hypotheses of Remarks 2 and 3 are fulfilled, then propositions (17), (18) of Theorem 4 remain valid with the distinction that x^* denotes a certain linear continuous functional over $\tilde{X}$ and M in (18) has to be replaced by any convex body $\tilde{M} \subset \tilde{X}$ with $M = \tilde{M} \cap X$.

As to an application of a previous version of this model refer to [1].

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BANACH CENTER PUBLICATIONS
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TIME OPTIMAL CONTROL PROBLEM FOR DIFFERENTIAL INCLUSIONS

V. I. BLAGODATSKIH

Mathematical Institute of USSR Academy of Sciences, Moscow, USSR

1. Introduction

Let E^n be a Euclidean space of state-vectors $x = (x_1, \ldots, x_n)$ with the norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and let $\Omega(E^n)$ be the metric space of all nonempty compact subsets of E^n with the Hausdorff metric

$$h(F, G) = \min\{d: F \subset S_d(G), G \subset S_d(F)\},$$

where $S_d(M)$ denotes a d-neighbourhood of a set M in the space E^n.

Let us consider an object with a behaviour described by the differential inclusion

$$\dot{x} \in F(t, x), \tag{1}$$

where $F: E^1 \times E^n \to \Omega(E^n)$ is a given mapping. The absolutely continuous function $x(t)$ is the solution of the inclusion (1) on the interval $[t_0, t_1]$ iff the condition $\dot{x}(t) \in F(t, x(t))$ is valid almost everywhere on this interval.

On the one hand, the differential inclusion is the extension of ordinary differential equations

$$\dot{x} = f(t, x), \tag{2}$$

when the function $f(t, x)$ is singlevalued. On the other hand, this extension is not formal: for many different problems can be transformed into differential inclusions and the development of differential inclusions permits the solution of those problems. For example, A. F. Filippov [1] investigated with the help of differential inclusions the solutions of differential equation (2) on the sets where the function $f(t, x)$ had discontinuities. N. N. Krasovski [6] used a differential inclusion for constructing a strategy in differential games. Let us consider the connection of differential inclusions with some other problems.

The optimal control problem was first considered by L. S. Pontryagin and others [8] for systems described by the equation

$$\dot{x} = f(t, x, u). \tag{3}$$

This problem may be transformed into the determination of the optimal solution $x(t)$ of the differential inclusion

$$\dot{x} \in f(t, x, U) = \{f(t, x, u)\colon u \in U\}.$$

Knowing the optimal solution $x(t)$ it is possible, with help of Filippov's implicit functions lemma [5], to construct for system (3) a control $u(t)$ which produces this optimal solution. Note that control system (3) may be transformed into a differential inclusion form even in the case where the set U depends on time and state, i.e., is of the form $U(t, x)$. On the other hand, inclusion (1) may be considered as a control system with changing control domain

$$\dot{x} = v, \quad v \in F(t, x).$$

It should be noted that optimal control problems have stimulated very much the development of the theory of differential inclusions.

The implicit differential equation

$$f(t, x, \dot{x}) = 0$$

can also be transformed into a differential inclusion form:

$$\dot{x} \in F(t, x) = \{v\colon f(t, x, v) = 0\}.$$

On the other hand, inclusion (1) may be considered as an implicit differential equation

$$\varrho\big(\dot{x}, F(t, x)\big) = 0;$$

here $\varrho(p, A)$ denotes the distance from a point p to a set A,

$$\varrho(p, A) = \min_{a \in A} \|p - a\|.$$

A system of differential inequalities

$$f_i(t, x, \dot{x}) \leqslant 0, \quad i = 1, \ldots, k,$$

may be transformed into the differential inclusion

$$\dot{x} \in F(t, x) = \{v\colon f^i(t, x, v) \leqslant 0,\ i = 1, \ldots, k\}.$$

On the other hand, inclusion (1) may be considered in the case of $F(t, x)$ being convex as an infinite system of differential inequalities

$$(\dot{x}, \psi) \leqslant c\big(F(t, x), \psi\big), \quad \psi \in S_1(0);$$

here $c(F, \psi)$ is the support function of the set F:

$$c(F, \psi) = \max_{f \in F} (f, \psi).$$

2. Time optimal control problem

Let M_0, M_1 be nonempty closed subsets of E^n. The solution $x(t)$ given on the interval $[t_0, t_1]$ transfers M_0 to M_1 in time $t_1 - t_0$ if the conditions $x(t_0) \in M_0$, $x(t_1) \in M_1$ are satisfied. The time optimal control problem is to determine the solution of the inclusion (1) transferring the set M_0 to the set M_1 in minimal time.

MAXIMUM PRINCIPLE. Assume that the support function $c(F(t, x), \psi)$ of the inclusion (1) is continuously differentiable in x and the solution $x(t)$, $t_0 \leqslant t \leqslant t_1$, transfers the set M_0 to the set M_1. We shall say that the solution $x(t)$ *satisfies the maximum principle on interval* $[t_0, t_1]$ if there exists such a nontrivial solution $\psi(t)$ of the adjoint system

$$\dot{\psi} = -\frac{\partial c\big(F(t, x(t)), \psi\big)}{\partial x} \tag{4}$$

that the following conditions are satisfied:

(A) *the maximum condition*

$$\big(\dot{x}(t), \psi(t)\big) = c\big(F(t, x(t)), \psi(t)\big)$$

is satisfied almost everywhere on the interval $[t_0, t_1]$;

(B) *the transversality condition on the set* M_0: vector $\psi(t_0)$ is the support vector for the set M_0 at the point $x(t_0)$, that is,

$$c\big(M_0, \psi(t_0)\big) = \big(x(t_0), \psi(t_0)\big);$$

(C) *the transversality condition on the set* M_1: vector $-\psi(t_1)$ is the support vector for the set M_1 at the point $x(t_1)$, that is,

$$c\big(M_1, -\psi(t_1)\big) = \big(x(t_1), -\psi(t_1)\big).$$

3. Necessary condition of optimality

A multivalued function $F\colon E^n \to \Omega(E^n)$ is called *measurable* if for any closed set $P \subset E^n$ the set $\{x\colon F(x) \cap P \neq \emptyset\}$ is Lebesgue measurable. The continuity and lipschitzianity of the multivalued function $F(x)$ is defined in the usual way. For example, the function $F(x)$ satisfies the Lipschitz condition with constant L if for any points $x, x' \in E^n$ the inequality

$$h\big(F(x), F(x')\big) \leqslant L \cdot \|x - x'\|$$

is valid. The number $|F| = h(\{0\}, F)$ is called the *modulus* of the set F.

THEOREM 1. *Let the multivalued function $F(t, x)$ of inclusion* (1) *be measurable in t and satisfy the Lipschitz condition in x with a summable constant $L(t)$ and $|F(t, x)| \leqslant g(t)$ where $g(t)$ is a summable function. Assume that the support function $c\big(F(t, x), \psi\big)$ is continuously differentiable in x, the sets M_0, M_1 are convex and the solution $x(t)$, $t_0 \leqslant t \leqslant t_1$, transferring the set M_0 to the set M_1 is optimal. Then this solution satisfies the maximum principle on the interval $[t_0, t_1]$. Moreover, the condition*

$$c\big(F(t_1, x(t_1)), \psi(t_1)\big) \geqslant 0$$

is valid.

The proof of Theorem 1 follows the plan suggested in [8] for systems of type (3). The main difficulty is to define the variation of the solution. Here instead of the classical theorem on differentiability of solution with respect to the initial condition (see, for example, [7]) it is necessary to use Theorem 2 stated below.

Let the function $f\colon E^n \to E^1$ satisfy the Lipschitz condition. The set of all partial limits of the gradient of this function at the point $x+h$ when $h \to 0$, that is,

$$\partial f(x) = \overline{\lim_{h\to 0}}\, \nabla f(x+h),$$

is called the *subdifferential* of the function $f(x)$ at the point x.

THEOREM 2. *Let $x(t)$, $t_0 \leqslant t \leqslant t_1$, be a solution of inclusion* (1) *with the initial condition x_0, let $\psi(t)$ be a solution of the adjoint system* (4) *corresponding to $x(t)$ and let $\delta x(t)$ be a solution of the differential inclusion*

$$\delta\dot{x} \in \partial_\psi\left[\frac{\partial c\big(F(t, x(t)), \psi(t)\big)}{\partial x}\,\delta x\right]$$

with the initial condition $\delta x(t_0) = h$, $\varepsilon > 0$. Then there exists such a $y_\varepsilon(t)$-solution of inclusion (1) *with the initial condition $y_\varepsilon(t_0) = x_0+\varepsilon h$ defined on the interval $[t_0, t_1]$ that the following condition is valid:*

$$y_\varepsilon(t) = x(t)+\varepsilon\,\delta x(t)+o(\varepsilon).$$

The idea of the proof is contained in [1].

Remark. The sets M_0, M_1 in Theorem 1 may be nonconvex. It is sufficient that there exist approximating cones to the sets M_0, M_1 at the points $x(t_0)$, $x(t_1)$, respectively. In this case conditions (B), (C) in the maximum principle have to be replaced by the conditions that the vectors $\psi(t_0)$, $-\psi(t_1)$, are supports to the respective approximating cones at the points $x(t_0)$, $x(t_1)$.

4. Convexity of the set of solutions

Naturally the question arises when the maximum principle is not only a necessary but also a sufficient condition of optimality. It seems very important to know when the set of solutions $\Sigma_{[t_0,t_1]}(P)$ of inclusion (1) with the initial condition $x(t_0) \in P$ is convex in the space $C_{[t_0,t_1]}$ of all continuous functions on the interval $[t_0, t_1]$. Denote by $Z(\tau)$ the intersection of the set $\Sigma_{[t_0,t_1]}(P)$ by the plane $t = \tau$. The set $Z(\tau)$ is the set of all points $x(\tau)$ where $x(t)$, $t_0 \leqslant t \leqslant t_1$, is any solution of inclusion (1) with the initial set P, at a moment τ.

The multivalued function $F(t, x)$ is *concave* in x on the set M if the condition

$$\alpha F(t, x)+\beta F(t, x') \subset F(t, \alpha x+\beta x')$$

is valid for any points $x, x' \in M$ and for any numbers $\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$.

THEOREM 3. *Assume that the initial set P is convex. Then the set of solutions $\Sigma_{[t_0,t_1]}(P)$ is convex iff the multivalued function $F(t, x)$ is concave in x on the sets $Z(\tau)$ for any $\tau \in [t_0, t_1]$.*

The idea of the proof is contained in [2].

5. Sufficient conditions of optimality

In order that the maximum principle should also be a sufficient condition of optimality of the solution $x(t)$ it is necessary to impose two additional conditions on the solution $x(t)$.

We shall say that the support function $c\big(F(t, x), \psi\big)$ is *concave* in x at the point x_0 in the direction ψ_0 if the condition

$$\left(\frac{\partial c\big(F(t, x_0), \psi_0\big)}{\partial x},\, x-x_0\right) \geqslant c\big(F(t, x), \psi_0\big)-c\big(F(t, x_0), \psi_0\big)$$

is valid for any point $x \in E^n$.

Note that the concavity of the multivalued function $F(t, x)$ yields the concavity of the support function $c\big(F(t, x), \psi\big)$ at any point and in any direction.

Let $x(t)$, $t_0 \leqslant t \leqslant t_1$, be a solution of inclusion (1) and let $\psi(t)$ be the corresponding solution of the adjoint system (4). We shall say that the solution $x(t)$ satisfies the *strong transversality condition on the set M_1 with the adjoint function $\psi(t)$* if the condition

$$c\big(M_1, -\psi(t)\big) < \big(x(t), -\psi(t)\big)$$

is valid for any moment t: $t_0 \leqslant t < t_1$.

THEOREM 4. *Assume that M_0, M_1 are nonempty closed subsets of E^n, the solution $x(t)$ of inclusion* (1) *transfers the set M_0 to the set M_1 on the interval $[t_0, t_1]$ and satisfies the maximum principle on this interval, and $\psi(t)$ is the corresponding solution of adjoint system* (4). *Assume that the support function $c\big(F(t, x), \psi\big)$ is concave in x at point $x(t)$ in the direction $\psi(t)$ for any $t \in [t_0, t_1]$ and the solution $x(t)$ satisfies the strong transversality condition on the set M_1 with adjoint function $\psi(t)$.*

Then the solution $x(t)$ is optimal.

The proof is contained in [3].

6. Uniqueness of optimal solution

In the case of continuous differentiability of the support function $c\big(F(t, x), \psi\big)$ in ψ the maximum condition (A) and the adjoint system (4) may be written as the system of differential equations

$$\dot{x}(t) = \frac{\partial c\big(F(t, x(t)), \psi(t)\big)}{\partial \psi},$$

$$\dot{\psi}(t) = -\frac{\partial c\big(F(t, x(t)), \psi(t)\big)}{\partial x}.$$

Initial conditions for solution $\big(x(t), \psi(t)\big)$ of this system may be determined from conditions (B), (C) of maximum principle and from the inclusions $x(t_0) \in M_0$, $x(t_1) \in M_1$. The question naturally arises in this case when, for given initial conditions, the unique solution $\big(x(t), \psi(t)\big)$ is determined from the maximum principle. The following theorem is an answer to this question.

THEOREM 5. *Assume that the support function* $c(F(t,x),\psi)$ *is measurable in* t *and continuously differentiable in* (x,ψ) *except for the points* $\psi = 0$, *and the derivatives* $c_x(F(t,x),\psi)$, $c_\psi(F(t,x),\psi)$ *satisfy the Lipschitz condition in* (x,ψ). *Then, for any given initial conditions, for functions* $(x(t),\psi(t))$ *the optimal solution is unique.*

7. Concluding remarks

The Pontryagin maximum principle was proved [8] for control system (3) for the function $f(t,x,u)$ continuously differentiable in x. These systems may be transformed into the differential inclusion form and the support function is in this case

$$c(F(t,x),\psi) = \max_{u\in U} (f(t,x,u),\psi).$$

Theorem 1 is applicable when this support function is continuously differentiable in x. Its not very difficult to show that there exists a function $f(t,x,u)$ continuously differentiable in x for which the corresponding support function $c(F(t,x),\psi)$ does not have this property and *vice versa*. That is, the Pontryagin maximum principle and Theorem 1 intersect over some class of control systems of type (3). It is possible to formulate as a hypothesis the following theorem, which includes Theorem 1 and the Pontryagin maximum principle.

THEOREM 6. *Suppose that the multivalued function* $F(t,x)$ *for inclusion* (1) *is measurable in* t *and satisfies the Lipschitz condition in* x, *and the sets* M_0, M_1 *are convex and the solution* $x(t)$, $t_0 \leqslant t \leqslant t_1$ *is optimal. Then there exists such a nontrivial absolutly continuous function* $\psi(t)$ *that the following conditions are valid*:

(A) $$\frac{d}{dt}\begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} \in \partial c_{(x,\psi)}\big(F(t,x(t)),\psi(t)\big)$$

almost everywhere on the interval $[t_0, t_1]$;

(B) $c(M_0, \psi(t_0)) = (x(t_0), \psi(t_0))$;

(C) $c(M_1, -\psi(t_1) = (x((t_1), -\psi(t_1))$.

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LOCAL CONTROLLABILITY OF ODD SYSTEMS

PAVOL BRUNOVSKÝ

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Czechoslovakia

1. Introduction

Consider a control system

(1) $$\dot{x} = f(x,u),$$

where $x \in R^n$ and $u \in U$ (we do not specify the set U at this point) and a set of admissible controls $\mathscr{U}$ which is a subset of the set of mappings u of intervals $[0, T(u)]$, $T(u) \geqslant 0$ into U, having the property that for any $u \in \mathscr{U}$ the solution $\varphi(t, x_0, u)$ of the differential equation

$$\dot{x} = f(x, u(t))$$

with $\varphi(0, x_0, u) = x_0$ is uniquely defined on $[0, T(u)]$. We denote by $\mathscr{R}(x_0)$ the reachable set of (1) from x_0, i.e., $\mathscr{R}(x_0) = \{\varphi(T(u), x_0, u) |\ u \in \mathscr{U}\}$ and call (1) *locally controllable at* x_0 if $x_0 \in \operatorname{int}\mathscr{R}(x_0)$.

The well-known theorem of Kalman gives necessary and sufficient conditions of local controllability at 0 for the class of linear systems $(f(x,u) = Ax+Bu)$ with U being a subset of R^m containing 0 in its interior (the bang-bang controllability theorem makes the choice of $\mathscr{U}$ irrevelant in this case). For nonlinear systems, sufficient conditions for local controllability at a rest point of the uncontrolled system are given by the theorem of Lee and Markus ([4]). However, since the theorem of Lee and Markus uses only the linearization of the system (in both x and u), it is not difficult to see that its sufficient condition is far from being necessary.

A more recent approach, going back to Hermann ([2], cf. also [5], [6]) relates the problem of controllability to the study of orbits of families of vector fields. Given a family of vector fields $\mathscr{X} = \{X^i |\ i \in I\}$ on R^n, the orbit of x_0 is defined as $\Omega(x_0) = \{\varphi_{t_p}^{i_p} \circ \ldots \circ \varphi_{t_1}^{i_1}(x_0) |\ p \geqslant 0,\ i_j \in I,\ t_j \in R,\ j = 1, \ldots, p\}$, where by φ^i we denote the flow of X^i. It is immediately seen that if $\mathscr{U}$ is taken as the set of piecewise constant controls and we associate with (1) the family of vectors fields $\mathscr{X} = \{X^u |\ u \in U\}$ defined by $X^u(x) = f(x,u)$, then $\mathscr{R}(x_0) = \Omega^+(x_0)$, were $\Omega^+(x_0)$ is the positive semiorbit of x_0, defined by $\Omega^+(x_0) = \{\varphi_{t_p}^{u_p} \circ \ldots \circ \varphi_{t_1}^{u_1}(x_0) |\ p \geqslant 0,\ u_i \in U,\ t_i \geqslant 0,\ i = 1, \ldots, p\}$.