THEOREM 5. *Assume that the support function* $c(F(t, x), \psi)$ *is measurable in* t *and continuously differentiable in* (x, ψ) *except for the points* $\psi = 0$, *and the derivatives* $c_x(F(t, x), \psi)$, $c_\psi(F(t, x), \psi)$ *satisfy the Lipschitz condition in* (x, ψ). *Then, for any given initial conditions, for functions* $(x(t), \psi(t))$ *the optimal solution is unique.*

7. Concluding remarks

The Pontryagin maximum principle was proved [8] for control system (3) for the function $f(t, x, u)$ continuously differentiable in x. These systems may be transformed into the differential inclusion form and the support function is in this case

$$c(F(t, x), \psi) = \max_{u \in U} (f(t, x, u), \psi).$$

Theorem 1 is applicable when this support function is continuously differentiable in x. Its not very difficult to show that there exists a function $f(t, x, u)$ continuously differentiable in x for which the corresponding support function $c(F(t, x), \psi)$ does not have this property and *vice versa*. That is, the Pontryagin maximum principle and Theorem 1 intersect over some class of control systems of type (3). It is possible to formulate as a hypothesis the following theorem, which includes Theorem 1 and the Pontryagin maximum principle.

THEOREM 6. *Suppose that the multivalued function* $F(t, x)$ *for inclusion* (1) *is measurable in* t *and satisfies the Lipschitz condition in* x, *and the sets* M_0, M_1 *are convex and the solution* $x(t)$, $t_0 \leqslant t \leqslant t_1$ *is optimal. Then there exists such a nontrivial absolutly continuous function* $\psi(t)$ *that the following conditions are valid*:

$$\text{(A)} \qquad \frac{d}{dt}\begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} \in \partial c_{(x,\psi)}\big(F(t, x(t)), \psi(t)\big)$$

almost everywhere on the interval $[t_0, t_1]$;

$$\text{(B)} \qquad c(M_0, \psi(t_0)) = (x(t_0), \psi(t_0));$$

$$\text{(C)} \qquad c(M_1, -\psi(t_1) = (x((t_1), -\psi(t_1)).$$

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LOCAL CONTROLLABILITY OF ODD SYSTEMS

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1. Introduction

Consider a control system

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in R^n$ and $u \in U$ (we do not specify the set U at this point) and a set of admissible controls $\mathscr{U}$ which is a subset of the set of mappings u of intervals $[0, T(u)]$, $T(u) \geqslant 0$ into U, having the property that for any $u \in \mathscr{U}$ the solution $\varphi(t, x_0, u)$ of the differential equation

$$\dot{x} = f(x, u(t))$$

with $\varphi(0, x_0, u) = x_0$ is uniquely defined on $[0, T(u)]$. We denote by $\mathscr{R}(x_0)$ the reachable set of (1) from x_0, i.e., $\mathscr{R}(x_0) = \{\varphi(T(u), x_0, u) \mid u \in \mathscr{U}\}$ and call (1) *locally controllable at* x_0 if $x_0 \in \operatorname{int} \mathscr{R}(x_0)$.

The well-known theorem of Kalman gives necessary and sufficient conditions of local controllability at 0 for the class of linear systems $(f(x, u) = Ax + Bu)$ with U being a subset of R^m containing 0 in its interior (the bang-bang controllability theorem makes the choice of $\mathscr{U}$ irrevelant in this case). For nonlinear systems, sufficient conditions for local controllability at a rest point of the uncontrolled system are given by the theorem of Lee and Markus ([4]). However, since the theorem of Lee and Markus uses only the linearization of the system (in both x and u), it is not difficult to see that its sufficient condition is far from being necessary.

A more recent approach, going back to Hermann ([2], cf. also [5], [6]) relates the problem of controllability to the study of orbits of families of vector fields. Given a family of vector fields $\mathscr{X} = \{X^i \mid i \in I\}$ on R^n, the orbit of x_0 is defined as $\Omega(x_0) = \{\varphi_{t_p}^{i_p} \circ \ldots \circ \varphi_{t_1}^{i_1}(x_0) \mid p \geqslant 0,\ i_j \in I,\ t_j \in R,\ j = 1, \ldots, p\}$, where by φ^i we denote the flow of X^i. It is immediately seen that if $\mathscr{U}$ is taken as the set of piecewise constant controls and we associate with (1) the family of vectors fields $\mathscr{X} = \{X^u \mid u \in U\}$ defined by $X^u(x) = f(x, u)$, then $\mathscr{R}(x_0) = \Omega^+(x_0)$, were $\Omega^+(x_0)$ is the positive semiorbit of x_0, defined by $\Omega^+(x_0) = \{\varphi_{t_p}^{u_p} \circ \ldots \circ \varphi_{t_1}^{u_1}(x_0) \mid p \geqslant 0,\ u_i \in U,\ t_i \geqslant 0,\ i = 1, \ldots, p\}$.

It is not true in general that $\Omega^+(x_0) = \Omega(x_0)$. The only simple (but rather restrictive) condition guaranteeing this is the symmetry condition: for every x and every $i \in I$ there exists a $j \in I$ such that $X^j = -X^i$ in some neighbourhood of x (cf. [5]).

For families of analytic vector fields, the classical theorem of Chow gives a certain rank condition (cf. Theorem 1 below), which is necessary and sufficient for $x_0 \in \operatorname{int} \Omega(x_0)$ and which, if applied to linear systems, is equivalent to the rank condition of Kalman (cf. [2], [5]). However, Chow's theorem does not yield a generalization of Kalman's one since the family of vector fields, associated with a linear system is not symmetric in general.

The aim of this paper is to prove the equivalence of Chow's rank condition to local controllability for systems exhibiting a different kind of symmetry which is satisfied for linear systems—a theorem which does contain Kalman's controllability theorem as its special case.

In § 2, we formulate the main theorem in the language of families of vector fields and three lemmas from which the proof of the theorem easily follows. The applications of the main theorem to local controllability of control systems are given in § 3 and § 4 contains the proof of Lemma 3.

2. Main theorem

We shall call a family of vector fields $\mathscr{X} = \{X^i \mid i \in I\}$ *odd* if for every $i \in I$ there exists a $j \in I$ such that $X^i(-x) = -X^j(x)$ for all x. An odd family of vector fields can always be indexed in such a way that I contains symbols $+i$ and $-i$ in such a way that $X^{-i}(x) = -X^i(-x)$ (sometimes X^i and X^{-i} may coincide). When dealing with an odd family of vector fields we shall always assume that it is indexed in this way.

Further, we shall always assume that all the vector fields under consideration are complete, i.e., that the domain of existence of their integral curves is R. This assumption, just as the assumption that the vector fields are defined and satisfy the oddness assumption over all R^n (instead of a neighbourhood of 0) is not essential and is made only for the sake of simplicity.

With a family $\mathscr{X}$ of C^∞ vector fields we associate the family $[\mathscr{X}]$, which is the smallest family of vector fields containing $\mathscr{X}$ and closed under the formation of Lie brackets (cf. [2], [5], [6]). We write $\mathscr{X}(x) = \{X(x) \mid X \in \mathscr{X}\}$.

THEOREM 1. *Let $\mathscr{X}$ be an odd family of analytic vector fields on R^n. Then $0 \in \operatorname{int} \Omega^+(0)$ if and only if Chow's rank condition is satisfied at* 0, *i.e.*, $\dim \operatorname{span} [\mathscr{X}](0) = n$.

For the proof we need the following three lemmas.

LEMMA 1. *Let $\mathscr{X} = \{X^i \mid i \in I\}$ be a family of C^∞ vector fields on $\mathbf{R}^n$ satisfying Chow's rank condition at x_0. Then, for every $\delta > 0$, there exist $i_1, \ldots, i_n \in I$, $s_1, \ldots, s_n \in [0, \delta)$ such that the map $(t_1, \ldots, t_n) \mapsto \varphi^{i_n}_{t_n} \circ \ldots \circ \varphi^{i_1}_{t_1}(x_0)$ is a local diffeomorphism at $(s_1, \ldots, s_n)$.*

For the proof, cf. [3].

LEMMA 2. *Let $\mathscr{X} = \{X^i \mid i \in I\}$ be a family of C^∞ vector fields, $y \in \operatorname{int} \Omega^+(x)$, $z \in \Omega^+(y)$. Then, $z \in \operatorname{int} \Omega^+(x)$.*

Proof. From $z = \varphi^{i_p}_{t_p} \circ \ldots \circ \varphi^{i_1}_{t_1}(y)$ it follows that

$$z \in \varphi^{i_p}_{t_p} \circ \ldots \circ \varphi^{i_1}_{t_1}(\operatorname{int} \Omega^+(x)) \subset \operatorname{int}\left(\varphi^{i_p}_{t_p} \circ \ldots \circ \varphi^{i_1}_{t_1}(\Omega^+(x))\right) \subset \operatorname{int} \Omega^+(x),$$

since $\varphi^{i_p}_{t_p} \circ \ldots \circ \varphi^{i_1}_{t_1}$ is a local diffeomorphism.

To make the formulation of Lemma 3 easier, we define for a given family of analytic vector fields, a *stream* on $V \subset \mathbf{R}^n$ open as an analytic map $\chi \colon (-\delta, \delta) \times V \to \mathbf{R}^n$, $\delta > 0$ (write $\chi_t(x) = \chi(t, x)$) such that

1. for every $t \in (-\delta, \delta)$, χ_t is a diffeomorphism $V \to \chi_t(V)$,
2. for all $x \in V$, $\chi_0(x) = x$,
3. for every $t \in [0, \delta)$ and all $x \in V$, $\chi_t(x) \in \Omega^+(x)$.

Note that if χ, ψ are streams on V and τ_1, τ_2 are analytic functions on a neighbourhood of 0 such that

$$\tau_1(0) = \tau_2(0) = 0 \quad \text{and} \quad \tau_1(t) > 0,\ \tau_2(t) > 0 \text{ for } t > 0, \tag{2}$$

then $t \mapsto \chi_{\tau_1(t)} \circ \psi_{\tau_2(t)}$ is also a stream on V. If for two streams χ, ψ there exists a stream η and analytic functions τ_1, τ_2 satisfying (2) such that $\psi_t = \eta_{\tau_1(t)} \circ \chi_{\tau_2(t)}$ for $|t|$ sufficiently small, we shall write $\chi \prec \psi$. The relation $\prec$ is obviously transitive.

LEMMA 3. *Let $\mathscr{X}$ be an odd family of analytic vector fields. Then for every stream χ there exists a stream $\vartheta \succ \chi$ such that $\vartheta_t(0) = 0$ for $t \geqslant 0$ sufficiently small.*

Proof of Theorem 1. *Sufficiency*. Write $\chi_t(x) = \varphi^{i_n}_{ts_n} \circ \ldots \circ \varphi^{i_1}_{ts_1}(x)$, where $i_1, \ldots, i_n$, $s_1, \ldots, s_n$ are chosen as in Lemma 1. Obviously, χ is a stream on some neighbourhood of 0. The Jacobian of χ_t at 0 is an analytic function of t which does not vanish for $t = 1$. Therefore, it must be non-zero for $t > 0$ sufficiently small. Consequently, $\chi_t(0) \in \operatorname{int} \Omega^+(0)$ for $t > 0$ sufficiently small. Let ϑ be as in Lemma 3. Then there exist analytic functions τ_1, τ_2 satisfying (2) and a stream η such that $\eta_{\tau_2(t)} \circ \chi_{\tau_1(t)}(0) = 0$ (which implies $0 \in \Omega^+(\chi_t(0))$) for $t > 0$ sufficiently small. By Lemma 2, $0 \in \operatorname{int} \Omega^+(0)$.

The *necessity* of Chow's condition follows from the fact that $\Omega^+(0) \subset \Omega(0)$ and that, if Chow's condition is not satisfied, $\Omega(0)$ is a submanifold of $\mathbf{R}^n$ of dimension $< n$ (cf. [5], [6]).

Let us note that although Theorem 1 is formulated in R^n, its nature is local. Thus, we can replace $\mathbf{R}^n$ by an n-dimensional analytic manifold, provided the oddness assumption is satisfied in some local chart at 0. This is the situation if e.g. Chow's condition is not satisfied and we consider the restriction of $\mathscr{X}$ to the orbit $\Omega(0)$, which is an analytic submanifold of R^n of dimension $< n$ (note that $\Omega(0)$ is symmetric with respect to 0 if $\mathscr{X}$ is odd!); cf. [5], [6]. Thus we have

THEOREM 2. *Let $\mathscr{X}$ be an odd family of analytic vector fields. Then $0 \in \operatorname{int} \Omega^+(0)$ in the topology of $\Omega(0)$.*

3. Application to control systems

Consider a control system

$$\dot{x} = f_0(x)+\sum_{i=1}^{p} u_i f_i(x), \quad u_i \in U_i = [-1, +1], \tag{3}$$

$U = U_1 \times U_2 \times \ldots \times U_p$. We associate with (3) the family of vector fields $\mathscr{X} = \{f_0 \pm f_i |\ i = 1, \ldots, p\}$. If we take as $\mathscr{U}$ the set of piecewise constant bang-bang controls (i.e., the set of piecewise constant controls with values ± 1), then obviously $\mathscr{R}(x) = \Omega^+(x)$. Let us also note that since $\mathscr{X}(x)$ and $\{f_i |\ i = 0, \ldots, p\}(x)$ span the same linear subspace, so do $[\mathscr{X}](x)$ and $[\{f_i |\ i = 0, \ldots, p\}](x)$. Thus we obtain the following corollary of Theorem 1:

THEOREM 3. *Let f_i, $i = 0, \ldots, p$ be analytic, let f_0 be odd, and let f_i, $i = 1, \ldots, p$, be odd or even. Then (3) is locally controllable at 0 if and only if* rank $[\{f_i |\ i = 0, \ldots, p\}](0)$ *is n.*

We omit the obvious reformulation of Theorem 2 in the language of control systems.

Let us note that the conditions of Theorem 3 are satisfied if f_0 is linear and f_i, $i = 1, \ldots, p$, are constant and so Kalman's controllability theorem is obtained as a special case of Theorem 3.

The perturbation theory of [1] allows us to extend the controllability result of Theorem 3 to "allmost odd" control systems:

THEOREM 4. *Given a system (3) satisfying the assumptions of Theorem 3 such that* dimspan $[\{f_i |\ i = 0, \ldots, p\}] = n$, *there exist $\varepsilon > 0$ and $\eta > 0$ such that for any function $g(x, u)$ which is Lipschitz continuous in x and continuous in u and satisfies $|g(x, u)| < \varepsilon$ for $|x| < \eta$ the system*

$$\dot{x} = f_0(x)+\sum_{i=1}^{p} u_i f_i(x)+g(x, u)$$

is locally controllable at 0.

The proof follows from [1], Proposition III-6. One has merely to note that the homogenity assumption is not essential in this proposition.

4. Proof of Lemma 3

For the sake of brevity we make the following convention: By a stream we shall always understand a stream on some neighbourhood of the origin. In statements concerning t we shall drop "for $|t|$ sufficiently small".

Let us note that if $\mathscr{X}$ is odd, for any stream χ the symmetric map χ^- defined by $\chi_t^-(x) = -\chi_t(-x)$ is also a stream. For the proof it suffices to note that $\chi_t(x) = \varphi_{t_p}^{i_p} \circ \ldots \circ \varphi_{t_1}^{i_1}(x)$ implies

$$\chi_t^-(x) = -\chi_t(-x) = -\varphi_{t_p}^{i_p} \circ \ldots \circ \varphi_{t_1}^{i_1}(-x) = \varphi_{t_p}^{-i_p} \circ \ldots \circ \varphi_{t_1}^{-i_1}(x) \in \Omega^+(x).$$

In the sequel we shall always denote pairs of symmetric streams by the same letter with superscripts $+$, $-$, sometimes dropping $+$. When dealing with them simultaneously we shall use the letter δ to indicate the signs; if multiplied, $+$, $-$ will be understood to behave like $+1$, -1.

In order to prove Lemma 3 we prove the following induction statement:

Let χ_i, $i = 1, \ldots, k$, ψ_k be streams such that

$$\chi_{i,t}(0) = a_i t^{p_i}+o(t^{p_i+1}), \quad \psi_{k,t}(0) = b_k t^{q_k}+o(t^{q_k+1}), \tag{4_k}$$

where a_i, $i = 1, \ldots, k$, are linearly independent and b_k does not belong to any subspace spanned by $k-1$ of the vectors a_i, $i = 1, \ldots, k$.

Then either there exists a stream $\psi_{k+1} \succ \psi_k$ such that $\psi_{k+1,t}(0) = 0$ or there exist streams χ_{k+1}, ψ_{k+1} such that $\psi_{k+1} \succ \psi_k$ and χ_i, $i = 1, \ldots, k+1$, and ψ_{k+1} satisfy (4_{k+1}).

The assertion of the lemma results from this induction statement as follows: If $\chi_t(0) \equiv 0$, we write $\vartheta = \chi$. Otherwise, (4_1) is satisfied for $\chi_1 = \psi_1 = \chi$. Using the induction statement we construct a sequence of streams $\psi_1 \prec \psi_2 \prec \ldots$ (and the auxiliary streams $\chi_1, \chi_2, \ldots$) until we reach k_0 such that $\psi_{k_0,t}(0) = 0$ and we write $\vartheta = \psi_{k_0}$. Since (4_{n+1}) is impossible, $k_0 \leqslant n+1$.

To prove the induction statement we write

$$\xi_{i,t} = \chi_{i,s_i(t)}, \quad \text{where} \quad s_i(t) = t^{p_1 \ldots p_{i-1}p_{i+1} \ldots p_k q_k},$$

$$\eta_{k,t} = \psi_{k,r_k(t)}, \quad \text{where} \quad r_k(t) = t^{p_1 \ldots p_k},$$

ξ_i, $i = 1, \ldots, k$, and η_k are streams, $\xi_i \succ \chi_i$, $\eta_k \succ \psi_k$ and

$$\xi_{i,t}(0) = a_i t^Q+O(t^{Q+1}),$$

$$\eta_{k,t}(0) = b_k t^Q+O(t^{Q+1}),$$

where $Q = p_1 \ldots p_k q_k$. Further we have

$$\begin{aligned} \xi_{i,t}(x) &= x+\sum_{j=1}^{Q} \alpha_{ij}(x) t^j+O(t^{Q+1}), \\ \eta_{k,t}(x) &= x+\sum_{j=1}^{Q} \beta_j(x) t^j+O(t^{Q+1}), \end{aligned} \tag{5}$$

where $\alpha_{ij}(x) = O(|x|)$, $\beta_j(x) = O(|x|)$ for $j = 0, \ldots, Q-1$ and $\alpha_{iQ}(x) = a_i+O(|x|)$, $\beta_Q(x) = b_k+O(|x|)$.

Assume that there exists no stream $\psi_{k+1} \succ \psi_k$ such that $\psi_{k+1,t}(0) = 0$. Write $T_k = \operatorname{span}\{a_i |\ i = 1, \ldots, k\}$ and choose such a complement S_k to T_k that if π_k denotes the projection onto T_k along S_k, then $\pi_k(b_k)$ does not lie in any subspace spanned by $k-1$ of the vectors a_i, $i = 1, \ldots, k$. This is possible owing to the assumption that b_k itself does not belong to any such subspace. We show that there exist functions $\tau_1, \ldots, \tau_k$ and signs $\delta_1, \ldots, \delta_k$ such that $\tau_i(0) = 0$, $\tau_i(0) > 0$ for $t > 0$, $i = 1, \ldots, k$, and

$$\pi_k \circ \xi_{1,\tau_1(t)}^{\delta_1} \circ \ldots \circ \xi_{k,\tau_k(t)}^{\delta_k} \circ \eta_{k,t}(0) = 0. \tag{6}$$

Denote $F^{\delta_1,\dots,\delta_k}\colon R^{n+1} \to T_k$ by

$$F^{\delta_1,\dots,\delta_k}(\tau_1, \dots, \tau_k, t) = \pi_k \circ \xi^{\delta_1}_{1,\tau_1} \circ \dots \circ \xi^{\delta_k}_{k,\tau_k} \circ \eta_{k,t}(0).$$

By (5) we have

$$F^{\delta_1,\dots,\delta_k}(\tau_1, \dots, \tau_k, t) = \sum_{i=1}^{k} \delta_i a_i \tau_i^{Q} + \pi_k(b_k) t^{Q} + \omega(\tau_1, \dots, \tau_k, t),$$

where ω is analytic and satisfies

(7) $$\omega(\tau_1, \dots, \tau_k, t) = o(|\tau_1|^{Q} + \dots + |\tau_k|^{Q} + |t|^{Q}).$$

Write

$$G^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = F^{\delta_1,\dots,\delta_k}(\sigma_1 t, \dots, \sigma_k t, t).$$

Then we have

$$G^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = t^{Q}\Big[\sum_{i=1}^{k} \delta_i a_i \sigma_i^{Q} + \pi_k(b_k)\Big] + \omega(\sigma_1 t, \dots, \sigma_k t, t).$$

By (7) we have $\partial^{j_1+\dots+j_{k+1}}\omega(\sigma_1 t, \dots, \sigma_k t, t)/\partial\sigma_1^{j_1} \dots \partial\sigma_k^{j_k}\partial t^{j_{k+1}}(0) = 0$ as soon as $j_{k+1} \leqslant Q$. Thus, by the Weierstrass preparation theorem, $\omega(\sigma_1 t, \dots, \sigma_k t, t) = t^{Q+1}\tilde{\omega}(\sigma_1, \dots, \sigma_k, t)$, where $\tilde{\omega}$ is analytic in $\sigma_1, \dots, \sigma_k, t$. Therefore,

$$G^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = t^{Q}\Big[\sum_{i=1}^{k} \delta_i a_i \sigma_i^{Q} + \pi_k(b_k) + t\tilde{\omega}(\sigma_1, \dots, \sigma_k, t)\Big]$$

and

$$G^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = 0 \quad \text{if} \quad H^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = 0,$$

where

$$H^{\delta_1,\dots,\delta_k}(\sigma_1, \dots, \sigma_k, t) = \sum_{i=1}^{k} \delta_i a_i \sigma_i^{Q} + \pi_k(b_k) + t\tilde{\omega}(\sigma_1, \dots, \sigma_k, t).$$

Since a_i, $i = 1, \dots, k$, form a basis of T_k and $\pi_k(b_k)$ does not belong to any subspace spanned by $k-1$ of the vectors a_i, there exists a unique k-tuple of reals γ_i, all of them $\neq 0$, such that $-\pi(b_k) = \sum_{i=1}^{k} a_i\gamma_i$. We write $\delta_i = \operatorname{sign}\gamma_i$ and choose $\sigma_i^* = (\delta_i\gamma_i)^{1/Q}$. Since $H^{\delta_1,\dots,\delta_k}(\sigma_1^*, \dots, \sigma_k^*, 0) = \sum_{i=1}^{k} a_i\gamma_i + \pi_k(b_k) = 0$ and $\partial H^{\delta_1,\dots,\delta_k}/\partial\sigma\,(\sigma_1^*, \dots, \sigma_k^*, 0) = Q(\delta_1\sigma_1^{*Q-1}a_1, \dots, \delta_k\sigma_k^{*Q-1}a_k)$ is nonsingular (here $\sigma = (\sigma_1, \dots, \sigma_k)$), there exists a unique k-tuple of analytic functions $\sigma_i(t)$ such that $\sigma_i(0) = \sigma_i^*$ and $H^{\delta_1,\dots,\delta_k}(\sigma_1(t), \dots, \sigma_k(t), t) = 0$. Moreover, $\sigma_i(t) \geqslant 0$. If we write $\tau_i(t) = t\sigma_i(t)$, $i = 1, \dots, k$, then τ_i will be analytic and will satisfy $\tau_i(0) = 0$, $\tau_i(t) > 0$ for $t > 0$ and $F^{\delta_1,\dots,\delta_k}(\tau_1(t), \dots, \tau_k(t), t) = 0$.

Write

$$\chi_{k+1,t}(x) = \xi^{\delta_1}_{1,\tau_1(t)} \circ \dots \circ \xi^{\delta_k}_{k,\tau_k(t)} \circ \eta_{k,t}.$$

Obviously $\chi_{k+1} \succ \psi_k$; thus $\chi_{k+1,t}(0) \neq 0$ by assumption. By (6) and the definition of χ_{k+1}, $\pi_k \circ \chi_{k+1,t}(0) = 0$, which implies that the first non-zero coefficient in the expansion of $\chi_{k+1,t}(0)$ must be linearly independent of the vectors a_i, $i = 1, \dots, k$. Therefore, $\chi_1, \dots, \chi_{k+1}$ satisfy (4_{k+1}).

To obtain ψ_{k+1} we choose another complement S_k' to T_k, the intersection of which with the span $\{a_i |\ i = 1, \dots, k+1\}$ does not lie in any subspace spanned by k of the vectors a_i, $i = 1, \dots, k+1$. We construct ψ_{k+1} by the same construction as χ_{k+1} with S_k replaced by S_k' and π_k' replaced by π_k', the projection onto T_k along S_k'. Since $\psi_{k+1,t}(0) \neq 0$, owing to the choice of S_k', χ_i, $i = 1, \dots, k+1$, and ψ_{k+1} will satisfy (4_{k+1}).

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