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MULTIPLE LAYER POTENTIALS FOR THE QUADRANT AND THEIR APPLICATION TO THE DIRICHLET PROBLEM IN PLANE DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

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0. Introduction

There is a vast literature on applications of double layer potentials to the Dirichlet problem for the Laplacian (cf. [5], [8]). G. Lauricella defined multiple layer potentials to solve the Dirichlet problem for the biharmonic equation. K. Schröder [17] has given a rigorous proof to Lauricella's ideas. S. Agmon [1] has defined multiple layer potentials for arbitrary homogeneous elliptic operators L with real constant coefficients in the plane. Both authors have dealt only with domains which have a sufficiently smooth boundary.

In this paper we modify Agmon's definition of multiple layer potentials in such a way that it gives Lauricella's potential in the special case of $L = \Delta^2$. If $\Omega \subset R^2$ is a simply connected domain with a piecewise smooth boundary, then the multiple layer potentials with respect to an operator L of order $2m$ define solutions

$$u \in C^{m-1}(\overline{\Omega}) \cap C^{2m}(\Omega)$$

of the equation $Lu = 0$ in Ω. In the case of higher order operators the multiple layer potentials are much more difficult to be handled than the double layer potentials for second order equations. For the operator Δ^2 and for some fourth order operators which do not differ from Δ^2 too much the Dirichlet problem for the quadrant can be solved by means of multiple layer potentials, i.e. the Poisson formulas can be given. It seems that for domains with a piecewise smooth boundary there is not yet a satisfying theory of the Dirichlet problem. It seems to be still an open question, whether the Dirichlet problem even for the biharmonic operator Δ^2 in a rectangle is solvable for each function $F \in C^1(R^2)$ and whether the following a priori estimate

$$\|u\|_{C^1(\bar{\Omega})} \leqslant \text{const} \max_{|\alpha|\leqslant 1} \|D^\alpha F\|_{C(\partial\Omega)} =: \text{const}\,\|F\|_{1,\partial\Omega} \tag{0.1}$$

does hold. Of course, there are existence theorems for boundary values which have additional properties (cf. [11], [12], [18]). Therefore an a priori estimate (0.1) would imply the existence of a solution for Dirichlet data induced by an arbitrary $F \in C^1(R^2)$. Using the Poisson formula for Δ^2 in the quadrant, we have tried to apply to biharmonic functions in a rectangle a method developed by C. Miranda, S. Agmon and others for domains with smooth boundaries (cf. [16]). This method led us to the problem, whether there are constants C_p ($p > 2$) such that for any $u \in \mathring{W}^2_p(\Omega)$ (closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^2_p(\Omega)$)

$$\|u\|_{W^2_p(\Omega)} \leqslant C_p \sup\{(\Delta u, \Delta v) \colon v \in C_0^\infty(\Omega) \text{ and } \|v\|_{W^2_{p'}(\Omega)} \leqslant 1\},$$

where $1/p+1/p' = 1$. From such an inequality the estimate (0.1) would follow. Such a Gårding inequality has been proved by C. Simader [19] under some smoothness assumptions on $\partial\Omega$.

One can say that multiple layer potentials are promising tools to treat the Dirichlet problem for higher order elliptic operators in domains with corners and edges and that it is a promising task to attack the problems which arise and which are sketched briefly in this article.

1. Multiple layer potentials

Let D_x and D_y denote the differentiation along the x- or y-axis, respectively, of the real plane R^2. We consider a homogeneous linear elliptic operator

$$L = \sum_{k=0}^{2m} a_{2m-k} D_x^k D_y^{2m-k}$$

of order $2m$ in the plane with real constant coefficients. For definiteness assume $a_0 = 1$.

Let Q^+ be the open first quadrant in R^2 and let $Q^- = R^2 \setminus \overline{Q^+}$ be the exterior of the closure of Q^+. A function $\hat{u} \in C^{m-1}(Q^+ \text{ or } \overline{Q^-})$ is defined by the m-tuple

$$u = \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{m-1} \end{bmatrix}$$

of its $(m-1)$st-order derivatives

$$u_k = D_x^{m-1-k} D_y^k \hat{u},$$

uniquely up to a polynomial of degree $< m$. Therefore, by a multiple layer potential (abbreviated: MLP) for L in $Q^+ \cup Q^-$ we do not mean a particular solution of $L\hat{u} = 0$, but the vector u of the $(m-1)$st-order derivatives of $\hat{u}$. In the same way, the Dirichlet problem and, correspondingly, the Poisson formula for Q^+ or Q^- are modified.

Densities of a MLP or *boundary data* for the Dirichlet problem for L in Q^+ or Q^- are defined as a $2m$-tuple

$$f = \left\{ \begin{bmatrix} f_0(s,0) \\ \cdot \\ \cdot \\ \cdot \\ f_{m-1}(s,0) \end{bmatrix}, \begin{bmatrix} f_0(0,s) \\ \cdot \\ \cdot \\ \cdot \\ f_{m-1}(0,s) \end{bmatrix} \right\} =: \big(f(\cdot, 0), f(0, \cdot)\big) \in \mathrm{CB}(\bar{R}_+)^{2m}$$

of bounded continuous real-valued functions on the closed positive coordinate half-axes satisfying, furthermore,

$$f(s,0)|_{s=0} = f(0,t)|_{t=0}.$$

We denote by $E^{(m)}$ the Banach space of all such $2m$-tuples equipped with the norm

$$\|f\| := \sup_{\substack{0\leqslant j<m \\ 0\leqslant s}} \{|f_j(s,0)|, |f_j(0,s)|\}.$$

The *Dirichlet problem* for L in $Q = Q^+$ or Q^- consists in finding an m-tuple $u \in C(\bar{Q})^m$ of $(m-1)$st-order derivatives of a solution $\hat{u}$ of $L\hat{u} = 0$ in Q such that

$$u(x,0) = f(x,0) \quad \text{and} \quad u(0,y) = f(0,y) \quad (x, y \in R_+).$$

We shall work out a special solution $u = Pf$ for the biharmonic operator Δ^2 in Q^+, and we shall call $u = Pf$ the *Poisson formula* for Δ^2 in Q^+. Instead of $\mathrm{CB}(\bar{R}_+)^{2m}$ and $C(\bar{Q})^m$ other function spaces can be taken as a basis of consideration.

The polynomial $z \mapsto L(z, 1) = \sum_{k=0}^{2m} a_{2m-k} z^k$ has exactly m roots $z_1, \ldots, z_m$ in the open lower complex half-plane. Following [1] we denote

$$M(x,y) := (x-z_1 y)(x-z_2 y)\ldots(x-z_m y) =: \sum_{k=0}^{m} b_{m-k} x^k y^{m-k},$$

$$\bar{M}(x,y) := \sum_{k=0}^{m} \bar{b}_{m-k} x^k y^{m-k},$$

where $\bar{b}_l$ is the complex conjugate of b_l,

$$M_j(x, y) := \sum_{k=0}^{j} b_{j-k} x^k y^{j-k} \qquad (0 \leqslant j < m)$$

and

$$P_j(x, y) := \sum_{k=0}^{j} b_{m-k} x^{k-j} y^k \qquad (0 \leqslant j < m).$$

Furthermore, we shall use the polynomials

$$N(z, 1) := (z - z_1^{-1}) \dots (z - z_m^{-1}) =: \sum_{k=0}^{m} c_{m-k} z^k$$

and

$$N_j(z, 1) := \sum_{k=0}^{j} c_{j-k} z^k \qquad (0 \leqslant j < m).$$

These polynomials are interrelated in the following way:

$$M(z, 1) = z^m b_m N(1/z, 1) \quad \text{and} \quad P_j(z, 1) = b_m N_j(1/z, 1).$$

We denote by $\boldsymbol{M}$, $\boldsymbol{P}$ and $\boldsymbol{N}$ the $m \times m$ matrices

$$\boldsymbol{M}(z) = \big(\!\big(M_{jk}(z)\big)\!\big) := \big(\!\big(z^{m-1-j} M_k(z, 1)\big)\!\big),$$
$$\boldsymbol{P}(z) = \big(\!\big(P_{jk}(z)\big)\!\big) := \big(\!\big(z^{m-1-j} P_{m-1-k}(z, 1)\big)\!\big)$$

and

$$\boldsymbol{N}(z) = \big(\!\big(N_{jk}(z)\big)\!\big) := \big(\!\big(z^j N_{m-1-k}(z, 1)\big)\!\big).$$

We shall apply the formula

$$\sum_{z_j} \operatorname{Res} \frac{\boldsymbol{M}(z)}{M(z, 1)} = \frac{1}{2\pi i} \int_\gamma \frac{\boldsymbol{M}(z)}{M(z, 1)}\, dz = \boldsymbol{I}, \tag{1.1}$$

where γ is any anti-clockwise oriented rectifiable Jordan contour in the lower complex half-plane enclosing all the z_j's in its interior and where $\boldsymbol{I}$ denotes the unit matrix (cf. [1], (2.10)). This formula also holds if M, $\boldsymbol{M}$ and z_j are replaced by $\bar{M}$, $\bar{\boldsymbol{M}}$ and $\bar{z}_j$ and if γ denotes a contour in the upper half-plane enclosing the z_j's. Substituting $z = 1/\zeta$ we obtain

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \frac{\boldsymbol{P}(z)}{M(z, 1)} = -\frac{1}{2\pi i} \int_C d\zeta \frac{\boldsymbol{N}(\zeta)}{N(\zeta, 1)} = -\boldsymbol{I}, \tag{1.2}$$

where C is the image of the oriented contour γ under the transformation $z \mapsto \zeta$.

LEMMA 1.1. *The formula*

$$\frac{1}{2\pi i} \int_\gamma \frac{\boldsymbol{M}(z)}{M(z, 1)(z+t)}\, dz = -\frac{\boldsymbol{M}(-t)}{M(-t, 1)} + \begin{bmatrix} 0 & 1 & -t & \dots & (-t)^{m-2} \\ 0 & 0 & 1 & \dots & (-t)^{m-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$= -\frac{(-t)^{m-1}}{M(-t, 1)} \boldsymbol{I} + O(|t|^{-2}) \quad \text{at } |t| = \infty$$

holds for $t \in R$.

Proof. For positive integers j we have

$$z^j = z^j - (-t)^j + (-t)^j = (z+t) \sum_{l=0}^{j-1} (-t)^{j-1-l} z^l + (-t)^j.$$

Applying (1.1) we get

$$\frac{1}{2\pi i} \int_\gamma \frac{\boldsymbol{M}(z)}{(z+t) M(z, 1)}\, dz = \frac{\boldsymbol{M}(-t)}{2\pi i} \int_\gamma \frac{dz}{(z+t) M(z, 1)} + \begin{bmatrix} 0 & 1 & -t & \dots & (-t)^{m-2} \\ 0 & 0 & 1 & \dots & (-t)^{m-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

If all the z_j's differ from each other, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{(z+t) M(z, 1)} = \sum_{j=1}^{m} \frac{1}{(z_j + t) \prod_{k \neq j} (z_j - z_k)}$$
$$= \frac{(-1)^m}{M(-t, 1)} \sum_{j=1}^{m} \prod_{k \neq j} \frac{z_k + t}{z_j - z_k}.$$

Now,

$$\prod_{k \neq j} (z_k + t) = \sum_{l=1}^{m-1} t^{m-1-l} \sum_{\substack{i_1 < \dots < i_l \\ i_\lambda \neq j}} z_{i_1} \dots z_{i_l} + t^{m-1}$$

and

$$\sum_{\substack{i_1 < \dots < i_l \\ i_\lambda \neq j}} z_{i_1} \dots z_{i_l} = \sum_{i_1 < \dots < i_l} z_{i_1} \dots z_{i_l} - z_j \sum_{\substack{i_1 < \dots < i_{l-1} \\ i_\lambda \neq j}} z_{i_1} \dots z_{i_{l-1}}$$
$$= \sum_{q=0}^{l-1} (-z_j)^q \sum_{i_1 < \dots < i_{l-q}} z_{i_1} \dots z_{i_{l-q}} + (-z_j)^l$$
$$= (-1)^l \sum_{q=0}^{l} b_{l-q} z_j^q = (-1)^l M_l(z_j, 1),$$

so that

$$\prod_{k\neq j}(z_k+t) = \sum_{l=0}^{m-1} t^{m-1-l}(-1)^l M_l(z_j, 1).$$

Since

$$\sum_{j=1}^{m} \frac{M_l(z_j, 1)}{\prod_{k\neq j}(z_j-z_k)} = \frac{1}{2\pi i}\int_\gamma \frac{M_l(z, 1)}{M(z, 1)}\,dz = \delta_{m-1,l},$$

we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{(z+t)M(z, 1)} = \frac{(-1)^m}{M(-t, 1)}t^{m-1-(m-1)}(-1)^{m-1}\cdot 1 = \frac{-1}{M(-t, 1)}.$$

This identity also holds, if some z_j's coincide.

The first assertion of the lemma is thus proved.

For $k<j+1$ we have

$$M_{jk}(-t) = (-t)^{m-1-j}\sum_{l=0}^{k} b_{k-l}(-t)^l = (-t)^{m-1-j+k} + \text{lower powers of } -t.$$

For $k\geqslant j+1$ we have

$$M_{jk}(-t) = (-t)^{m-1-j}\sum_{l=0}^{k} b_l(-t)^{k-l} = (-t)^{k-j-1}\sum_{l=0}^{k} b_l(-t)^{m-l}$$

$$= (-t)^{k-j-1}M(-t, 1)-(-t)^{k-j-1}\sum_{l=k+1}^{m} b_l(-t)^{m-l},$$

i.e.

$$\frac{M_{jk}(-t)}{M(-t, 1)} = (-t)^{k-j-1}+O(|t|^{-2}) \quad \text{at} \quad |t| = \infty.$$

The lemma is fully proved.

COROLLARY. *Considering* (1.2) *one can prove that*

$$\frac{1}{2\pi i}\int_\gamma \frac{\boldsymbol{P}(z)}{M(z, 1)(tz+1)}\,dz$$

$$= \frac{\boldsymbol{N}(-t)}{N(-t, 1)} - \begin{bmatrix} 0 & 0 & 0 & \dots & 0\\ 1 & 0 & 0 & \dots & 0\\ -t & 1 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ (-t)^{m-2} & (-t)^{m-3} & & \dots & 0\end{bmatrix} = \frac{(-t)^{m-1}}{N(-t, 1)}\boldsymbol{I}+O(|t|^{-2})$$

at $|t| = \infty$.

DEFINITION. For $(x, y)\in Q^+\cup Q^-$ and any $f\in E^{(m)}$ the m-tuple

$$Uf(x, y) := \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{1}{2\pi i}\int_\gamma \frac{\boldsymbol{P}(\zeta)\,d\zeta f(s, 0)}{M(\zeta, 1)[(x-s)\zeta+y]}\,ds + $$

$$+\frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{1}{2\pi i}\int_\gamma \frac{\boldsymbol{M}(\zeta)\,d\zeta f(0, t)}{M(\zeta, 1)(x\zeta+y-t)}\,dt$$

of functions $(Uf)_j$ is called the *multiple layer potential with respect to* L.

By direct computations it follows that the $(Uf)_j$ are the $(m-1)$st-order derivatives of a solution u of $Lu = 0$ in $Q^+\cup Q^-$.

PROPOSITION 1.1. *For $f\in E^{(m)}$ the restrictions $Uf|Q^+$ and $Uf|Q^-$ of a MLP Uf can be continuously extended onto the closures $\overline{Q^+}$ and $\overline{Q^-}$, respectively. The boundary values are given by*

$$(1.3_\pm)\qquad \lim_{x\to\pm 0} Uf(x, y) = \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{N}(s)}{N(s, 1)}f(sy, 0)\,ds \pm f(0, y) \quad (y>0)$$

and

$$(1.4_\pm)\qquad \lim_{y\to\pm 0} Uf(x, y) = \pm f(x, 0)-\frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{M}(t)}{M(t, 1)}f(0, tx)\,dt \quad (x>0).$$

Proof. At first we prove $(1.3_\pm)$. From Lemma 1.1 and its corollary it follows that

$$\operatorname{Im}\frac{\boldsymbol{M}(-t)}{M(-t, 1)},\ \operatorname{Im}\frac{\boldsymbol{N}(-t)}{N(-t, 1)} = O(|t|^{-2}) \quad \text{at} \quad |t| = \infty.$$

Therefore the limits can be taken under the integral sign, i.e. we get

$$\lim_{x\to\pm 0} Uf(x, y) = \frac{1}{\pi}\int_0^\infty \lim_{x\to 0}\operatorname{Im}\frac{\boldsymbol{N}((s-x)/y)}{N((s-x)/y, 1)}f(s, 0)\frac{ds}{y} + $$

$$+\frac{1}{\pi}\lim_{x\to\pm 0}\int_0^\infty -\operatorname{Im}\frac{\boldsymbol{M}((t-y)/x)}{M((t-y)/x, 1)}f(0, t)\frac{dt}{x}$$

$$= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{N}(s)}{N(s, 1)}f(sy, 0)\,ds \mp \lim_{x\to\pm 0}\frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{M}((t-y)/x)f(0, t)}{M((t-y)/x, 1)}\frac{dt}{\pm x}$$

$$= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{N}(s)}{N(s, 1)}f(sy, 0)\,ds \mp \lim_{x\to\pm 0}\frac{1}{\pi}\int_{\mp y/x}^\infty \operatorname{Im}\frac{\boldsymbol{M}(\pm t)f(0, y\pm tx)}{M(\pm t, 1)}\,dt$$

$$= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\boldsymbol{N}(s)}{N(s, 1)}f(sy, 0)\,ds \mp \frac{1}{\pi}\int_{-\infty}^\infty \operatorname{Im}\frac{\boldsymbol{M}(\pm t)}{M(\pm t, 1)}\,dt\, f(0, y).$$

The proof of $(1.3_\pm)$ is completed by the observation

$$\int_{-\infty}^{\infty} \operatorname{Im} \frac{\boldsymbol{M}(t)}{\boldsymbol{M}(t,1)}\,dt = \int_{-\infty}^{\infty} \frac{1}{2i}\,\frac{\boldsymbol{M}(t)\overline{\boldsymbol{M}}(t,1)-\overline{\boldsymbol{M}}(t)\boldsymbol{M}(t,1)}{\boldsymbol{M}(t,1)\overline{\boldsymbol{M}}(t,1)}\,dt$$

$$= \pi \sum_{z_j} \operatorname{Res} \frac{-\boldsymbol{M}(\zeta)}{\boldsymbol{M}(\zeta,1)} = -\pi \boldsymbol{I}$$

(cf. (1.1)). The identities $(1.4_\pm)$ are proved in the same way.

Now we investigate the restriction $Uf|Q^+$ near the origin. For $x, y > 0$ we have

$$(1.5)\qquad Uf(x,y) = \frac{1}{\pi}\int_{-x/y}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)} f(x+sy,0)\,ds - \frac{1}{\pi}\int_{-y/x}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(t)}{\boldsymbol{M}(t,1)} f(0,tx+y)\,dt$$

$$= \frac{1}{\pi}\int_{-x/y}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}[f(x+sy,0)-f(0,0)]\,ds + \frac{1}{\pi}\int_{-y/x}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(t)}{\boldsymbol{M}(t,1)}[-f(0,tx+y)+f(0,0)]\,dt + \frac{1}{\pi}\left[\int_{-x/y}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}\,ds - \int_{-y/x}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(t)}{\boldsymbol{M}(t,1)}\,dt\right] f(0,0).$$

The first and the second summand on the right-hand side tend to 0 if $x, y \to +0$. We will show that

$$(1.6)\qquad \boldsymbol{C}(t) := \int_{-1/t}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}\,ds - \int_{-t}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(s)}{\boldsymbol{M}(s,1)}\,ds$$

does not depend on t, $0 \leqslant t \leqslant \infty$ (with $\int_{-1/0}^{\infty} = \int_{-\infty}^{\infty}$ and $\int_{-1/\infty}^{\infty} = \int_{0}^{\infty}$). Then it is obvious that the extended MLP $Uf|\overline{Q^+}\setminus\{(0,0)\}$ can be continuously extended to the origin by $\boldsymbol{C}(t)f(0,0)/\pi$.

We have

$$\int_{-1/t}^{0} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}\,ds = \int_{-t}^{-\infty} \operatorname{Im}\frac{\boldsymbol{N}(1/r)}{\boldsymbol{N}(1/r,1)}\,\frac{-dr}{r^2} = \int_{\infty}^{-t} \operatorname{Im}\frac{\boldsymbol{P}(r)}{\boldsymbol{M}(r,1)}\,\frac{dr}{r}$$

and

$$\boldsymbol{C}(t) = \int_{-\infty}^{-t} \operatorname{Im}\left[\frac{\boldsymbol{P}(s)}{s\boldsymbol{M}(s,1)} + \frac{\boldsymbol{M}(s)}{\boldsymbol{M}(s,1)}\right] ds + \int_{0}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}\,ds - \int_{-\infty}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(s)}{\boldsymbol{M}(s,1)}\,ds.$$

Since

$$s^{-1}\boldsymbol{P}_{jk}(s)+\boldsymbol{M}_{jk}(s) = s^{m-1-j}[s^{-1}\boldsymbol{P}_{m-1-k}(s,1)+\boldsymbol{M}_k(s,1)]$$

$$= s^{m-1-j}\Big(\sum_{l=0}^{m-1-k} b_{m-l}s^{l-m+k} + \sum_{l=0}^{k} b_{k-l}s^l\Big) = s^{k-1-j}\,\boldsymbol{M}(s,1),$$

we have

$$\boldsymbol{C}(t) = \int_{0}^{\infty} \operatorname{Im}\frac{\boldsymbol{N}(s)}{\boldsymbol{N}(s,1)}\,ds - \int_{-\infty}^{\infty} \operatorname{Im}\frac{\boldsymbol{M}(s)}{\boldsymbol{M}(s,1)}\,ds,$$

independently of $t \in [0, +\infty]$.

The continuity of $Uf|\overline{Q^-}\setminus\{(0,0)\}$ at the origin is proved in a similar way, but one has to consider Uf on each single open quadrant and on all coordinate half-axes separately.

The proposition is thus proved.

In view of this proposition it seems reasonable to apply the classical method of potentials to solve the Dirichlet problem. For given boundary values $f \in E^{(m)}$ we seek solutions Ug_+ in Q^+ and $-Ug_-$ in Q^- with densities $g_\pm \in E^{(m)}$. For each sign + or − the relations (1.3) and (1.4) result in a system of $2m$ integral equations

$$(1.7)\qquad \begin{aligned} f(0,y) &= g_\pm(0,y) \pm \mathfrak{N}[g_\pm(\cdot,0)](y),\\ f(x,0) &= g_\pm(x,0) \pm \mathfrak{M}[g_\pm(0,\cdot)](x). \end{aligned}$$

Applying the operator $\mathfrak{M}$ to the first equation of (1.7) and combining the result with the second equation, one sees that (1.7) is solvable if and only if the system

$$(1.8)\qquad f = g + \mathfrak{M}\mathfrak{N}g$$

of m integral equations is solvable. Furthermore, (1.7) has a unique solution if and only if (1.8) has a unique solution.

The concept of a MLP may be defined with densities in other function spaces and also the systems (1.7) or (1.8) of integral equations may be considered in various function spaces. A measurable function K on R_+^2 is called a *Hardy kernel* for some $L^p(R_+)$ if K is homogeneous of degree -1 and if

$$\|K\| := \int_0^\infty |K(1, y)|\, y^{-1/p} dy < \infty .$$

If K is a Hardy kernel for some $L^p(R_+)$, then the integral operator

$$Kf(x) := \int_0^\infty K(x, y) f(y)\, dy = \int_0^\infty K(1, y) f(xy)\, dy \tag{1.9}$$

is a continuous linear map in $L^p(R_+)$ with the norm $\|K\|$. The product of two integral operators with Hardy kernels in the same $L^p(R_+)$ is again an integral operator with a Hardy kernel in $L^p(R_+)$. Integral operators with Hardy kernels in a common $L^p(R_+)$ commute with each other. This follows from (1.9) and from the Fubini theorem. The spectrum of an integral operator with a Hardy kernel K can be described by the Mellin transform

$$\hat{K}(z) = \int_0^\infty x^{z-1} K(x, 1)\, dx$$

of $K(\cdot, 1)$ (cf. [6]).

The kernels of the integral operators in (1.7) and in (1.8) are Hardy kernels for $L^p(R_+)$, $1 < p \leqslant \infty$. Since the integral operators in the matrix $\mathfrak{M}\mathfrak{N}$ in (1.8) commute mutually, one can define

$$\det(I + \mathfrak{M}\mathfrak{N}) =: I + \mathscr{L},$$

where I on the left denotes the identity in $L^p(R_+)^m$, whereas I on the right denotes the identity in $L^p(R_+)$. The operator $\mathscr{L}$ in $L^p(R_+)$ has a Hardy kernel, just as well. Properties of the operators $I + \mathfrak{M}\mathfrak{N}$ or $\mathfrak{M}\mathfrak{N}$ can be characterized by properties of $I + \mathscr{L}$ or $\mathscr{L}$, e.g. $I + \mathfrak{M}\mathfrak{N}$ is invertible if and only if $I + \mathscr{L}$ is invertible.

Although the operators

$$\mathfrak{A} = \begin{bmatrix} I & \mp\mathfrak{M} \\ \pm\mathfrak{N} & I \end{bmatrix}$$

in $E^{(m)}$ or $L^p(R_+)^{2m}$ have several nice properties, we have not been able to decide whether $\mathfrak{A}^{-1}$ does exist in general.

2. Poisson formula in Q^+

If Uf denotes the double layer potential for the Laplacian Δ in $Q^+ \cup Q^-$ with a density $f \in E^{(1)} =: E'$, then it is well known that

$$Pf(x, y) := Uf(x, y) + Uf(-x, -y)$$

is a Poisson formula for Δ in Q^+. For Δ^2, however, the function

$$\begin{aligned} Vf(x, y) &:= Uf(x, y) + Uf(-x, -y) \\ &= \frac{2}{\pi} \int_{-y/x}^\infty \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} \frac{f(0, y + tx)}{(1+t^2)^2}\, dt + \frac{2}{\pi} \int_{y/x}^\infty \begin{bmatrix} -t^2 & t \\ t & -1 \end{bmatrix} \frac{f(0, -y + tx)}{(1+t^2)^2}\, dt + \\ &\quad + \frac{2}{\pi} \int_{-x/y}^\infty \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} \frac{f(x + sy, 0)}{(1+s^2)^2}\, ds + \frac{2}{\pi} \int_{x/y}^\infty \begin{bmatrix} -1 & s \\ s & -s^2 \end{bmatrix} \frac{f(sy - x, 0)}{(s^2+1)^2}\, ds \end{aligned} \tag{2.1}$$

on Q^+ has the boundary values

$$g(x, 0) = \frac{4}{\pi} \int_0^\infty \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} \frac{f(0, tx)}{(1+t^2)^2}\, dt + f(x, 0)$$

and

$$g(0, y) = f(0, y) + \frac{4}{\pi} \int_0^\infty \begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix} \frac{f(sy, 0)}{(s^2+1)^2}\, ds,$$

i.e. $g =: f + Bf$. The operator B maps E'' continuously into itself and its norm is

$$\|B\| \leqslant \frac{4}{\pi} \int_0^\infty \frac{t}{(1+t^2)^2}\, dt = \frac{2}{\pi} < 1 .$$

Therefore we get the Poisson formula for Δ^2 in Q^+ as an infinite series, namely

$$Pf = V(I+B)^{-1} f = V \sum_{j=0}^\infty (-B)^j f = V(I - B) \sum_{j=0}^\infty B^{2j} f. \tag{2.2}$$

This formula can be handled quite well, since the operator B^2 has the simple form

$$B^2 f(x) = \frac{16}{\pi^2} \int_0^\infty \int_0^\infty f(stx) \frac{s\, ds}{(1+s^2)^2} \frac{t\, dt}{(1+t^2)^2}$$

and the norm of B^2 satisfies

$$\|B^2\| = 4/\pi^2 < 1/2 .$$

For further purposes we need some weak regularity properties of the Poisson formula Pf. Therefore we introduce the function spaces

$$F_{1,\alpha} := \mathrm{CB}_\alpha(\overline{R_+}) \cap C^1_\alpha(R_+) \qquad (0 < \alpha < 1)$$

of bounded Hölder continuous real-valued functions φ on R_+ which are continuously differentiable in R_+ and whose derivatives φ' satisfy the condition

$$|\varphi|_{1,-a} := \sup_{x>0} x^a |\varphi'(x)| < \infty .$$

LEMMA 2.1. *For $a \in]0,1[$ the linear space $F_{1,a}$ with the norm*

$$\|\varphi\|_{1,a} := \sup_{x\geq 0} |\varphi(x)| + \sup_{\substack{x,y\geq 0\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^a} + |\varphi|_{1,-a} =: \|\varphi\|_0 + H_a(\varphi) + |\varphi|_{1,-a}$$

is a Banach space.

Proof. Let (φ_j) be a Cauchy sequence in $F_{1,a}$. Then (φ_j) converges to φ in $\mathrm{CB}(\overline{R_+})$. Since (φ_j) is a Cauchy sequence also with respect to H_a, φ belongs to $\mathrm{CB}_a(\overline{R^+})$ and (φ_j) converges to φ with respect to H_a. Since (φ_j) is also a Cauchy sequence with respect $|\cdot|_{1,-a}$, the derivatives φ_j' of φ_j converge uniformly on each compact subset of R_+, i.e. $\varphi \in C^1(R_+)$ and $\varphi_j' \to \varphi'$ locally uniformly and $|\varphi_j - \varphi|_{1,-a}$ converges to 0. Thus $\varphi \in F_{1,a}$ and $\|\varphi_j-\varphi\|_{1,a} \to 0$.

LEMMA 2.2. *The operator E defined by*

$$E\varphi(x) := \frac{4}{\pi}\int_0^\infty \frac{t\varphi(tx)}{(1+t^2)^2}\,dt \qquad (x \geq 0)$$

maps $F_{1,a}$ continuously into itself and its norm satisfies

$$\|E\|_a < 1.$$

Proof. Let $\|E\|$, $\|E\|_{H_a}$ and $\|E\|_{1,-a}$ denote the operator norms of E with respect to the seminorms $\|\cdot\|_0$, H_a and $|\cdot|_{1,-a}$, respectively. Then

$$\|E\|_a \leq \max\{\|E\|,\ \|E\|_{H_a},\ \|E\|_{1,-a}\}.$$

Now, firstly, we have $\|E\| \leq 2/\pi$. Furthermore, the estimate

$$|E\varphi(x)-E\varphi(y)| \leq \frac{4}{\pi}\int_0^\infty \frac{tH_a(\varphi)|xt-yt|^a}{(1+t^2)^2}\,dt = \frac{4}{\pi} H_a(\varphi)|x-y|^a \int_0^\infty \frac{t^{1+a}}{(1+t^2)^2}\,dt$$

holds, i.e.

$$\|E\|_{H_a} \leq \frac{4}{\pi}\int_0^\infty \frac{t^{1+a}}{(1+t^2)^2}\,dt =: \frac{4}{\pi} I(a).$$

From the mean value theorem and the Lebesgue theorem on dominated convergence we see that $E\varphi \in C^1(R_+)$ for all $\varphi \in F_{1,a}$ and

$$(E\varphi)'(x) = \frac{4}{\pi}\int_0^\infty \frac{t^2\varphi'(tx)}{(1+t^2)^2}\,dt.$$

Consequently

$$|(E\varphi)'(x)| \leq \frac{4}{\pi}\int_0^\infty \frac{t^2|\varphi'(tx)|}{(1+t^2)^2}\,dt \leq \frac{4}{\pi}\,\frac{|\varphi|_{1,-a}}{|x|^a}\, I(1-a).$$

We have thus obtained

$$\|E\|_a \leq \frac{4}{\pi}\max\{I(0),\ I(a),\ I(1-a)\}.$$

Since $I(0) = 1/2$, $I(1) = \pi/4$, and

$$I'(a) = \int_0^\infty \frac{t^{1+a}\ln t}{(1+t^2)^2}\,dt = \int_0^1 \frac{t^{1+a}\ln t}{(1+t^2)^2}\,dt + \int_1^0 \frac{t^{-1-a}(-\ln t)}{(t^2+1)^2}\, t^4\left(-\frac{dt}{t^2}\right)$$

$$= \int_0^1 \frac{t\ln t}{(1+t^2)^2}(t^a - t^{-a})\,dt > 0 \qquad (0<a<1),$$

the function I increases monotonously on $[0,1]$, and so

$$\|E\|_a \leq \frac{4}{\pi} I(\max\{a, 1-a\}) < \frac{4}{\pi} I(1) = 1 \qquad (0<a<1).$$

Now we introduce the Banach space

$$E''_{1,a} := \left\{ f = \begin{bmatrix} f(\cdot,0)\\ f(0,\cdot)\end{bmatrix} \in \begin{matrix} F^2_{1,a}\\ \oplus\\ F^2_{1,a}\end{matrix} :\ f(x,0)|_{x=0} = f(0,y)|_{y=0} \right\}$$

normed by

$$\|f\|_{1,a} := \max_{k=0,1}\{\|f_k(\cdot,0)\|_{1,a},\ \|f_k(0,\cdot)\|_{1,a}\}.$$

PROPOSITION 2.1. *Let $p \geq 1$ and $a \in]0, 1/p[$ be given. Then for each $f \in E''_{1,a}$ the function Pf satisfies*

$$Pf \in C^\infty(Q^+)^2 \cap W^1_{p,\mathrm{loc}}(\overline{Q^+})^2 \cap C^{0+a}_{\mathrm{loc}}(\overline{Q^+})^2.$$

Proof. It is obvious that $Pf \in C^\infty(Q^+)^2$. We apply a Privalov type theorem in order to show that

$$Pf \in C^{0+a}_{\mathrm{loc}}(\overline{Q^+})^2.$$

Obviously, for constant f we have

$$g := (I-B)\sum_{j=0}^{\infty} B^{2j} f = (I-B)\sum_{j=0}^{\infty} B^{2j} f = \text{const}$$

and $Pf = Vg = \text{const}$ (cf. the proof of Proposition 1.1). Therefore we may assume $f(0,0) = 0$. This implies $g(0,0) = 0$. To get Hölder continuity of Pf, we extend g by 0 to the negative coordinate half-axes. Then the integration in the formula for Vg may be extended to the whole coordinate axes. The kernels $x^{3-k}y^k(x^2+y^2)^{-2}$ or $x^k y^{3-k}(x^2+y^2)^{-2}$ $(k = 0, 1, 2)$ fulfil the conditions ([2], (3.2) with $\alpha' = 1$ and (3.3)) with respect to the half-plane $x > 0$ or $y > 0$, respectively. We write g as a sum $g = \tilde{g}+h$, where $\tilde{g}$ vanishes in a sufficiently large open bounded neighbourhood G of the origin and h vanishes outside a sufficiently large compact set. The Privalov type theorem ([1], Th. 3.1) is then applied to Vh and the function $V\tilde{g}$ is C^∞ on each compact set $K' \subset G \cap \overline{Q^+}$.

From $f(0,0) = 0$ and $f'(0,s)$, $f'(s,0) = O(s^{-\alpha})$ at $s = 0$ and at $s = \infty$ it follows that $g(0,0) = 0$ and $g'(0,s)$, $g'(s,0) = O(s^{-\alpha})$ at $s = 0$ and at $s = \infty$. If we differentiate the integrals in (2.1) with respect to the parameter x, the lower limits of the integrals yield no term to the derivative $D_x Vg$, since $g(0,0) = 0$. From

$$|g'(0,s)|,\ |g'(s,0)| \leqslant Cs^{-\alpha}$$

it follows that

$$D_x Vg(x,y) = \frac{2}{\pi}\int_{-y/x}^{\infty}\begin{bmatrix} t^3 & t^2 \\ t^2 & t\end{bmatrix}\frac{g'(0,tx+y)}{(1+t^2)^2}\,dt + \tag{2.3}$$

$$+\frac{2}{\pi}\int_{y/x}^{\infty}\begin{bmatrix} -t^3 & t^2 \\ t^2 & -t\end{bmatrix}\frac{g'(0,tx-y)}{(1+t^2)^2}\,dt + \frac{2}{\pi}\int_{-x/y}^{\infty}\begin{bmatrix} 1 & t \\ t & t^2\end{bmatrix}\frac{g'(x+ty,0)}{(1+t^2)^2}\,dt +$$

$$+\frac{2}{\pi}\int_{x/y}^{\infty}\begin{bmatrix} 1 & -t \\ -t & t^2\end{bmatrix}\frac{g'(ty-x,0)}{(1+t^2)^2}\,dt$$

and $D_y Vg(x,y)$ is written analogously. This is checked in the same way as we now show the absolute convergence of the integrals in (2.3).

The integrals in (2.3) are dominated by the quantities

$$\text{const}\cdot x^{-\alpha}\int_{\lambda}^{\infty}\frac{|t|^k(t-\lambda)^{-\alpha}}{(1+t^2)^2}\,dt \qquad (\lambda = \pm y/x \in R\setminus\{0\},\ k = 0,1,2,3)$$

and by analogous expressions with x, y interchanged. For $\lambda > 0$ we have

$$\int_{\lambda}^{\infty}\frac{t^k(t-\lambda)^{-\alpha}}{(1+t^2)^2}\,dt \leqslant \frac{2^k\lambda^k}{(1+\lambda^2)^2}\,\frac{\lambda^{1-\alpha}}{1-\alpha} + 2^\alpha\int_{2\lambda}^{\infty}\frac{t^{k-\alpha}}{(1+t^2)^2}\,dt \leqslant c < \infty,$$

for $-1 \leqslant \lambda < 0$ we have

$$\int_{\lambda}^{\infty}\frac{|t|^k(t-\lambda)^{-\alpha}}{(1+t^2)^2}\,dt \leqslant 2^k\int_0^{2-\lambda} t^{-\alpha}\,dt + \int_2^{\infty} t^{k-4-\alpha}\,dt \leqslant c < \infty,$$

but for $\lambda < -1$ we get

$$\int_{\lambda}^{\infty}\frac{|t|^k(t+|\lambda|)^{-\alpha}}{(1+t^2)^2}\,dt \leqslant \int_0^{\infty}\frac{t^{k-\alpha}}{(1+t^2)^2}\,dt + \int_0^{1/2}\frac{t^k(|\lambda|-t)^{-\alpha}}{(1+t^2)^2}\,dt +$$

$$+\int_{1/2}^{|\lambda|/2}\frac{t^k(|\lambda|-t)^{-\alpha}}{(1+t^2)^2}\,dt + \int_{|\lambda|/2}^{|\lambda|}\frac{t^k(|\lambda|-t)^{-\alpha}}{(1+t^2)^2}\,dt$$

$$\leqslant c_1 + 2^\alpha\int_0^{1/2} t^k\,dt + \left(\frac{2}{|\lambda|}\right)^{\alpha}\int_{1/2}^{|\lambda|/2} t^{k-4}\,dt + |\lambda|^{k-3-\alpha}\int_{1/2}^{1} t^{k-4}(1-t)^{-\alpha}\,dt \leqslant c < \infty.$$

In all the three cases the constants c on the right-hand sides can be chosen independently of λ. We have

$$|D_x(Vg)_j(x,y)|,\ |D_y(Vg)_j(x,y)| \leqslant c(x^{-\alpha}+y^{-\alpha}).$$

Then we get for $v := (Vg)_j$ and $p \geqslant 1$

$$\iint_{\substack{x^2+y^2<a^2\\ x,y>0}} |D_x v|^p\,dx\,dy,$$

if $\alpha < 1/p$. The same estimate holds also for $D_y v$. Thus the proposition is proved.

3. A priori estimates for solutions of the Dirichlet problem for Δ^2 in a rectangle

Let Ω be a rectangle and let $p > 2$. Let $W_p^2(\Omega)$ denote the Sobolev space of those distributions u on Ω whose derivatives $D^\beta u$ of order $|\beta| \leqslant 2$ belong to $L^p(\Omega)$. According to the Sobolev imbedding theorem, this space is continuously imbedded in $C^1(\overline{\Omega})$. For any given biharmonic function $u \in C^2(\overline{\Omega})$ with Dirichlet data $(D^\beta u|\partial\Omega)_{|\beta|\leqslant 1}$ we construct a function $\tilde{u} \in W_p^2(\Omega)$ with the Dirichlet data of u and such that

$$\|\tilde{u}\|_{C^1(\overline{\Omega})} \leqslant c_1\|u\|_{1,\partial\Omega}, \tag{3.1}$$

where c_1 does not depend on u. For all $w \in C_0^\infty(\Omega)$ the identity

$$(u-\tilde{u}, \Delta^2 w) := \iint_{\Omega}(u-\tilde{u})\,\Delta^2 w\,dx\,dy = (\Delta\tilde{u}, \Delta w)$$

holds, and its right-hand side can be estimated by

(3.2) $$|(\Delta\tilde{u}, \Delta w)| \leqslant c_2\|u\|_{1,\partial\Omega}\|w\|_{W^2_{p'}(\Omega)},$$

where c_2 depends neither on u nor on w and here $p' \in]1, 2[$ is defined by $1/p+1/p' = 1$.

Since the Dirichlet data of $u-\tilde{u} \in W^2_p(\Omega)$ vanish, this function belongs to the closure $\mathring{W}^2_p(\Omega)$ of $C^\infty_0(\Omega)$ in $W^2_p(\Omega)$ (cf. [20], p. 59; the proof given there can be adapted). For bounded domains G with C^2-boundary C. Simader [19] has proved the existence of a constant $c(p)$ such that the estimate

(3.3) $$\|f\|_{W^2_p(G)} \leqslant c(p)\sup\{|(\Delta f, \Delta g)|: \ g \in C^\infty_0(G) \text{ and } \|g\|_{W^2_{p'}(G)} \leqslant 1\}$$

holds for all $f \in \mathring{W}^2_p(G)$. If the estimate (3.3) holds for $G = \Omega$, then there will be a constant c_4 such that for biharmonic functions $u \in C^2(\overline{\Omega})$ the estimate

(3.4) $$\|u\|_{C^1(\overline{\Omega})} \leqslant c_4\|u\|_{1,\partial\Omega}$$

is valid. From this estimate of Agmon–Miranda type it would follow by an approximation argument that for any $F \in C^1(R^2)$ there is a biharmonic function $u \in C^1(\overline{\Omega})$ with the same Dirichlet data

$$D^\beta u = D^\beta F \quad \text{on } \partial\Omega \quad (|\beta| \leqslant 1).$$

If for any two positive numbers a_0, b_0 there are neighbourhoods $U, V \subset R_+$ of a_0 and b_0, respectively, such that for all $\Omega =]0, a[\times]0, b[$, $a \in U$ and $b \in V$, the estimate (3.4) holds with the same constant c_4, then $u = 0$ will be the only biharmonic function in $C^1(\overline{\Omega})$ which satisfies the homogeneous Dirichlet conditions.

LEMMA 3.1. *Let $a_0, b_0 > 0$ and $p > 2$ be arbitrary and let $\Omega =]0, a[\times\times]0, b[$, where $a, b > 0$. There are neighbourhoods $U \subset R_+$ of a_0 and $V \subset R_+$ of b_0 and a constant c_1 such that for any $a \in U$, $b \in V$ and any function $F \in C^2(R^2)$ there is a function $v \in W^2_p(\Omega) \subset C^1(\overline{\Omega})$ which satisfies the Dirichlet conditions*

$$D^\beta v = D^\beta F \quad \text{on } \partial\Omega \quad (|\beta| \leqslant 1)$$

and which satisfies the estimate

(3.5) $$\|v\|_{C^1(\overline{\Omega})} \leqslant c_1\|F\|_{1,\partial\Omega}.$$

Proof. The vertices of Ω are $A = (0, 0)$, $B = (a, 0)$, $C = (a, b)$, and $D = (0, b)$. We denote by Q_X, $X \in \{A, \ldots, D\}$, the open quadrant which has its vertex in X and which contains Ω. We denote by P_X the Poisson formula for Δ^2 in Q_X which was derived in the preceding section for $X = A$, $Q_A = Q^+$.

Let us denote $\Omega^0 =]0, a_0[\times]0, b_0[$ and let $(\chi_X)_X$ be a modified smooth partition of unity for a neighbourhood of $\partial\Omega^0$, i.e. a family of functions $\chi_X \in C^\infty_0(R^2)$ with $\chi_X \geqslant 0$, $= 1$ in a neighbourhood of X, and such that $\sum_X \chi_X^2 = 1$ in a neighbourhood of $\partial\Omega^0$. Furthermore, we assume that $\operatorname{supp}\chi_X$ intersects $\partial\Omega^0$ only in the two sides of Ω^0 adjacent to X and that χ_X is constant in the normal direction near the boundary $\partial\Omega^0$. The neighbourhoods U and V are chosen in such a way that the partition of unity retains all of its properties when considered with respect to Ω.

Let now $F \in C^2(R^2)$ be given and let us denote $D_xF = f_0$ and $D_yF = f_1$. We apply P_A to the boundary values

$$f_A(x, 0) = \begin{bmatrix} \chi_A(x,0)f_0(x,0)+D_x\chi_A(x,0)\int\limits_0^x f_0(s,0)\,ds \\ \chi_A(x,0)f_1(x,0) \end{bmatrix}$$

and analogously to $f_A(0, y)$. The result P_Af_A is a pair of derivatives of a biharmonic function u_A in Q_A which is uniquely determined by $u_A(A) = 0$. In the same way we define functions u_X for the remaining vertices X. The function

$$v := \sum_X \chi_X u_X\Big|\overline{\Omega} + \sum_X F(X)\chi_X^2\Big|\overline{\Omega}$$

fulfils all the assertions of the lemma. We have

$$\|v\|_{C^1(\overline{\Omega})} \leqslant c_5\Big[\sum_{X,j}\|(P_Xf_X)_j\|_{C(\overline{\Omega})} + \sum_X|F(X)|\Big],$$

where the constant c_5 depends on the partition of unity (χ_X). On account of (2.2), Lemma 2.2 and (2.1), the summands on the right-hand side can be estimated so as to obtain (3.5).

LEMMA 3.2. *Let $u \in C^2(\overline{\Omega})$ be a biharmonic function and let $\tilde{u}$ denote the function which was constructed in the preceding lemma for $F = u$. Then* (3.2) *holds.*

Proof. Since

$$\Delta\tilde{u} = \sum_X \chi_X\Delta u_X + 2\sum_X \nabla u_X\nabla\chi_X + \sum_X u_X\Delta\chi_X + \sum_X u(X)\Delta\chi_X^2,$$

there is a constant c_6, which depends on the χ_X's, such that

$$|(\Delta\tilde{u}, \Delta w)| \leqslant \sum_X |(\chi_X\Delta u_X, \Delta w)| + c_6\|u\|_{1,\partial\Omega}\|w\|_{W^2_{p'}(\Omega)}$$

for all $w \in C^\infty_0(\Omega)$. For $w \in C^\infty_0(\Omega)$ we have

$$(\chi_X\Delta u_X, \Delta w) = \big(\Delta(\chi_X\Delta u_X), w\big).$$

Since $\Delta^2 u_X = 0$, there are derivatives $D^\beta u_X$ up to order 3 only in $\Delta(\chi_X \Delta u_X)$. We proceed by integrating twice by parts to get an integrand which contains derivatives of u_X up to order 1, of w up to order 2 and of χ_X. Then

$$|(\chi_X \Delta u_X, \Delta w)| \leqslant c_7 \|u_X\|_{C^1(\bar{\Omega})} \|w\|_{W^2_{p'}(\Omega)}$$

is evident. As in the proof of the preceding lemma, one can estimate

$$\|u_X\|_{C^1(\bar{\Omega})} \leqslant c_8 \|u\|_{1,\partial\Omega},$$

and this proves the lemma.

4. Remarks

For the construction of the Poisson formula $P = P_+$ for Δ^2 in Q^+, it was only convenient but not essential to use Vf instead of the MLP Uf. The MLP $Uf|Q^+$ has boundary values $h = : f - A_+ f$. The operator A_+ maps E'' into itself and has a norm $\|A_+\| \leqslant 1/2 + 1/\pi < 1$. Therefore, the series

$$P_+ f = U \sum_j A_+{}^j f \tag{4.1}$$

converges. The use of the MLP instead of the modified MLP Vf has the advantage that one can construct the Poisson formula P_- for Δ^2 quite analogously in the exterior Q^- of the first quadrant.

Unfortunately, the formula (4.1) cannot be applied in the same way as formula (2.2) because there are terms like $I(2-a)$ (cf. the proof of Lemma 2.2) occurring in the estimate of $\|A_+\|$, so that the Neumann series $\sum A_+{}^j f$ seems to diverge in $E''_{1,a}$. One might propose a function space $F_{1,a,\beta} := \mathrm{CB}_a(\overline{R^+}) \cap C^1_{-\beta}(R_+)$ instead of $F_{1,a}$. If one chooses β near 1, the Neumann series will converge in the corresponding space $E''_{1,a,\beta}$. For β near 1, however, we have $Pf \in W^1_{p,\mathrm{loc}}(\overline{Q^+})^2$ only for p near 1, i.e. we cannot conclude $Pf \in C^1(\overline{Q^+})^2$ by Sobolev's imbedding theorem.

The operator Δ^2 is characterized by $z_1 = z_2 = -i$ (cf. Section 1). It is to be expected that there is a neighbourhood $D \subset C^2$ of $(-i, -i)$ such that the Dirichlet problem can be treated for operators L, $L(z, 1) = (z - z_1)(z - z_2)(z - \bar{z}_1)(z - \bar{z}_2)$ and $(z_1, z_2) \in D$, in Q^+ in the same way as for Δ^2. We have not checked this conjecture yet. At any rate, the details will be a little bit more complicated.

The construction of Poisson formulas by modified MLP's and by Neumann series in $E^{(m)}$ is not a general method. It does not work, for instance, either for any fourth order operator L, nor does it for Δ^3 in Q^+. Therefore it remains an open and interesting question whether the operators $\mathfrak{A}$ in $E^{(m)}$ which were discussed in Section 1 have an inverse.

It is clear that the results of the third section are also valid for some slightly more general domains and operators. As long as we do not have the Poisson formula for any operator L in a quadrant, however, the whole approach is limited to rather special cases of domains with piecewise smooth boundaries. By the way, the use of MLP is not limited to the dimension $n = 2$, but in higher dimensions the analogue of the operator $\mathfrak{A}$ seems to be essentially more complicated.

Finally, we want to point out that the question whether (3.3) does hold for domains with piecewise smooth boundary may be posed in the same generality as it was answered by C. Simader [19] for domains with sufficiently smooth boundary.

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NEW IDEAS ON COMPLETE INTEGRABILITY

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The notion of complete integrability for nonlinear Hamiltonian systems is based on a theorem of Liouville. This result states that if a Hamiltonian system defined on $\boldsymbol{R}^{2N}$ has N independent first integrals in involution, then the system is completely integrable (i.e. can be integrated by quadrature). This notion has been extended to infinite dimensional Hamiltonian systems recently be Faddeev, Gardenar, Lax, Novikov (and others) who have shown that certain partial differential equations are integrable in this sense provided one lets $N \to \infty$. In particular, these authors study the Korteweg–de Vries equation

$$u_t = uu_x - u_{xxx}. \tag{1}$$

However, this notion of complete integrability seems limited to two-dimensional partial differential equations. Moreover, the methods developed in those studies, mentioned above, totally break down when a system "nearby" a given integrable system is examined. Finally, these methods do not seem to apply to nonlinear elliptic boundary value problems.

In this article we define a new type of complete integrability for nonlinear elliptic boundary value problem (in fact, for nonlinear continuous mappings between Banach spaces), and we show that this new notion does not suffer from the defects described above.

1. The nonlinear boundary value problem

For explicitness we shall study the following nonlinear elliptic Dirichlet problem:

$$\begin{cases} \Delta u + f(u) = g, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ with boundary $\partial\Omega$ and $f(u)$ is a C^k