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ON THE MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROCESSES DESCRIBED BY HAMMERSTEIN INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNELS

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1

Let X be a real Banach space with norm $||\cdot||$, X_0 an open subset of X, and W a certain set. Let $T\colon X_0 \times W \to X$ be a transformation and $f\colon X_0 \times W \to R$ a real-valued functional. We consider the following *optimal control problem*:

(1) $$\operatorname{Min} f(x, w), \qquad (x, w) \in X_0 \times W$$

subject to the process equation

(2) $$x = T(x, w).$$

For this problem L. Bittner [1] has developed a simple exact method of deriving maximum principles. He has applied his method especially to optimal control processes governed by Urysohn integral equations with variable integration domains and continuous kernels. I want to give here at first a modification of Bittner's basic theorem for this model of an optimal control process. Then I will apply this modification to a derivation of a maximum principle in the case where the process equation is a Hammerstein integral equation with a completely continuous kernel in a space of continuous functions.

2

Let Y be another real Banach space with norm $||\cdot||_y$, which has the following properties:

(a) *X is a dense subset of Y,*

(b) *the embedding operator $X \to Y$ is continuous*, i.e., *there exists a constant $C > 0$ such that*

$$||x||_y \leqslant C||x|| \qquad \forall x \in X.$$

THEOREM. *Let the following assumptions hold*:

(i) *$T(x, w)$ has a partial Fréchet derivative $T_x(x, w)$ at every point (x, w) of the set $B(x_0, r_0) \times W_0$, $r_0 > 0$, where $B(x_0, r_0)$ denotes the ball with centre x_0 and* *radius r_0 in X, and W_0 is a fixed subset of W. $T_x(x, w)$ is a linear bounded operator in X, which can be extended to such an operator in Y.*

For every fixed $w \in W_0$ the relation

$$(3)\qquad \|T(x_1, w)-T(x, w)-T_x(x, w)(x_1-x)\|_y = o(\|x_1-x\|_y)\quad \forall x, x_1 \in B(x_0, r_0),\ \|x_1-x\| \to 0$$

is valid. Besides

$$(4)\qquad \begin{aligned} &\sup_{w\in W_0} \|T_x(x_0, w)-T_x(x, w)\| \to 0 \quad \text{if} \quad \|x-x_0\| \to 0,\\ &\sup_{w\in W_0} \|T_x(x_0, w)-T_x(x, w)\|_y \to 0 \quad \text{if} \quad \|x-x_0\| \to 0. \end{aligned}$$

(ii) *$[I-T_x(x_0, w_0)]^{-1}$ exists as a linear bounded operator in X and can be extended to such an operator in Y.*

(iii) *$f(x, w)$ has a partial Fréchet derivative $f_x(x, w)$ at every point $(x, w) \in B(x_0, r_0)\times W_0$. $f_x(x, w)$ is a linear bounded functional on X and can be extended to such a functional on Y. We have*

$$(5)\qquad \sup_{w\in W_0} \|f_x(x_0, w)-f_x(x, w)\|_y \to 0 \quad \text{if} \quad \|x-x_0\| \to 0.$$

Then for an optimal pair $(w_0, x_0) \in W_0\times X_0$ of the problem (1), (2) *the following variational inequality is valid*:

$$(6)\qquad \delta f_x+\delta f \geqslant 0$$

for every pair of limits

$$(7)\qquad \delta f_x = \lim_k \frac{1}{\gamma_k} f_x(x_0, w_0)\Delta x_k,$$

where

$$\Delta x_k = [I-T_x(x_0, w_0)]^{-1}[T(x_0, w_k)-T(x_0, w_0)],$$

and

$$(8)\qquad \delta f = \lim_k \frac{1}{\gamma_k}[f(x_0, w_k)-f(x_0, w_0)].$$

Here γ_k is a sequence of positive numbers which tends to 0, *and w_k is a sequence of elements of W_0 with the following properties*:

$$(9)\qquad \|T(x_0, w_k)-T(x_0, w_0)\| \to 0,$$

$$(10)\qquad \begin{aligned} &\|T_x(x_0, w_k)-T_x(x_0, w_0)\| \to 0,\\ &\|T_x(x_0, w_k)-T_x(x_0, w_0)\|_y \to 0, \end{aligned}$$

$$(11)\qquad \frac{1}{\gamma_k}\|f_x(x_0, w_k)-f_x(x_0, w_0)\| \quad \text{is bounded},$$

$$(12)\qquad \frac{1}{\gamma_k}\|T(x_0, w_k)-T(x_0, w_0)\|_y \quad \text{is bounded}.$$

Remark. For $Y = X$ we get the basic theorem of Bittner [1]. In this case condition (11) can be weakened. But in some applications conditions (11), (12) for a suitable chosen space Y are easier to fulfil than assumption (12) for X. This is because in the applications which we have in mind, roughly speaking, the functional f behaves better than the operator T.

3

If the weak limit in Y

$$(13)\qquad \frac{1}{\gamma_k}[T(x_0, w_k)-T(x_0, w_0)] \rightharpoonup \delta T \in Y$$

exists, it follows that

$$(14)\qquad \delta f_x = f_x(x_0, w_0)\,\delta x,$$

where $\delta x \in Y$ is the unique solution of the *variational equation*

$$(15)\qquad \delta x = T_x(x_0, w_0)\,\delta x+\delta T \text{ in } Y.$$

Because of (ii) the inverse operator $[I-T_x^*(x_0, w_0)]^{-1}$ in Y^* exists, where * means the adjoint operator and space, respectively. Therefore

$$(16)\qquad \delta f_x = \mathcal{S}(\delta T),$$

where $\mathcal{S}$ is the unique solution of the *adjoint variational equation*

$$(17)\qquad \mathcal{S} = T_x^*(x_0, w_0)\mathcal{S}+f_x(x_0, w_0) \text{ in } Y^*.$$

Assumption (ii) is valid if the operator $T_x(x_0, w_0)$ is completely continuous and the equation $[I-T_x(x_0, w_0)]y = \theta$ in Y has the trivial solution only.

4

Let D be a compact subset of a Euclidean space. Further, let U be a given subset of the real axis R and define W as the set of all $w = u(\cdot)$ which are bounded measurable or piecewise continuous functions on D, where

$$(18)\qquad u(t) \in U \quad \forall t \in D.$$

(For simplicity we only deal with the scalar case.) We consider the following *integral equation of Hammerstein type*:

$$(19)\qquad x(s) = \int_D k(s, t)G\big(t, x(t), u(t)\big)\,dt, \quad s \in D,$$

where $G(t, x, u)$ is a given function with the continuous partial derivative $G_x(t, x, u)$ and the kernel $k(s, t)$ satisfies the conditions

$$(20)\qquad \sup_{s\in D}\int_D |k(s, t)|^2 dt < \infty, \quad \sup_{t\in D}\int_D |k(s, t)|^2 ds < \infty,$$

and

(21)
$$\lim_{s\to\sigma}\int_D |k(s,t)-k(\sigma,t)|^2 dt = 0 \quad \forall\sigma\in D,$$
$$\lim_{t\to\sigma}\int_D |k(s,t)-k(s,\sigma)|^2 ds = 0 \quad \forall\sigma\in D.$$

Further, the following *functional* is to be minimized:

(22)
$$\varphi\Big(\int_D F\big(t, x(t), u(t)\big)\,dt\Big) = \text{Min}\,!,$$

where $\varphi(z)$ is a given function with the Lipschitzian derivative $\varphi'(z)$ and $F(t, x, u)$ is a given function with the continuous partial derivative $F_x(t, x, u)$.

We treat problem (19), (22) in the space $X = C(D)$ of continuous functions on D and put $Y = L_2(D)$, the space of quadratic summable functions on D. Let $\big(u_0(s), x_0(s)\big)$ be a pair of optimal functions for this problem. We make the *assumption* that the homogeneous linear integral equation

(23)
$$h(s) = \int_D k(s,t)\,G_x\big(t, x_0(t), u_0(t)\big)\,h(t)\,dt$$

possesses in $C(D)$ only the trivial solution. Then we can define the H-function

(24)
$$H(s,u) = -\varphi'\Big(\int_D F\big(t, x_0(t), u_0(t)\big)\,dt\Big)\cdot F\big(s, x_0(s), u\big) - $$
$$-G\big(s, x_0(s), u\big)\cdot\int_D k(t,s)\mathscr{S}(t)\,dt \qquad (s\in D,\ u\in U),$$

where $\mathscr{S}(t)\in L_\infty(D)$ is the solution of the linear integral equation

(25)
$$\mathscr{S}(s) = G_x\big(s, x_0(s), u_0(s)\big)\cdot\int_D k(t,s)\mathscr{S}(t)\,dt + $$
$$+\varphi'\Big(\int_D F\big(t, x_0(t), u_0(t)\big)\,dt\Big)\cdot F_x\big(s, x_0(s), u_0(s)\big).$$

For this function the *maximum condition*

(26)
$$\operatorname*{Max}_{v\in U} H(\sigma, v) = H\big(\sigma, u_0(\sigma)\big)$$

is valid for every point $\sigma\in D$ which is a point of continuity of $u_0(t)$.

5

For details and some extensions the reader is referred to the following paper of the author [„Mathem. Operationsforschung und Statistik" 6 (1975), pp. 609 ff].

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