i(x, \eta)\big)(\xi_i - \eta_i) \geqslant 0 \quad \text{for all } \xi, \eta \in \boldsymbol{R}^q.$$

Since the estimate $g(u) \leqslant \bar{G}^{-1}\big(G(u)\big)$ takes place (see [9]), then using (2.2) and the convexity of $\bar{G}(u)$ we estimate

$$\bar{G}\left(\frac{A_i(x, \xi)}{2C_1 q}\right) \leqslant \frac{1}{2}\bar{G}\left(\frac{a(x)}{qC_1}\right) + \frac{1}{2}\bar{G}\left(\sum_{|j| \leqslant m} \frac{1}{q}\bar{G}^{-1}\big(G(C_2\xi_j)\big)\right)$$
$$\leqslant \frac{1}{2}\bar{G}\left(\frac{a(x)}{qC_1}\right) + \frac{1}{2q}\sum_{|j| \leqslant m} G(C_2\xi_j).$$

Hence we see that the operator $A_i\big(x, \xi(u)\big)$ $(|i| \leqslant m)$ maps $W_0^m E_G$ into the Orlicz space $L_{\bar{G}}$. Moreover, this operator is bounded on a small ball in $W_0^m E_G$ centered at 0. Thus, by means of the form

$$\langle Au, v\rangle = \sum_{|i| \leqslant m} \int_\Omega D^i v A_i\big(x, \xi(u)\big)\, dx,$$

we define an operator $A\colon Y = W_0^m L_G \to Z = W^{-m} L_{\bar{G}}$, its domain being

$$D(A) = \{u \in W_0^m L_G;\ A_i\big(x, \xi(u)\big) \in L_{\bar{G}} \text{ for all } |i| \leqslant m\}.$$

The inclusion $D(A) \supset Y_0 = W_0^m E_G$ is obvious. The properties (a) and (c) of A are evidently fulfilled. In essence, the properties (b) and (d) of A are proved in [1] (Theorem 4.1). Thus the operator A is of type $(\bar{\mathrm{M}})$ with respect to the complementary system

$$(W_0^m L_G, W_0^m E_G;\ W^{-m} L_{\bar{G}}, W^{-m} E_{\bar{G}}).$$

Remark 2. Let (2.1)–(2.3) be satisfied. Then $Au \in Z$ for some $u \in Y$ if and only if $A_i\big(x, \xi(u)\big) \in L_{\bar{G}}$ for all $|i| \leqslant m$. Indeed, we have

$$\sum_{|i| \leqslant m} \int_\Omega w_i A_i\big(x, \xi(u)\big)\, dx$$
$$\leqslant \langle Au, u\rangle + \sum_{|i| \leqslant m} \int_\Omega w_i A_i(x, w)\, dx - \sum_{|i| \leqslant m} \int_\Omega D^i u A_i(x, w)\, dx$$

for all $w \in \Pi E_G$ and hence $A_i\big(x, \xi(u)\big) \in L_{\bar{G}}$, because the operators $A_i(x, w)$ $(|i| \leqslant m)$ are bounded mappings from a small ball in ΠE_G into $L_{\bar{G}}$.

The algebraic condition which ensures the coerciveness of the operator A is

(2.4) $$\sum_{|i| \leqslant m} \xi_i A_i(x, \xi) \geqslant C_1 \sum_{i=m} \xi_i g\left(\frac{\xi_i}{r}\right) - C_2,$$

where $\xi \in \boldsymbol{R}^q$ and $r > 1$ is arbitrary.

LEMMA 5. *Let $\bar{G}$ satisfy the Δ_2-condition. If* (2.4) *holds, then the assumption* I *of Theorem* 1 *is satisfied.*

Proof. In view of (2.4) we have

(2.5) $$\langle Au, u\rangle \geqslant C_1 \sum_{|i|=m} \int_\Omega G\left(\frac{D^i u}{r}\right) dx - C_2 \geqslant C_3 \sum_{|i| \leqslant m} \int_\Omega G\left(\frac{D^i u}{r_1}\right) dx - C_4,$$

since the Poincaré inequality

$$\sum_{|j| < m} \int_\Omega G(D^j u)\, dx \leqslant C_5 \sum_{|i|=m} \int_\Omega G(sD^i u)\, dx$$

is true for a sufficiently big s and for all $u \in W_0^m L_G$ (see [1], Lemma 5.7). Δ_2-condition for $\bar{G}$ implies $E_{\bar{G}} = L_{\bar{G}}$. Then from [1] (Lemma 3.14) we obtain

$$\|u\|_G^{-1} \int_\Omega G(u)\, dx \to \infty \quad \text{for } \|u\|_G \to \infty,\ u \in L_G.$$

Hence and from (2.5) we obtain Lemma 5.

LEMMA 6. *If* (2.4) *holds, then the assumption* II *of Theorem* 1 *is satisfied.*

Proof. From (2.5) we obtain (K_1) (see [8]). Let be $\bar{f} \in W^{-m} E_{\bar{G}} + L_2$, $\bar{f} = \bar{f}_1 + \bar{f}_2$, where $\bar{f}_1 \in W^{-m} E_{\bar{G}}$ $(\bar{f}_1 = (\bar{f}_{1i};\ |i| \leqslant m))$ and $\bar{f}_2 \in L_2$. We consider the set of all $f \in W^{-m} L_{\bar{G}} + L_2$ $(f = f_1 + f_2, f_1 \in W^{-m} L_{\bar{G}}, f_1 = (f_{1i};$ $|i| \leqslant m)$ and $f_2 \in L_2)$ for which

$$\|f_{1i} - \bar{f}_{1i}\|_{\bar{G}} < \frac{2r_1}{C_3}, \qquad \|f_2 - \bar{f}_2\| < 1$$

hold for all $|i| \leqslant m$, where $r_1 > 1$ and $C_3 < 1$ are from (2.5). This set generates a neighbourhood $U_{\bar{f}}$ in $W^{-m} L_{\bar{G}} + L_2$. Let us chose K such that

$$\int_\Omega \bar{G}\left(\frac{2r_1}{C_3}\bar{f}_{1i}\right) dx \leqslant K \quad \text{for } |i| \leqslant m.$$

Let $f \in U_{\bar{f}}$ and let

(2.6) $$\langle Au, v\rangle + (\lambda u, v) = \langle f_1, v\rangle + (f_2, v)$$

for all $v \in W_0^m E_G \cap L_2$. Since $W_0^m E_G$ is dense in $W_0^m L_G$ with respect to the $\sigma(\Pi L_G, \Pi L_{\bar{G}})$ topology, we obtain (2.6) for all $v \in W_0^m L_G \cap L_2$. Thus we have

$$\langle Au, u\rangle + (\lambda u, u) = \int_\Omega \sum_{|i|\leqslant m} f_{1i} D^i u\, dx + \int_\Omega f_2 u\, dx$$

$$\leqslant \sum_{|i|\leqslant m} \int_\Omega \bar G\left(\frac{r_1}{C_3} f_{1i}\right) dx + \sum_{|i|\leqslant m} \int_\Omega G\left(C_3 \frac{D^i u}{r_1}\right) dx +$$

$$+ \frac{\lambda}{2} \int_\Omega u^2\, dx + \frac{1}{2\lambda} \int_\Omega f_2^2\, dx.$$

Since

$$\int_\Omega G\left(C_3 \frac{D^i u}{r_1}\right) dx \leqslant C_3 \int_\Omega G\left(\frac{D^i u}{r_1}\right) dx$$

and

$$\int_\Omega \bar G\left(\frac{r_1}{C_3}\right) f_{1i}\, dx \leqslant \frac{1}{2} \int_\Omega \bar G\left(\frac{2r_1}{C_3} \bar f_{1i}\right) dx + \frac{1}{2} \int_\Omega \bar G\left(\frac{2r_1}{C_3} (f_{1i} - \bar f_{1i})\right) dx \leqslant K+1$$

(see [8]), from (2.5) we conclude that

$$\frac{C_3}{2} \sum_{|i|\leqslant m} \int_\Omega G\left(\frac{D^i u}{r_1}\right) dx + \frac{\lambda}{2} \int_\Omega u^2\, dx \leqslant C(f, \bar f, \lambda)$$

which implies Lemma 6.

Now, let us consider the equation (E_1) with the initial and boundary conditions

(I_0) $$u(x, 0) = u_0(x),$$

(B) $D_\nu^k u(x, t) = 0$ on $\partial\Omega$ for $k = 0, 1, \ldots, m-1$ and for a.e. $t \in (0, T)$,

where ν is the outward normal to $\partial\Omega$.

Applying the results of Section 1 we obtain

THEOREM 2. *Let* (2.1)–(2.4) *be satisfied. We assume that* $f \in C(\langle 0, T\rangle, L_2(\Omega))$ *with* $\bigvee_{\langle 0,T\rangle} (f; L_2(\Omega)) < \infty$, $u_0 \in D(A) \cap L_2(\Omega)$ *and* $Au_0 \in L_2(\Omega)$. *Then there exists a unique solution* $u \in L_\infty(\langle 0, T\rangle, W_0^m L_G \cap L_2)$ *of* (E_1), (I_0), (B) *in the following sense:*

(i) $u(x, t) = u(t)\colon \langle 0, T\rangle \to L_2(\Omega)$ *is Lipschitz continuous and* $u(x, 0) = u_0(x)$ *(in* L_2*)*;

(ii) $\dfrac{\partial u}{\partial t} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$, $Au \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$;

(iii) *The equality*

$$\left(\frac{\partial u}{\partial t}, v\right) + \sum_{|i|\leqslant m} \int_\Omega D^i v A_i(x, \xi(u))\, dx = \int_\Omega f v\, dx$$

holds for every $v \in W_0^m L_G \cap L_2$ *and for a.e.* $t \in (0, T)$.

Theorem 1 can be applied also to the anisotropic situation. Let us assume that there exist $g_i(u) \in M$ $(|i| \leqslant m)$ such that $g_i(u) \leqslant g_j(u)$ or $g_j(u) \leqslant g_i(u)$ holds for all $u \geqslant u_1$, where u_1 is sufficiently large. Then the growth conditions are of the form

$$|A_i(x, \xi)| \leqslant a_i(x) + b \sum_{|j|\leqslant m} \min(|g_i(C\xi_j)|, |g_j(C\xi_j)|) \tag{2.7}$$

for all $|i| \leqslant m$, where b, C are suitable constants, $a_i \in E_{\bar G_i}$ and G_i are the N-functions corresponding to $g_i(u)$. Using the $G_i(u)$ $(|i| \leqslant m)$ we construct the Orlicz–Sobolev space $W^m L_{\vec G} = W^m L_{\vec G}(\Omega)$:

$$W^m L_{\vec G} = \{u \in L_1(\Omega);\ D^i u \in L_{G_i} \text{ for all } |i| \leqslant m\}$$

with the norm

$$\|u\|_{m,\vec G} = \Big(\sum_{|i|\leqslant m} \|D^i u\|_{G_i}^2\Big)^{1/2}.$$

The space $W^m L_G$ can be canonically imbedded into the space $\prod_{|i|\leqslant m} L_{G_i} \equiv \Pi L_{G_i}$. Let $W_0^m L_{\vec G}$ be the closure of $C_0^\infty(\Omega)$ in ΠL_{G_i} in $\sigma(\Pi L_{G_i}, \Pi E_{G_i})$ topology. We write $G_i(u) \prec G_j(u)$ whenever there exist $k > 0$ and $u_1 > 0$ such that $G_i(u) \leqslant G_j(ku)$ for all $u \geqslant u_1$. Under the assumption

$$G_i(u) \prec G_j(u) \quad \text{for all } j \leqslant i \quad (\text{i.e. } j_k \leqslant i_k \text{ for } k = 1, \ldots, n) \tag{2.8}$$

the following density result holds (see [1]):

$W_0^m L_{\vec G}$ is the closure of $C_0^\infty(\Omega)$ in $\sigma(\Pi L_{G_i}, \Pi L_{\bar G_i})$ topology.

In the anisotropic case we have

$$Y = W_0^m L_{\vec G}, \quad Y_0 = W_0^m E_{\vec G} = W_0^m L_{\vec G} \cap \Pi E_{G_i}, \quad Z = W^{-m} L_{\vec G},$$

where

$$W^{-m} L_{\vec G} = \Big\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i|\leqslant m} (-1)^{|i|} D^i f_i,\ f_i \in L_{\bar G_i}\Big\}$$

and

$$Z_0 = W^{-m} E_{\vec G},$$

where

$$W^{-m} E_{\vec G} = \Big\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i|\leqslant m} (-1)^{|i|} D^i f_i,\ f_i \in E_{\bar G_i}\Big\}.$$

Since $C_0^\infty(\Omega)$ is a dense set in $L_2(\Omega) = H$, $E_{\bar{G}_i}$ (for all $|i| \leqslant m$) and the set $\{f \in \mathscr{D}'(\Omega);\ f = \sum_{|i|\leqslant m} (-1)^{|i|} D^i f_i,\ f_i \in C_0^\infty(\Omega)\}$ is dense in Z_0 and H, we conclude that (1.1) is satisfied. The norm $\|\cdot\|_{m,\vec{G}}$ is admissible in $W_0^m L_{\vec{G}}$ (see [1]).

Owing to the inequality (see [9])

$$\min(|g_i(u)|, |g_j(u)|) \leqslant 2\bar{G}_i^{-1}(G_j(u)),$$

analogously to the previous case we can prove that the operator $A_i(x, \xi(u))$ ($|i| \leqslant m$) maps $W_0^m E_{\vec{G}}$ into the Orlicz space $L_{\bar{G}_i}$. These operators are bounded on a small ball in $W_0^m E_{\vec{G}}$ centered at 0. Thus, by means of the form

$$\langle Au, v\rangle = \sum_{|i|\leqslant m} \int_\Omega D^i v A_i(x, \xi(u))\,dx$$

for $u \in D(A) \subset W_0^m L_{\vec{G}}$, $v \in W_0^m E_G$ we define the operator A from its domain $D(A) \subset Y$ into Z, where

$$D(A) = \{u \in W_0^m L_{\vec{G}};\ A_i(x, \xi(u)) \in L_{\bar{G}_i} \text{ for all } |i| \leqslant m\}.$$

According to [1] (Theorem 4.1), the operator A is of type $(\overline{\mathrm{M}})$ with respect to the complementary system $(W_0^m L_{\vec{G}}, W_0^m E_{\vec{G}}; W^{-m} L_{\vec{\bar{G}}}, W^{-m} E_{\vec{\bar{G}}})$. The coerciveness of the operator A is ensured by the following algebraic condition:

$$\sum_{|i|\leqslant m} \xi_i A_i(x, \xi) \geqslant \sum_{|i|\leqslant m} C_i \xi_i g_i(\xi_i/r) - C \tag{2.9}$$

where $\xi \in \boldsymbol{R}^q$, $C_i > 0$ for $|i| \leqslant m$. If for each i with $|i| < m$ there exists a j with $|j| = m$ such that $G_i \prec G_j$, then in (2.9) one can take $C_i = 0$.

The proofs that under condition (2.9) the assumptions I, II (of Theorem 1) are fulfilled are analogous to the proofs of Lemmas 5 and 6. Consequently we have

THEOREM 3. *Let* (2.1), (2.2), (2.7) *and* (2.9) *be fulfilled. We assume that* $u_0 \in W_0^m L_{\vec{G}} \cap L_2$ *and* $Au_0 \in L_2(\Omega)$. *If* $\bar{G}_i(u)$ ($|i| \leqslant m$) *do not satisfy* Δ_2*-condition, we assume also* (2.8). *Let* $f \in C(\langle 0, T\rangle, H)$ *with* $\bigvee_{\langle 0,T\rangle}(f; H) < \infty$. *Then there exists a unique (weak) solution of the problem* (E_1), (I_0), (B) *with the properties* (i), (ii) *and* (iii) (*where* $W_0^m L_G$ *is replaced by* $W_0^m L_{\vec{G}}$) *of Theorem* 2.

Remark 3. The nonhomogeneous Dirichlet boundary value problem can be reduced by a standard transformation to the homogeneous Dirichlet boundary value problem. More general boundary value problems can be solved by the same method using the corresponding subspace Q,

$$W_0^m L_G \subset Q \subset W^m L_G \quad (\text{resp. } W_0^m L_{\vec{G}} \subset Q \subset W^m L_{\vec{G}}).$$

The results obtained can be applied e.g. to the following equations.

EXAMPLE 1.

$$\frac{\partial u}{\partial t} + \sum_{|i|\leqslant m} (-1)^{|i|} D^i g_i(D^i u) = f,$$

where $g_i(u) \in M$ for all $|i| \leqslant m$ satisfy $g_i(u) \leqslant g_j(u)$ or $g_i(u) \geqslant g_j(u)$ for $u \geqslant u_1$ and $|i|, |j| \leqslant m$. If $\bar{G}_i(u)$ ($|i| \leqslant m$) does not satisfy Δ_2-condition, we assume (2.8).

EXAMPLE 2.

$$\frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} \exp\left(\sum_{i=1}^N C_i \left(\frac{\partial u}{\partial x_i}\right)^2\right)\right),$$

where $C_i \geqslant 0$, $i = 1, \ldots, N$.

EXAMPLE 3.

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\left(\operatorname{sgn}\left(\frac{\partial u}{\partial x}\right) \ln\left(\left|\frac{\partial u}{\partial x}\right| + 1\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} \exp\left(\left(\frac{\partial u}{\partial y}\right)^2\right)\right) = f.$$

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КРАЕВЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ СМЕШАННОГО ТИПА В МНОГОМЕРНЫХ ОБЛАСТЯХ

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1

Пусть D — ограниченная область пространства E_{m-1}, $m \geqslant 2$, точек $x' = (x_1, x_2, \ldots, x_{m-1})$ с кусочно-гладкой границей ∂D. Обозначим $G = \{x = (x', x_m) \in E_m;\ x' \in D,\ \varphi_2(x') < x_m < \varphi_1(x')\}$, где $\varphi_i(x') \in C^2(\bar{D})$, $i = 1, 2$; Γ_i: $x_m = \varphi_i(x')$, $i = 1, 2$, $x' \in \bar{D}$; Γ_3 — боковая поверхность G (Γ_3 или некоторая ее часть может отсутствовать); $n = (n_1, n_2, \ldots, n_m)$ — единичный вектор внешней нормали к $\partial G = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Будем предполагать, что $n_m > 0$ на Γ_1 и $n_m < 0$ на Γ_2.

Рассмотрим в области G уравнение

$$(1)\qquad Lu \equiv a^{ij}(x)u_{x_ix_j} + k(x)u_{x_mx_m} + b^i(x)u_{x_i} + b^m(x)u_{x_m} + c(x)u = f(x),$$

где $a^{ij}(x) \in C^2(\bar{G})$, $a^{ij} = a^{ji}$, $a^{ij}(x)\xi_i\xi_j \geqslant \lambda \sum_{i=1}^{m-1} \xi_i^2$ в $\bar{G}$, для любого вектора $(\xi_1, \ldots, \xi_{m-1})$, $\lambda = \text{const} > 0$; $k(x) \in C^2(\bar{G})$; $b_i(x) \in C^1(\bar{G})$, $i = 1, \ldots, m$; $c(x) \in C(\bar{G})$, $c_{x_m}(x) \in C(\bar{G})$ (по повторяющимся индексам предполагается суммирование от 1 до $m-1$). Будем предполагать, что $H = a^{ij}n_in_j + kn_m^2 = 0$ на $\Gamma_1 \cup \Gamma_2$.

Уравнение (1) эллиптико-параболическое при $k(x) \geqslant 0$ в $\bar{G}$ и гиперболо-параболическое при $k(x) \leqslant 0$ в $\bar{G}$. Если функция $k(x)$ меняет знак в области $\bar{G}$, то уравнение (1) является уравнением смешанного типа.

Краевые задачи для некоторых уравнений смешанного типа вида (1) рассматривались в работах А. В. Бицадзе [1], [2], Г. Д. Каратопраклиева [3]–[6], Н. Г. Сорокиной [7], [8], В. Н. Врагова [9], [10], Г. Д. Дачева [11], [12] и других авторов.

В настоящей статье рассматриваются две краевые задачи для уравнения (1).