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TANGENT SETS AND DIFFERENTIABLE FUNCTIONS

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The purpose of the present paper is to define some tangent sets (see Definitions 1, 2, 3, 4) and to discuss the transformation of these tangent sets (see Theorems 1, 2) by means of differentiable functions (see Definitions 5, 6).

Tangent sets

Let X be a real topological vector space. We shall denote by $\mathcal{U}$ the family of all neighbourhoods of 0 in X.

Let $X_0 \subseteq X$, and let $x_0 \in X$.

Definition 1. $L_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: there are $r > 0$, $U \in \mathcal{U}$ such that if $s \in (0, r)$, $u \in U$, then $x_0 + s(x+u) \in X_0$.

Definition 2. $l_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: there is a $U \in \mathcal{U}$ such that for every $r > 0$ there is an $s \in (0, r)$ such that if $u \in U$, then $x_0 + s(x+u) \in X_0$.

Definition 3. $k_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: for every $U \in \mathcal{U}$ there is an $r > 0$ such that for every $s \in (0, r)$ there is a $u \in U$ such that $x_0 + s(x+u) \in X_0$.

Definition 4. $K_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: for every $r > 0$, $U \in \mathcal{U}$ there are $s \in (0, r)$, $u \in U$, such that $x_0 + s(x+u) \in X_0$.

There exist many definitions, and notations for each of the above tangent sets. Let us recall that Definition 4 is equivalent to the definition in [1]. For other references see [2].

In the sequel we shall denote by $\operatorname{comp} X_0$ the complement of X_0. The subsequent three propositions are direct consequences of the above definitions.

Proposition 1. *The following equalities hold*:

$$\operatorname{comp} L_{X_0}(x_0) = K_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} l_{X_0}(x_0) = k_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} k_{X_0}(x_0) = l_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} K_{X_0}(x_0) = L_{\operatorname{comp} X_0}(x_0).$$

PROPOSITION 2. *The following inclusions hold:*

$$L_{X_0}(x_0) \subseteq l_{X_0}(x_0) \subseteq K_{X_0}(x_0);$$
$$L_{X_0}(x_0) \subseteq k_{X_0}(x_0) \subseteq K_{X_0}(x_0).$$

PROPOSITION 3. *Let* $X_1 \subseteq X_2 \subseteq X$. *Then:*

$$L_{X_1}(x_0) \subseteq L_{X_2}(x_0);$$
$$l_{X_1}(x_0) \subseteq l_{X_2}(x_0);$$
$$k_{X_1}(x_0) \subseteq k_{X_2}(x_0);$$
$$K_{X_1}(x_0) \subseteq K_{X_2}(x_0).$$

PROPOSITION 4. *Let* $X_1 \subseteq X$, $X_2 \subseteq X$. *Then:*

$$L_{X_1}(x_0) \cap L_{X_2}(x_0) \subseteq L_{X_1 \cap X_2}(x_0);$$
$$L_{X_1}(x_0) \cap l_{X_2}(x_0) \subseteq l_{X_1 \cap X_2}(x_0);$$
$$L_{X_1}(x_0) \cap k_{X_2}(x_0) \subseteq k_{X_1 \cap X_2}(x_0);$$
$$L_{X_1}(x_0) \cap K_{X_2}(x_0) \subseteq K_{X_1 \cap X_2}(x_0).$$

Proof. We shall assume that $x \in L_{X_1}(x_0) \cap M_{X_2}(x_0)$, and we shall show that $x \in M_{X_1 \cap X_2}(x_0)$ whenever $M \in \{L, l, k, K\}$. In any case, there are $r_1 > 0$, $U_1 \in \mathcal{U}$ such that $x_0+s(x+U_1) \subseteq X_1$ for all $s \in (0, r_1)$ (Definition 1).

First, let $x \in L_{X_1}(x_0) \cap L_{X_2}(x_0)$. There are $r_2 > 0$, $U_2 \in \mathcal{U}$ such that $x_0+s(x+U_2) \subseteq X_2$ for all $s \in (0, r_2)$ (Definition 1). Write $r = \min(r_1, r_2)$, $U = U_1 \cap U_2$. Then $x_0+s(x+U) \subseteq X_1 \cap X_2$ for all $s \in (0, r)$, i.e., $x \in L_{X_1 \cap X_2}(x_0)$ (Definition 1).

Next, let $x \in L_{X_1}(x_0) \cap l_{X_2}(x_0)$. There is a $U_2 \in \mathcal{U}$ such that for every $r_2 > 0$ there is an $s \in (0, r_2)$ such that $x_0+s(x+U_2) \subseteq X_2$ (Definition 2). Write $U = U_1 \cap U_2$, and let $r > 0$. Consider $r_2 = \min(r, r_1)$, and choose $s \in (0, r_2)$ as above. Then $s \in (0, r)$ and $x_0+s(x+U) \subseteq X_1 \cap X_2$, i.e., $x \in l_{X_1 \cap X_2}(x_0)$ (Definition 2).

Further, let $x \in L_{X_1}(x_0) \cap k_{X_2}(x_0)$, and let $U \in \mathcal{U}$. Corresponding to $U_2 = U \cap U_1$, there is an $r_2 > 0$ such that $(x_0+s(x+U_2)) \cap X_2 \neq \emptyset$ for all $s \in (0, r_2)$ (Definition 3). Write $r = \min(r_1, r_2)$. Then $(x_0+s(x+U)) \cap X_1 \cap X_2 \neq \emptyset$ for all $s \in (0, r)$, i.e., $x \in k_{X_1 \cap X_2}(x_0)$ (Definition 3).

Finally, let $x \in L_{X_1}(x_0) \cap K_{X_2}(x_0)$, and let $r > 0$, $U \in \mathcal{U}$. Corresponding to $r_2 = \min(r, r_1)$, $U_2 = U \cap U_1$ there is an $s \in (0, r_2)$ such that $(x_0+s(x+U_2)) \cap X_2 \neq \emptyset$ (Definition 4). Then $s \in (0, r)$ and $(x_0+s(x+U)) \cap X_1 \cap X_2 \neq \emptyset$, i.e., $x \in K_{X_1 \cap X_2}(x_0)$ (Definition 4).

PROPOSITION 5. *Let* $X_1 \subseteq X$, $X_2 \subseteq X$. *Then:*

$$l_{X_1}(x_0) \cap k_{X_2}(x_0) \subseteq K_{X_1 \cap X_2}(x_0).$$

Proof. Let $x \in l_{X_1}(x_0) \cap k_{X_2}(x_0)$, and let $r > 0$, $U \in \mathcal{U}$. There is a $U_1 \in \mathcal{U}$ such that for every $r_1 > 0$ there is an $s \in (0, r_1)$ such that $x_0+s(x+U_1) \subseteq X_1$ (Definition 2). Write $U_2 = U \cap U_1$. There is an $r_2 > 0$ such that $(x_0+s(x+U_2)) \cap X_2 \neq \emptyset$ for all $s \in (0, r_2)$ (Definition 3). Consider $r_1 = \min(r, r_2)$, and choose $s \in (0, r_1)$ as above. Then $s \in (0, r)$ and $(x_0+s(x+U)) \cap X_1 \cap X_2 \neq \emptyset$, i.e., $x \in K_{X_1 \cap X_2}(x_0)$ (Definition 4).

Differentiable functions

Let Y be a real topological vector space. We shall denote by $\mathcal{V}$ the family of all neighbourhoods of 0 in Y.

Let $x_0 \in X_0$, and let $f: X_0 \to Y$.

DEFINITION 5. The function f is said to be *differentiable at the point* x_0 if for every $x \in K_{X_0}(x_0)$ there is at least one $y \in Y$ satisfying the condition: for every $V \in \mathcal{V}$ there are $r > 0$, $U \in \mathcal{U}$ such that if $s \in (0, r)$, $u \in U$, and $x_0+s(x+U) \in X_0$, then $f(x_0+s(x+u)) \in f(x_0)+s(y+V)$.

LEMMA. *Let* Y *be separated, and let* f *be differentiable at* x_0. *Then for every* $x \in K_{X_0}(x_0)$ *there is at most one* $y \in Y$ *satisfying the condition in the statement of Definition* 5.

Proof. Let $x \in K_{X_0}(x_0)$. Assume, by contradiction, that there are at least two $y_1 \in Y$, $y_2 \in Y$ satisfying the condition in the statement of Definition 5. There is a $V \in \mathcal{V}$ such that $(y_1+V) \cap (y_2+V) = \emptyset$. Also, there are $r_i > 0$, $U_i \in \mathcal{U}$ such that $f(x_0+s(x+u)) \in f(x_0)+s(y_i+V)$ whenever $s \in (0, r_i)$, $u \in U_i$ and $x_0+s(x+u) \in X_0$ ($i \in \{1, 2\}$). Write $r = \min(r_1, r_2)$, $U = U_1 \cap U_2$. There are $s \in (0, r)$, $u \in U$ such that $x_0+s(x+u) \in X_0$ (Definition 4). It follows that $(1/s)\big(f(x_0+s(x+u)) - f(x_0)\big) \in (y_1+V) \cap (y_2+V)$, and we get a contradiction.

Henceforth we shall suppose Y separated.

DEFINITION 6. Let f be differentiable at x_0. The *differential* of f at x_0 is the function $D_f(x_0)\colon K_{X_0}(x_0) \to Y$, which assigns $y \in Y$ to each $x \in K_{X_0}(x_0)$ in the manner described in Definition 5.

For other properties of differentiable functions we refer to [3].

Transformation of tangent sets

Let $Y_0 \subseteq Y$. In what follows we shall denote by $f^{-1}(Y_0)$ the counterimage of Y_0 by f.

PROPOSITION 6. *Let* f *be differentiable at* x_0. *Then:*

$$(D_f(x_0))^{-1}\big(L_{Y_0}((x_0))\big) \subseteq K_{X_0}(x_0) \cap L_{f^{-1}(Y_0) \cup \operatorname{comp} X_0}(x_0);$$
$$K_{f^{-1}(Y_0)}(x_0) \subseteq (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big).$$

Proof. Let $x \in (D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big)$, that is, $x \in K_{X_0}(x_0)$ and $D_f(x_0)(x) \in L_{Y_0}(f(x_0))$. There are $\tilde{r} > 0$, $V \in \mathcal{V}$ such that $f(x_0)+s(D_f(x_0)(x)+V) \subseteq Y_0$ for all $s \in (0, \tilde{r})$ (Definition 1). There are $\bar{r} > 0$, $U \in \mathcal{U}$ such that $f(x_0+s(x+u)) \in f(x_0)+s(D_f(x_0)(x)+V)$ whenever $s \in (0, \bar{r})$, $u \in U$ and $x_0+s(x+u) \in X_0$ (Definition 6). Write $r = \min(\tilde{r}, \bar{r})$, and let $s \in (0, r)$. If $u \in U$, then either $x_0+s(x+u) \in X_0$, therefore $f(x_0+s(x+u)) \in Y_0$, or $x_0+s(x+u) \notin X_0$. Consequently, $x_0+s(x+U) \subseteq f^{-1}(Y_0) \cup \operatorname{comp} X_0$, which is equivalent to $x \in L_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0)$ (Definition 1).

The second inclusion follows from the first inclusion by replacing Y_0 by comp Y_0 and using Proposition 1.

PROPOSITION 7. *Let f be differentiable at* x_0. *Then*:

$$(D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big) \subseteq K_{X_0}(x_0) \cap l_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0);$$

$$k_{f^{-1}(Y_0)}(x_0) \subseteq (D_f(x_0))^{-1}\big(k_{Y_0}(f(x_0))\big).$$

Proof. Let $x \in (D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big)$, that is, $x \in K_{X_0}(x_0)$ and $D_f(x_0)(x) \in l_{Y_0}(f(x_0))$. There is a $V \in \mathscr{V}$ such that for every $\tilde{r} > 0$ there is an $s \in (0, \tilde{r})$ such that $f(x_0)+s(D_f(x_0)(x)+V) \subseteq Y_0$ (Definition 2). There are $\bar{r} > 0$, $U \in \mathscr{U}$ such that $f(x_0+s(x+u)) \in f(x_0)+s(D_f(x_0)(x)+V)$ whenever $s \in (0, \bar{r})$, $u \in U$ and $x_0+s(x+u) \in X_0$ (Definition 6). Let $r > 0$. Consider $\tilde{r} = \min(r, \bar{r})$, and choose $s \in (0, \tilde{r})$ as above. If $u \in U$, then either $x_0+s(x+u) \in X_0$, and therefore $f(x_0+s(x+u)) \in Y_0$, or $x_0+s(x+u) \notin X_0$. Consequently, $x_0+s(x+U) \subseteq f^{-1}(Y_0) \cup \operatorname{comp} X_0$, which is equivalent to $x \in l_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0)$ (Definition 2).

The second inclusion follows from the first inclusion by replacing Y_0 by comp Y_0 and using Proposition 1.

THEOREM 1. *Let f be differentiable at* x_0. *Then*:

(a) $$L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big) \subseteq L_{f^{-1}(Y_0)}(x_0) \subseteq L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(k_{Y_0}(f(x_0))\big);$$

(b) $$l_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big) \subseteq l_{f^{-1}(Y_0)}(x_0) \subseteq l_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big);$$

(c) $$k_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big) \subseteq k_{f^{-1}(Y_0)}(x_0) \subseteq k_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(k_{Y_0}(f(x_0))\big);$$

(d) $$(D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big) \subseteq K_{f^{-1}(Y_0)}(x_0) \subseteq (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big).$$

Proof. First, if $x \in L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(L_{Y_0}(f(x_0))\big)$, then $x \in L_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0)$ (Proposition 6) and $x \in L_{f^{-1}(Y_0)}(x_0)$ (Propositon 4); thus the first inclusion of (a) is verified. The first inclusions of (b), (c), (d) can be proved in a similar way.

Next, if $x \in L_{f^{-1}(Y_0)}(x_0)$, then $x \in L_{X_0}(x_0)$ (Proposition 3), $x \in k_{f^{-1}(Y_0)}(x_0)$ (Proposition 2) and $x \in (D_f(x_0))^{-1}\big(k_{Y_0}(f(x_0))\big)$ (Proposition 7); thus the last inclusion of (a) is established. The last inclusion of (c) follows from Propositions 3 and 7.

Finally, if $x \in l_{f^{-1}(Y_0)}(x_0)$, then $x \in l_{X_0}(x_0)$ (Proposition 3), $x \in K_{f^{-1}(Y_0)}(x_0)$ (Proposition 2) and $x \in (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big)$ (Proposition 6); thus the last inclusion of (b) is proved. The last inclusion of (d) is the same as in Proposition 6.

THEOREM 2. *Let f be differentiable at* x_0. *Then*:

(e) $$L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big) \subseteq L_{X_0}(x_0) \cap l_{f^{-1}(Y_0)}(x_0) \subseteq L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big);$$

(f) $$k_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big) \subseteq k_{X_0}(x_0) \cap K_{f^{-1}(Y_0)}(x_0) \subseteq k_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big).$$

Proof. First, if $x \in L_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big)$, then $x \in l_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0)$ (Proposition 7) and $x \in l_{f^{-1}(Y_0)}(x_0)$ (Proposition 4); so the first inclusion of (e) is proved.

Next, if $x \in k_{X_0}(x_0) \cap (D_f(x_0))^{-1}\big(l_{Y_0}(f(x_0))\big)$, then $x \in l_{f^{-1}(Y_0)\cup \operatorname{comp} X_0}(x_0)$ (Proposition 7) and $x \in K_{f^{-1}(Y_0)}(x_0)$ (Proposition 5); so the first inclusion of (f) is verified.

Further, if $x \in L_{X_0}(x_0) \cap l_{f^{-1}(Y_0)}(x_0)$, then $x \in K_{f^{-1}(Y_0)}(x_0)$ (Proposition 2) and $x \in (D_f(x_0))^{-1}\big(K_{Y_0}(f(x_0))\big)$ (Proposition 6); so the last inclusion of (e) is established.

Finally, the last inclusion of (f) follows from Proposition 6.

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