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ON THE CONTROL OF STABILITY IN NONLINEAR MECHANICS

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In this paper I show how to control stability on some typical examples of nonlinear mechanics. Clearly, in general, this problem seems to be of considerable interest because of the increasing demands on material in all technical branches. Interesting examples of nonlinear systems losing stability are the buckling of plates and shells or kicking of bars and rods under increasing boundary and volume forces.

To deal with the first example denote by $G+S$ in the x, y-plane a simply connected domain with regular (1) boundary $S \subset C^{4+\mu}$ occupied by the plate under boundary forces $K_x(s)$, $K_y(s)$, $s \subset S$, acting within the x, y-plane. On a portion $\hat{S}$ of S the boundary forces $\hat{K}_x(s)$, $\hat{K}_y(s)$, $s \subset \hat{S}$, may be varied arbitrarily, while on $S-\hat{S}$ the forces $K_x(s)$, $K_y(s)$ are fixed. Let G_0 be an interior subdomain of G. After a general theorem concerning the solutions of elliptic boundary value problems, I proved in [3] that an arbitrary consistent stress distribution

$$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \quad \text{on } G_0$$

may be approximated uniformly in C^0-norm by the solutions of the first boundary value problem for the biharmonic equation (1) by suitable choices of the boundary forces along $\hat{S}$ only. A similar result holds in the general three-dimensional case for the solutions of the classical boundary value problems in linear elasticity, [6]. For convenience we note the representation formula for the first boundary value problem for the plate:

$$(1) \qquad \Delta\Delta u = f, \quad u(s) = \varphi(s), \quad \frac{\partial u}{\partial n}(s) = \psi(s),$$

$$f(x, y) \subset C^{\mu}, \quad \varphi(s),\ \psi(s) \in C^{4+\mu}(S);$$

$$(2) \qquad \frac{\partial u}{\partial x} = \int_{P_0} K_x(s)\,ds, \quad -\frac{\partial u}{\partial y} = \int_{P_0} K_y(s)\,ds, \quad P_0 \subset S,$$

(1) $H_{m,2}$ are the well-known Sobolev spaces with $H_{0,2} = L_2$, and $C^{m+\nu}$, $0 < \nu < 1$, the Banach spaces of m times $\nu-H$ continuously differentiable functions with the usual norms $\|u\|_{m,2}$ and $\|u\|_{m+\nu}$, respectively.

(3) $$u(\xi,\eta) = U(K_x, K_y)+B(f),$$

$$U(\xi,\eta,K_x,K_y) = \frac{1}{8\pi}\int_S \varphi(s)\frac{\partial \Delta G}{\partial n}(x,y,\xi,\eta)-\psi(s)\Delta G(x,y,\xi,\eta)\,ds,$$

$$B(\xi,\eta,f) = \frac{1}{8\pi}\int_G G(x,y,\xi,\eta)f(\xi,\eta)\,dx\,dy;$$

(4) $$u_{xx} = \sigma_{yy},\quad u_{xy} = -\sigma_{xy},\quad u_{yy} = \sigma_{xx},$$

$G(x, y, \xi, \eta)$ being the Green function, and $u(x, y)$ = the Airy stress function. Clearly, boundary forces may be allowed to vary on a bounded set $U(\hat{S})$ along the window $\hat{S}$ only in practice. Let the control domain $U(\hat{S}) \supset \hat{K}_x, \hat{K}_y$ be a bounded closed convex set of a sufficient regular Sobolev space, of a reflexive Banach space or of a $C^{m+\mu}$-space; then we have the theorem, [4]:

THEOREM 1. *For an arbitrarily given stress distribution*

$$\sigma'_{xx}, \sigma'_{xy}, \sigma'_{yy} \text{ on } G_0$$

there exist optimal control forces $\tilde{K}_x, \tilde{K}_y \subset U(\hat{S})$ *such that the corresponding stresses* $\tilde{\sigma}_{xx}, \tilde{\sigma}_{xy}, \tilde{\sigma}_{yy}$ *calculated along* (1), (2), (4) *solve the minimum problem*

$$\min_{K_x,K_y\subset U(\hat{S})}\ \max_{x,y\in G_0}\{|\sigma_{xx}-\sigma'_{xx}|,\ldots,|\sigma_{yy}-\sigma'_{yy}|\}.$$

The proof is classical, using the weak compactness of $U(\hat{S})$ in the first two cases and complete continuity arguments concerning the linear transformation $K_x, K_y \to \{\sigma_{xx}, \sigma_{xy}, \sigma_{yy}\}$ along (1), (2), (3), (4).

Remark. In the case $U(\hat{S}) \subset C^{k+\mu}(\hat{S})$, $k \geqslant 2$, $U(\hat{S})$ being then a compact set in the C^k-norm, we can choose $G_0 = G$ in our theorem using the bounded dependence of a solution of (1) in $C^{4+\mu}$ on its boundary values in $C^{4+\mu}(S)$ and on $f(x, y) \subset C^\mu$ because of the fundamental Schauder estimates.

We now arrive at our main problems:

The buckling of a thin plate can be described—as is well known—by a nonlinear bifurcation problem corresponding to the von Karman–Föppl system

(5) (a) $$\Delta\Delta u+w_{xy}^2-w_{xx}w_{yy} = 0,$$

(b) $$\gamma h^2\Delta\Delta w+w_{xx}u_{yy}-2w_{xy}u_{xy}+w_{yy}u_{xx} = 0$$

under the boundary conditions

(6) (a) clamped plate: $$w(s) = \frac{\partial w}{\partial n}(s) = 0,\quad s\in S,$$

(b) supported plate: $$w(s) = \Delta w(s) = 0,$$

h = thickness of the plate, $\frac{1}{12(1-\nu)} = \gamma^2$, ν being the Poisson number, $w(x, y)$ the displacement of the middle surface of the plate in the z-direction and $u(x, y)$ the Airy stress function.

If

(*) $$K_x(s) \to \lambda K_x(s);\quad K_y(s) \to \lambda K_y(s),$$

the plate buckles for discrete critical forces $\lambda_j K_x$, $\lambda_j K_y$, $j = 0, 1, 2, \ldots$, λ_j, being the characteristic values of the nonlinear eigenvalue problem

(7) $$Tw+N(w)w+\lambda Lw = 0$$

under the boundary conditions (6a) or (6b)

$$Tw = \Delta\Delta w,$$

$$Lw = \frac{1}{\gamma^2 h}\left(\frac{\partial}{\partial x}(U_{yy}w_x-U_{xy}w_y)+\frac{\partial}{\partial y}(U_{xx}w_y-U_{xy}w_x)\right),$$

$$N(w)w = L(B(w), w).$$

To get (7) we solve equation (5a) by (3) and substitute it in (5b), setting $f = w_{xy}^2-w_{xx}w_{yy}$ and introducing (*).

(8) $$Tw+\lambda Lw = 0$$

under (6b) or (6a) is the Fréchet eigenvalue problem corresponding to (7) with enumerable infinite discrete eigenvalues λ_i, $-\infty < \lambda_i < \infty$. (8) can be put in variational form under (6):

(9) $$(Tw, w)_0-\lambda(Lw, w)_0 \to \text{Extr}.$$

$$(Lw, w)_0 = \int_E U_{yy}w_x^2-2U_{xy}w_xw_y+U_{xx}w_y^2\,dx\,dy,$$

$$(Tw, w)_0 = \int_E (\Delta w)^2\,dx\,dy.$$

The bifurcation problem (7) has been solved in [5], where the authors use the theory of critical points, proving the eigenvalues λ_i to be bifurcations points of (7).

To control the stability of our plate on $U(\hat{S})$ means in practice to control the absolute least eigenvalue $\bar{\lambda}$ of (8)

(10) $$\frac{1}{\bar{\lambda}} = \max\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right),$$

$$\max(Lw, w)_0 = \frac{1}{\mu_0},\qquad \min(Lw, w)_0 = \frac{1}{\mu_1},$$

under the constraints $(Tw, w)_0 = 1$ and (6a) or (6b). We apply the remark to Theorem 1 and simple variational arguments for (10) to prove, [4]:

THEOREM 2. *There exist optimal control forces* $\hat{K}_x, \hat{K}_y \subset U(\hat{S}) \subset C^{k+\mu}(\hat{S})$, $k \geqslant 2$, *for which the least eigenvalue* $\bar{\lambda}$ *in* (10) *attains its maximum, so that our plate is most stable with respect to all boundary forces* $K_x, K_y \subset S-\hat{S}$, $\hat{K}_x, \hat{K}_y \subset U(\hat{S})$.

Remark. According to our general theorem mentioned before, one may approximate an arbitrary consistent stress distribution U_{xx}, U_{xy}, U_{yy} on G_0 uniformly by suitable choices of K_x, K_y along $\hat{S}$ only. It is easily proved that therefore the least eigenvalue $\bar{\lambda}$ in (10) can be made arbitrarily small, the plate becoming highly instable in this case.

On the other hand, if we assume the small strip $G-G_0$ to be of extreme stiffness or clamped but so that displacement in the x, y-plane are not reduced, we can prove the opposite statement in the same manner: our plate can be made extremely stable if suitably controlled by the forces along $\hat{S}$ only, [4].

All that I have said about the control of stability can be generalized to plates which are supported over partial subdomains G_i, $G_j' \subset G$, so that the plate can only turn towards the free side. The corresponding buckling problem has recently been solved by my pupil, Miersemann, [8]. To construct the branch of bifurcation here, one has to solve the variational problem

(11) $$(Lw, w)_0 \to \max$$

under the boundary conditions (6) and the constraints

(11a) $$(Tw, w)_0 - \tfrac{1}{2}(N(w)w, w)_0 = s > 0, \quad w \geqslant 0 \text{ on } G_i, \quad w \leqslant 0 \text{ on } G_j'.$$

Since we can apply our control mechanism to (11) from $U(\hat{S})$ as before, it is possible to control the branch of bifurcation along (11a), $s > 0$, and thus its limit, proving that all the remarks remain valid as stated.

To control stability in general systems A of nonlinear elasticity we introduce the important formula (12) of E. Trefftz [9] for the second variation of the elastic potential of A in a deformed state Z derived under transparent assumptions (13):

(12) $$Q_2 = \frac{1}{2}\int_A \Big(\sum_{\mu,\nu,k} \sigma_{\mu\nu}(x)\frac{\partial u_k}{\partial x_\mu}\frac{\partial u_k}{\partial x_\nu} + 2a\Big)dx;$$

a means here the classical expression of the elastic potential in linear theory of elasticity and $\sigma_{\mu\nu}$ is the stress tensor of A in the state Z. In his paper [9] E. Trefftz, in order to derive (12), splits the stress tensor in the neighbouring state Z', $Z \to Z'$: $k_{\mu\nu}$, into two parts,

(13) $$k_{\mu\nu} = \sigma_{\mu\nu} + \tau_{\mu\nu},$$

and proposes to calculate $\tau_{\mu\nu}$, the additional stresses arising from $Z \to Z'$, within the framework of the linear theory of elasticity.

The state Z is stable iff $Q_2 > 0$ for all admissible displacements $u(x)$. The least positive eigenvalue λ_0 of the classical eigenvalue problem

(14) $$\min_{\|u\|_{0,2}=1} Q_2 = \lambda_0 > 0,$$

relevant to the elliptic case, measures stability to some extent. The same is true for the greatest lower bound λ_0' of

(15) $$\min_{\|u\|_{1,2}=1} Q_2 = \lambda_0' > 0$$

such a positive bound follows from a theorem of Hestenes in the positive elliptic case (14). Conversely, ellipticity can be derived from (15).

In my paper [2], I gave a rigorous mathematical description of the deformation of an elastic body $G+S \subset C^{2+\mu}$ under body forces $X \subset C^{\mu}$ and given boundary displacements $\varphi(\sigma) \subset C^{2+\mu}$, $\sigma \in S$, in nonlinear elasticity under the assumptions (12), (13) of Trefftz. Our proof remains valid in far more general cases concerning the second variation (12) and the transition relations (13) up to the so-called *fading memory models*.

Replacing X, φ by λX, $\lambda\varphi$, $0 \leqslant \lambda \leqslant 1$, we first describe approximately in our theory [2] the deformation of $G+S$ when passing the curve $\lambda X \times \lambda\varphi \in C^{\mu} \times C^{2+\mu}$ as follows: For an arbitrary subdivision $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = 1$, $\mathrm{Max}(\lambda_l - \lambda_{l-1}) = \varepsilon$, the corresponding deformation chain $G \to G_1 \to G_2 \to \ldots \to G_n$ may inductively be defined by the transformations

(16) $$x^{l+1} = x^l + v^l(x^l), \qquad l = 0, 1, 2 \ldots$$

The strains $v^l(x^l)$ over G^l describing the transition $G^l \to G^{l+1}$ are defined as the solutions of the variational problems

(17) $$\frac{1}{2}\int_{E^l}\Big(\sigma_{\mu\nu}(x^l)\frac{\partial v_k^l}{\partial x_\mu^l}\frac{\partial v_k^l}{\partial x_\nu^l} + 2a\Big)dx^l + (\lambda_{l+1}-\lambda_l)(X, v^l)_0 \to \min,$$

(18) $$(\lambda_{l+1}-\lambda_l)\varphi(\sigma^l) = v^l(\sigma^l), \qquad \sigma^l \in S^l;$$

then the new stresses over G^{l+1} are given by (²)

(19) $$\sigma_{\mu\nu}^{l+1}(x^{l+1}) = \sigma_{\mu\nu}^l(x^l) + \tau_{\mu\nu}^l(x^l),$$

$$\tau_{\mu\nu}^l = M\Big(\frac{\partial v_\nu^l}{\partial x_\mu^l} + \frac{\partial v_\mu^l}{\partial x_\nu^l}\Big)\ldots \qquad (M = \text{shear modulus}),$$

according to (13). Clearly, we have to assume that our variational problems (17) are elliptic and the transformations $G^l \to G^{l+1}$ remain topologic. Since there is no loss of regularity, we may proceed to $\lambda_n = 1$. Along our chain the fundamental estimates of Schauder type are valid [1]:

(20) $$\|v^l\|_{2+\mu} \leqslant \Delta\lambda L(H_{1+\mu}, c, \lambda_0, R_{2+\mu})\big(\|X\|_\mu(G^l) + \|\varphi\|_{2+\mu}(G^l)\big),$$

the constant L depending on

$H_{1+\mu}$ = bound of the $C^{1+\mu}$-norms of the coefficients $\sigma_{\mu\nu}^l$ in (17),
$R_{2+\mu}$ = bound of the $C^{2+\mu}$-norms for the boundary representations of S^l,
c = ellipticity constant,
λ_0 = least positive eigenvalue in (14) under zero boundary conditions.

Putting (16) together from $l = 0, 1, 2, \ldots, k$, we receive

(21) $$x_j^k = f_j^k(x), \quad x \in G \to G^k, \qquad j = 1, 2, 3.$$

In [2] I proved the convergence of the deformation process (20) for $\varepsilon \to 0$ in the Banach space $C^{2+\mu}(G) \times C^0$, receiving a rigorous description, $x^\lambda = f(\lambda, x)$, for the deformation $G \to G^\lambda$ when passing the data curve over states of equilibrium. The

(²) In (17) we have used the summation convention.

convergence proof is first limited to an interval I_s: $0 \leqslant \lambda \leqslant s$, but we may proceed further till at least one of the following three cases occurs.

(1) For $\lambda \to s_a$ the deformation tends to instability $\lambda_0 \to 0$, or the ellipticity constant c tends to zero,

(2) $C^{1+\mu}$-norms of stress components $\sigma_{\mu\nu}^{\lambda}$ tend to infinity,

(3) the surface ∂G^{λ} of G^{λ} tends to one with twofold points, [2].

After this report on my paper [2] we are able to formulate and prove our optimal control problem in general nonlinear elasticity theory.

Let G' be an elastic body in a deformed state Z with the stress distribution $\sigma_{\mu\nu}(x) \subset C^{1+\mu}(G')$ arising from G by solving the first boundary value problem for boundary displacements $\varphi_1(\sigma)$ and volume forces X according to our theory just mentioned. $\varphi^*(\sigma^*) \in U(S^*) \subset C^{2+\mu'}(S^*)$, $0 < \mu < \mu' < 1$, $\operatorname{supp} \varphi^*(\sigma^*) \subset S^* \subset S'$, may be a bounded convex set of boundary displacements, containing the zero point, over an arbitrary part of S'. The state Z over G' is assumed to be stable according to (14), (15) with stability constants λ_0, λ_0'. For simplicity, $U(S^*)$ may lie in a sufficiently small ball, so that our existence proof for the continuation of the deformation $G \to G'$ from G' under the boundary displacements

$$\varphi(\sigma') = \begin{cases} \varphi^*(\sigma^*), & \sigma' = \sigma^* \in S^* \subset S', \\ 0, & \sigma' \in S' - S^* \end{cases} \tag{22}$$

remains valid along the whole curve $\lambda\varphi(\sigma')$, $0 \leqslant \lambda \leqslant 1$, for all $\varphi(\sigma') \in U(S^*)$. There is no difficulty in proving this choice, [2].

THEOREM 3. *There exists at least one optimal control displacement* $\varphi^0(\sigma') \in U(S^*)$, *so that the corresponding equilibrium state with* (22) *has a maximum stability constant* λ_0 *with respect to* $U(S^*)$.

Proof. We consider the totality of equilibrium states corresponding to displacements (22). The stability constant λ_0 in (14) can be proved to be a continuous function of $\varphi(\sigma')$ over $U(S^*)$. This follows from the fundamental Schauder estimates (20), the transition relations (13), and general continuous perturbation theory or simple variational considerations, [2]. Our proof is now an easy consequence of the compactness of $U(S^*)$ in the $C^{2+\mu}$-norm.

In a similar way we can solve the problem of constructing a displacement curve of "optimal directions" in the variety of curves

$$\varphi(\lambda, \sigma^*) = \sum_{i=1}^{r} \varphi_i^*(\sigma^*) \Delta\lambda_i, \quad \varphi_i^*(\sigma^*) \in U(S^*), \tag{23}$$

and their limits for $\Delta\lambda_j \to 0$, $\sum_{i=1}^{r} \Delta\lambda_i = s_1$, corresponding to an arbitrary division of a fixed interval I_{s_1}: $0 \leqslant \lambda \leqslant s_1 < 1$. To approximate the desired curve we determine for each i, $0 \leqslant i \leqslant r$, according to our inductive calculation scheme (17), (19), $\varphi_i^*(\sigma^*)$ in (23) as a solution of controlling the stability constant λ_0 in the following state Z_{i+1} optimally in the way just mentioned. By the compactness of $U(S^*)$ our existence proof directly leads to the desired solution.

So far we have considered the first boundary value problem. Only some modifications are needed to generalize our theorems to the second boundary value problem, in which boundary forces are prescribed.

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