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UNIQUENESS CONDITIONS FOR OPTIMAL TIME SOLUTIONS IN NONLINEAR CONTROL SYSTEMS

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It is known that one can formulate a control system in orientor form [3]. The optimal time solutions are trajectories of an orientor field, contained in the boundary of the emission zone [1]. In this note we consider only the emission zone of a single point and we give sufficient conditions for any two trajectories tangent at this point to be identical. For uniqueness at regular points see [2]. The problem was suggested by Professor A. Pliś. I am grateful for his valuable advice.

LEMMA 1. *Let ψ be an absolutely continuous function on a closed interval* $[0, l]$ *with values in* $\boldsymbol{R}^n$ *tangent to zero-function at the point zero. We assume that there exists a non-negative integrable function M on interval* $[0, 1]$ *such that the inequality*

$$t|\psi'(t)| \leqslant [1+tM(t)]\,|\psi(zt)|$$

holds for almost every $t \in$ $[0, l]$.

Then we have $\psi \equiv 0$.

Proof. Let φ be a function defined by the formula $\varphi(t) = \int_0^t |\psi'(s)|\,ds$. We are going to prove that $\varphi(t) = 0$ for $t \in [0, l]$, which implies our thesis. Suppose that $\varphi \not\equiv 0$. Then there exists an $a \in [0, l]$ such that $\varphi(a) = 0$ and $\varphi(t) > 0$ for $t \in (a, l]$. We get the inequalities:

$$t\varphi'(t) \leqslant [1+tM(t)]\varphi(t) \qquad \text{for a.e. } t \in [0, l],$$

$$\frac{\varphi(\beta)}{\beta} \leqslant \frac{\varphi(\alpha)}{\alpha}\exp\left[\int_\alpha^\beta M(t)\,dt\right] \qquad \text{for } a < \alpha < \beta < l,$$

$$\frac{\varphi(\beta)}{\beta} \leqslant \left[1+\int_0^l M(t)\,dt\right]\sup_{t\in[a,\alpha]}\frac{|\psi(t)|}{t}\exp\left[\int_0^l M(t)\,dt\right] \qquad \text{for } a < \alpha < \beta < l.$$

The right side of the last inequality tends to zero if α tends to a, and so $\varphi(\beta) = 0$. Thus we must have $\varphi \equiv 0$.

Notation and definitions. Let X be a Euclidean space. By an *orientor* we mean a nonempty, compact, and convex subset of X, by a *strictly convex orientor* — an orientor such that its boundary does not contain a non-empty open interval.

Let $A \subset X$, $x \in X$; we are going to define two subsets of X: S_+ and S_-:

$$z \in S_+(x, A) \Leftrightarrow \exists \varepsilon > 0\colon \operatorname{int conv}\{\{x\} \cup \{y \in X\colon |y-x-\varepsilon(z-x)| = \varepsilon^2\} \subset A,$$

$$z \in S_-(x, A) \Leftrightarrow \exists \varepsilon > 0\colon \operatorname{int conv}\{\{x\} \cup \{y \in X\colon |y-x-\varepsilon(z-x)| = \varepsilon^2\} \subset X \setminus A.$$

We say that A is *regular* if $S_+(x, A)$ is convex and an equality $\operatorname{cl}[S_+(x, A)] \cup S_-(x, A) = X$ holds for every $x \in X$.

Let $A \subset X$ be a regular set. For $a \in \partial A$ (boundary of A) we put:

$$V(a, A) = \{v \in X\colon |v| = 1,\ v \cdot S_+(a, A) \leqslant 0\}.$$

Let B and C be strictly convex orientors in X. We write

$$r(B, C) = \max\{\sup_{x \in B} \inf_{y \in C} |x-y|,\ \sup_{x \in C} \inf_{y \in B} |x-y|\}.$$

Function r is a Hausdorff metric.

$$s(B, C) = \sup\{|y-x|\colon x \in \partial B,\ y \in \partial C\colon V(x, B) \cap V(y, C) \neq \emptyset\}.$$

Function s is a metric introduced by A. Pliś.

$$p(B, A) = \sup\Big\{\frac{|\bar{x}-\bar{y}|}{|x-y|}\colon (x, y) \in \partial A \times \partial A,\ x \neq y,$$
$$(\bar{x}, \bar{y}) \in \partial B \times \partial B\colon V(x, A) \cap V(\bar{x}, B) \neq \emptyset, \text{ and } V(y, A) \cap V(\bar{y}, B) \neq \emptyset\Big\}.$$

PROPERTY 1. *For strictly convex B and C and for a regular A the following implication holds*:

$$[V(b, B) \cap V(a, A) \neq \emptyset,\ V(c, C) \cap V(a', A) \neq \emptyset,\ a \neq a'] \Rightarrow$$
$$\Rightarrow |b-c| \leqslant p(B, A)|a-a'| + s(B, C).$$

Proof. Let $v \in V(c, C) \cap V(a', A)$; there exists a b' such that $v \in V(b', B)$. Then

$$|b-b'| \leqslant p(B, A)|a-a'| \quad \text{and} \quad |b'-c| \leqslant s(B, C).$$

By the *orientor field* we mean the function $F\colon \boldsymbol{R} \times \boldsymbol{R}^n \to \mathscr{P}(\boldsymbol{R}^n)$ such that $F(t, x)$ is an orientor for $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n$. We say that an absolutely continuous function x given on a closed interval L is a *trajectory* of the orientor field F if $x'(t) \in F(t, x(t))$ for a.e. $t \in L$.

Let E be the union of all trajectories of the orientor field F, on the closed interval $[0, l]$ (with $l > 0$), such that $x(0) = 0$. We call it the *right-hand emission zone* of point $(0, 0) \in \boldsymbol{R} \times \boldsymbol{R}^n$, We put

$$E(t) = \{x \in \boldsymbol{R}^n\colon (t, x) \in E\} \quad \text{for} \quad t \in [0, l].$$

In what follows we use the above notations.

LEMMA 2. *We assume that*:

(a) *F is continuous in the Hausdorff metric and $F(t, x)$ are strictly convex.*

(b) *E is regular.*

Let $(t, x(t)) \in \partial E$ for $t \in [0, l]$. Then:

(1) *$x(t) \in \partial[E(t)]$ and $E(t)$ is regular for $t \in (0, l)$.*

(2) *$x'(t) \in \partial F(t, x(t))$ for* a.e. *$t \in (0, l)$.*

(3) *$V[x(t), E(t)] \cap V[x'(t), F(t, x(t))] \neq \emptyset$ for* a.e. *$t \in (0, l)$.*

Proof. (2) follows easily from (a). (1) follows from (b) if we notice that $\partial[E(t)] = (\partial E)(t)$ for $t \in (0, 1)$. Proof of (3): We fix $t \in (0, l)$ such that $x'(t) \in \partial F(t, x(t))$. We assume that $F(t, x(t))$ has interior points. (In the opposite case $F(t, x(t))$ is reduced to a single point and (3) holds.) For $v = (v_1, v_2) \in V[(t, x(t)), E]$ we denote by H the subspace of $\boldsymbol{R} \times \boldsymbol{R}^n$ passing through the point $(t, x(t))$ and perpendicular to v. It follows from (a) that the cone with the top in $(t, x(t))$ and with the section at the point $t+1$ equal to $x(t)+F(t, x(t))$ is contained in the closed half-space, determined by H, which does not contain the point $(t, x(t))+v$. We can choose v such that $(t, x(t))+(1, x'(t)) \in H$; then $v_2 \neq 0$. We put $u = v_2/|v_2|$, and then

$$u \in V(x(t), E(t)) \cap V[x'(t), F(t, x(t))].$$

THEOREM. *We assume* (a) *and* (b) *from Lemma* 2, *and that there exists a non-negative, integrable function M on the interval* $[0, l]$ *such that*

$$s(F(t, x), F(t, y)) \leqslant M(t)|x-y|$$

and

$$p\Big(F(t, x), \frac{1}{t}E(t)\Big) \leqslant 1 + tM(t)$$

for $(t, x) \in E$ and $(t, y) \in E$. Then any two trajectories of F on $[0, l]$ *contained in ∂E tangent at zero are identical.*

Proof. Let x and y be such trajectories. We apply Lemma 2 to x and y, and according to Property 1 we get the inequality

$$|x'(t)-y'(t)| \leqslant p\big(F(t, x(t)),$$
$$\frac{1}{t}E(t)\Big|\frac{x(t)}{t} - \frac{y(t)}{t}\Big| + s\big(F(t, x(t)), F(t, y(t))\big)$$

for a.e. $t \in s[0, 1]$; it follows that $t|x'(t)-y'(t)| \leqslant (1+2M(t)t)|x(t)-y(t)|$ for a.e. $t \in (0, 1)$.

We apply Lemma 1 to the function $x-y$.

The proof is thus complete.

References

[1] A. Pliś, *On trajectories of orientor fields*, Bull. Acad. Polon. Sci., Sér. math. astr. phys. 13 (1965), pp. 559–561.

[2] —, *Uniqueness of optimal trajectories for nonlinear control systems*, to appear in Annales Polonici Mathematici.

[3] T. Ważewski, *On an optimal control problem*, Proc. of the Conference on Diff. Equations and Their Applications, held in Prague in September 1962.

BANACH CENTER PUBLICATIONS
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TANGENT SETS AND DIFFERENTIABLE FUNCTIONS

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The purpose of the present paper is to define some tangent sets (see Definitions 1, 2, 3, 4) and to discuss the transformation of these tangent sets (see Theorems 1, 2) by means of differentiable functions (see Definitions 5, 6).

Tangent sets

Let X be a real topological vector space. We shall denote by $\mathscr{U}$ the family of all neighbourhoods of 0 in X.

Let $X_0 \subseteq X$, and let $x_0 \in X$.

DEFINITION 1. $L_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: there are $r > 0$, $U \in \mathscr{U}$ such that if $s \in (0, r)$, $u \in U$, then $x_0+s(x+u) \in X_0$.

DEFINITION 2. $l_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: there is a $U \in \mathscr{U}$ such that for every $r > 0$ there is an $s \in (0, r)$ such that if $u \in U$, then $x_0+s(x+u) \in X_0$.

DEFINITION 3. $k_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: for every $U \in \mathscr{U}$ there is an $r > 0$ such that for every $s \in (0, r)$ there is a $u \in U$ such that $x_0+s(x+u) \in X_0$.

DEFINITION 4. $K_{X_0}(x_0)$ is the set of all $x \in X$ satisfying the condition: for every $r > 0$, $U \in \mathscr{U}$ there are $s \in (0, r)$, $u \in U$, such that $x_0+s(x+u) \in X_0$.

There exist many definitions, and notations for each of the above tangent sets. Let us recall that Definition 4 is equivalent to the definition in [1]. For other references see [2].

In the sequel we shall denote by $\operatorname{comp} X_0$ the complement of X_0. The subsequent three propositions are direct consequences of the above definitions.

PROPOSITION 1. *The following equalities hold*:

$$\operatorname{comp} L_{X_0}(x_0) = K_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} l_{X_0}(x_0) = k_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} k_{X_0}(x_0) = l_{\operatorname{comp} X_0}(x_0);$$
$$\operatorname{comp} K_{X_0}(x_0) = L_{\operatorname{comp} X_0}(x_0).$$