$$\dot{y} = u(t)Y(y) - tu(t)(\mathrm{ad}X, Y)(y(t)) + \tfrac{1}{2}t^2u(t)(\mathrm{ad}X, Y)(y(t)) \ldots$$

with initial data $y(0) = p$. If $\mathscr{B}(t, p)$ denotes the set of points attainable at time t by solutions of (3) initiating from p at time 0, then $\mathscr{A}(t, p, \mathscr{D}) = T^X(t)\mathscr{B}(t, p)$. Since, for fixed $t > 0$, T^X is a homeomorphism, $p \in \mathrm{int}\,\mathscr{A}(t, p, \mathscr{D})$ if and only if $p \in \mathrm{int}\,\mathscr{B}(t, p)$. One may thus study the (non-autonomous) equation (3).

We next discuss still another approach to the general problem. Let $t_1 > 0$. One may show that for each integer $i = 0, 1, 2, \ldots$, there is an $\varepsilon_i > 0$ and a C^1 map $\gamma^i\colon (-\varepsilon_i, \varepsilon_i) \to \mathscr{A}(t_1, p, \mathscr{D})$ such that $\gamma^i(0) = p$ and $\dot{\gamma}^i(0) = (\mathrm{ad}^iX, Y)(p)$. We next follow ideas of Krener, [5], and recall that if $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$ and $\dot{\gamma}(0) = 0$ then $\ddot{\gamma}(0) \in TM_p$. Precisely, for $s \geqslant 0$ if $\xi(s) = 2\gamma(\sqrt{s})$ then $\lim_{s \to +0} \dot{\xi}(s) = \ddot{\gamma}(0)$. If $\mathscr{A}(t_1, p, \mathscr{D})$ has interior but p belongs to its boundary, the tangent space to $\mathscr{A}(t, p, \mathscr{D})$ at p is a "half space" hence to obtain it, one must consider curves γ through p with $\dot{\gamma}(0) = 0$. One next must show that, say, if $Y(p)$, $[X, Y](p)$ are linearly dependent, then there exists a map $\gamma\colon (-\varepsilon, \varepsilon) \to \mathscr{A}(t, p, \mathscr{D})$ such that

$$\gamma(0) = p, \quad \dot{\gamma}(0) = 0 = \alpha Y(p) + \beta[X, Y](p), \quad \text{while } \ddot{\gamma}(0) = 0 = [Y, [X, Y]](p).$$

One associates with each term of the form $(\mathrm{ad}^kX, (\mathrm{ad}^mY, X))$ a coefficient $s^m\tau^{k+1}$, where $(k+1)$ indicates the number of times X appears in the product and m the number of times Y appears. (This parallels the proof of Theorem 1, [4].) It is possible, therefore, that $[Y, [X, Y]](p)$ and $[Y, (\mathrm{ad}^kX, Y)](p)$ (or $[(\mathrm{ad}^mX, Y), (\mathrm{ad}^kX, Y)](p)$) are negatives of each other but this cannot occur with the same coefficient, i.e. the first has coefficient $s^2\tau$, the second $s^2\tau^k$ and if these are negatives of each other $k \neq 1$. Using this approach (but with a full proof not completed at this time) it seems one may obtain the following result for the general n-dimensional system (1).

Let $\mathscr{S}^0 = \{Y, (\mathrm{ad}X, Y), (\mathrm{ad}^2X, Y), \ldots\}$; define $\mathscr{S}^1$ to be the set of all products of pairs of elements of $\mathscr{S}^0$; $\mathscr{S}^2$ to be the products of all three element subsets of $\mathscr{S}^0$ (repeated usage of an element possible) etc. Assume $\dim \tau(\mathscr{S}^0)_p = n$ so $\mathscr{A}(t, p, \mathscr{D})$ has nonempty interior for all $t > 0$. Then a necessary and sufficient condition that $T^X(t)p \in \mathrm{int}\,\mathscr{A}(t, p, \mathscr{D})$ for all $t > 0$ is that

$$\dim\mathrm{span} \bigcup_{i=0}^{m} \mathscr{S}_p^i = \dim\mathrm{span} \bigcup_{i=0}^{m+1} \mathscr{S}_p^i \quad \text{for all even } m.$$

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CONTROLLABLE TWO DIMENSIONAL NEUTRAL SYSTEMS

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Consider the following functional differential equation

(1) $$\dot{x}(t) = L(t, x_t) + B(t)u(t)$$

where

$$L(t, \varphi) = \int_{-h}^{0} d_\theta\eta(t, \theta)\varphi(\theta)$$

and $\eta(t, \theta)$ is an $n\times n$ matrix for $t, \theta \in \mathbf{R}$ which satisfies the regularity conditions in [3]. In particular, $\theta \to \eta(t, \theta)$, $\theta \in \mathbf{R}$ is of bounded variation. The function $t \to B(t)$, $t \in \mathbf{R}$ is continuous and $B(t)$ is an $n\times m$ matrix for each $t \in \mathbf{R}$. Just as in [1], [2] we use $X = W_2^{(1)}([-h, 0], \mathbf{R}^n)$ as our state space, and $\mathscr{U} = L_2([t_0, t_1], \mathbf{R}^m)$, $t_1 > t_0 + h$ as our admissible controllers. Given $\varphi \in X$ and $u \in \mathscr{U}$, there is a unique absolutely continuous function $t \to x(t) \equiv x(t, t_0, \varphi, u)$, $t \in [t_0, t_1]$ satisfying (1) a.e. on $[t_0, t_1]$ and the initial condition

(2) $$x_{t_0} = \varphi,$$

where $x_{t_0} \in X$ stands for $x_{t_0}(\theta) = x(t_0+\theta)$, $-h \leqslant \theta \leqslant 0$. System (1) is *controllable on* $[t_0, t_1]$ means that for each $\varphi, \psi \in X$ there is a $u \in \mathscr{U}$ such that

$$x_{t_1}(\cdot, t_0, \varphi, u) = \psi.$$

Systems (1) which are controllable were characterized in [2] as follows:

In order that (1) *be controllable on* $[t_0, t_1]$, $t_1 > t_0 + h$ *it is necessary and sufficient that the following two conditions be satisfied*:

(i) $\mathrm{Rank}\,B(t) = n$ *a.e. on* $[t_1 - h, t_1]$,

(ii) $t \to B^*(t)\,(B(t)B^*(t))^{-1}$, $t \in [t_1 - h, t_1]$ *is essentially bounded.*

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This shows that controllability will often be too restrictive an assumption in many physical problems. It was shown in [1], [4] that much of what a controllability hypothesis achieves in an optimal control problem can be accomplished equally well if one can merely establish that the attainable set

$$\mathscr{A}(t_0, t_1, \varphi) = \{\psi \in X \mid \psi = x_{t_1}(\cdot, t_0, \varphi, u) \text{ for some } u \in \mathscr{U}\}$$

is closed in X. In [2] it was shown that *if* $\mathscr{A}(t_0, t_1, \varphi)$ *is closed, then the generalized (or Moore–Penrose) inverse of* $B(t)$ *(see* [6] *for the definition) defines a function* $t \to B^\dagger(t)$, $t \in [t_1-h, t_1]$ *which is essentially bounded on* $[t_1-h, t_1]$. This is not, however, sufficient to assure $\mathscr{A}(t_0, t_1, \varphi)$ is closed (see [2] for an example). A partial converse is possible. Let

$$\tilde{\eta}(t, \theta) = \begin{cases} \eta(t, \theta), & \theta \neq 0, \\ \eta(t, 0^-), & \theta = 0. \end{cases}$$

Then if one makes the added assumption that

$$\operatorname{Im} \tilde{\eta}(t, \theta) \subset \operatorname{Im} B(t)$$

for a.e. $t \in [t_1-h, t_1]$, and each $\theta \in [-h, 0]$, it was shown in [2] that $\mathscr{A}(t_0, t_1, \varphi)$ is closed in X. A. Olbrot (1) and S. Kurcyusz have recently obtained necessary and sufficient conditions for the attainable set $\mathscr{A}(t_0, t_1, \varphi)$ of the system

(3) $$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B(t)u(t)$$

to be closed in X. In case A_0, A_1, B do not depend on t, their results state that $\mathscr{A}(t_0, t_1, \varphi)$ is closed in X if and only if

$$\operatorname{Im}(A_1 A_0^k B) \subset \operatorname{Im}(B), \quad k = 0, 1, \ldots, n-1.$$

In the present paper we will show that for neutral systems an assumption of controllability is more reasonable than in the retarded case mentioned above. We will confine our attention here to the special two dimensional system ($n = 2$)

(4) $$\dot{x}(t) = A_{-1}\dot{x}(t-h) + A_0 x(t) + A_1 x(t-h) + Bu(t)$$

where

$$A_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}, \quad i = \pm 1, 0, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and u is a scalar control. Controllability turns out to be a *generic property* of systems (4) in the sense that systems which are controllable form an open dense subset of all such systems, cf. [5]. We shall only sketch the proofs of the main propositions. Complete proofs for the general n-dimensional case will be published elsewhere. The analysis is based on methods developed in [2], [7].

Write (4) in the operational form

(5) $$Q(D, S)x = Bu$$

(1) This was communicated to us in a special evening seminar at the Zakopane Conference on Optimal Control, cf. [4].

where

$$Q(D, S) = I_2 D - A_{-1} DS - A_0 - A_1 S$$

and operators D and S are defined by

$$(Dx)(t) = \dot{x}(t), \quad (Sx)(t) = x(t-h).$$

Since (4) is autonomous, we can assume $t_0 = 0$, and discuss controllability on the interval $J = [0, t_1]$, $t_1 > 2h$.

Observe that (4) is controllable on J if and only if

(6) $$\mathscr{A}(0, t_1, 0) = X.$$

Thus the initial data in (4) or (5) can be taken to be the zero function without any loss of generality, and we abbreviate $x(t, 0, 0, u)$ by $x(t, u)$. For this reason it will be convenient to have the following notation:

$W_{2,0}^{(k)}(\tau, R^2) = \{x\colon (-\infty, \tau] \to R^2 \mid x(t) = 0,\ t < 0,\ x^{(k-1)} \in C((-\infty, \tau]),\ x^{(k)} \mid [0, \tau] \in L_2\}$, $k = 0, 1, 2, \ldots$, $\tau > 0$, and $\mathscr{U}_0 = W_{2,0}^{(0)}(t_1, R)$ is the set of admissible controllers.

Treating D and S formally as scalar multipliers we can define the 2×2 operator matrices

(7) $$P(D, S) = \operatorname{adj} Q(D, S) = P_0(D) + P_1(D)S, \qquad K(D) = [P_0(D)B, P_1(D)B],$$

where adj A denotes the transposed matrix of cofactors.

In fact, since $A \mapsto \operatorname{adj} A$ is linear on the space of 2×2 matrices (this is false for higher dimensions) $P_0(D)$, $P_1(D)$ are readily seen to be

(8) $$P_0(D) = DI_2 - \operatorname{adj} A_0, \quad P_1(D) = -[\operatorname{adj} A_1 + D \operatorname{adj} A_{-1}].$$

Hence $K(D)$ in (7) can be written as

(9) $$\begin{aligned} K(D) &= [B, -(\operatorname{adj} A_{-1})B]D - [(\operatorname{adj} A_0)B, (\operatorname{adj} A_1)B] \\ &= C_2[-\operatorname{adj} A_{-1}, B]D - [(\operatorname{adj} A_0)B, (\operatorname{adj} A_1)B] \end{aligned}$$

where we have used the notation $C_2[A, B]$ for the *controllability matrix* $[B, AB]$.

When we specialize Theorem 5.2 of [2] to the present situation in (4) we get following result.

Let $t_1 > 2h$, $\psi \in X$. *There is a* $u \in W_{2,0}^{(0)}(t_1, R)$ *such that*

(10) $$x_{t_1}(\cdot, u) = \psi$$

if and only if there is an $\omega \in W_2^{(2)}([t_1-h, t_1], R^2)$ *such that*

(11) $$[K(D)\omega](t) = \psi(t-t_1), \quad t_1 - h \leqslant t \leqslant t_1,$$

and

(12) $$\omega_1(t_1-h) = \omega_2(t_1), \quad \dot{\omega}_1(t_1-h) = \dot{\omega}_2(t_1),$$

$\omega = [\omega_1, \omega_2]^*$.

Taking advantage of the expression (9) for the operator $K(D)$ we write (11) in the form

(13) $$C_2[-\operatorname{adj} A_{-1}, B]\dot{\omega}(t) - [(\operatorname{adj} A_0)B, (\operatorname{adj} A_1)B]\omega(t) = \psi(t-t_1),$$

$t_1-h \leqslant t \leqslant t_1$. Therefore if (6) is to be satisfied, then (13) has a solution $\omega \in W_2^{(2)}([t_1-h, t_1], R^2)$ for every $\psi \in X = W_2^{(1)}([-h, 0], R^2)$. Clearly, this implies that

$$\text{Rank } C_2[-\text{adj } A_{-1}, B] = 2. \tag{14}$$

One can then verify that (14) is satisfied if and only if

$$\text{Rank } C_2[A_{-1}, B] = 2. \tag{15}$$

Now define

$$\mathscr{A}(0, t_1, \varphi)(\theta) = \{x \in R^2 \mid x = \psi(\theta) \textit{ for some } \psi \in \mathscr{A}(0, t_1, \varphi)\}, \quad -h \leqslant \theta \leqslant 0.$$

Clearly, (6) implies that

$$\mathscr{A}(0, t_1, 0)(-h) = R^2. \tag{16}$$

One might expect that (15) and (16) would imply (6). However, this turns out to be false. Indeed, if $\beta_0 \neq 0$, and $B = [0, 1]^*$, then the ordinary differential equation

$$\dot{y}(t) = A_0 y(t) + Bu(t)$$

is controllable. By Remark 3.3 of [1], this implies (16). Thus, if $\beta_0 \neq 0$, $\beta_{-1} \neq 0$, then both (16) and (15) are true when $B = [0, 1]^*$. The results given below can be used to show that if $\alpha_{-1} = \alpha_0 = \alpha_1 = 0$, $\beta_{-1} = \beta_0 = 1$, $\beta_1 = -1$, and $B = [0, 1]^*$, then (6) is false although both (15) and (16) are true.

We shall now indicate how the operational methods developed in [2] can be used to establish precisely the conditions that are both necessary and sufficient for (4) to be controllable on $[0, t_1]$, $t_1 > 2h$.

Define the following matrices

$$\tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 1 \\ -\det A_{-1} & \text{tr } A_{-1} \end{bmatrix}, \quad F = C_2[A_{-1}, B], \quad H = C_2[\Lambda, \tilde{B}]. \tag{17}$$

If (15) is true then the change of variable $x = FH^{-1}y$ in (4) leads to the system

$$\dot{y}(t) = \tilde{A}_{-1}\dot{y}(t-h) + \tilde{A}_0 y(t) + \tilde{A}_1 y(t-h) + \tilde{B}u(t) \tag{18}$$

where $\tilde{A}_i = HF^{-1}A_i FH^{-1}$, $i = \pm 1, 0$, and in particular $\tilde{A}_{-1} = \Lambda$. The matrices $\tilde{A}_i$, $i = 0, 1$, will be denoted by

$$\tilde{A}_i = \begin{bmatrix} \tilde{\alpha}_i & \tilde{\beta}_i \\ \tilde{\gamma}_i & \tilde{\delta}_i \end{bmatrix}, \quad i = 0, 1.$$

Since (6) implies (15), it suffices for us to consider only the necessary and sufficient conditions for the controllability of (18). Now replace A_i, $i = \pm 1, 0$, B in (1.13) with $\tilde{A}_i$ and $\tilde{B}$, respectively, and write the resulting equation in the following equivalent form:

$$\dot{\omega}(t) = A\omega(t) + \tilde{\psi}(t), \quad t_1-h \leqslant t \leqslant t_1, \tag{19}$$

where

$$A = [C_2[-\text{adj } \tilde{A}_{-1}, \tilde{B}]]^{-1}[(\text{adj } \tilde{A}_0)\tilde{B}, (\text{adj } \tilde{A}_1)\tilde{B}],$$

$$\tilde{\psi}(t) = [C_2[-\text{adj } \tilde{A}_{-1}, \tilde{B}]]^{-1}\psi(t-t_1), \quad t_1-h \leqslant t \leqslant t_1. \tag{20}$$

The variation of constants formula gives

$$\omega(t) = e^{At}\Big[e^{-A(t_1-h)}\omega(t_1-h) + \int_{t_1-h}^{t} e^{-As}\tilde{\psi}(s)\,ds\Big], \quad t_1-h \leqslant t \leqslant t_1, \tag{21}$$

whereas the two boundary conditions in (12) require that

$$e_1^*[\Pi\dot{\omega}(t_1) - \dot{\omega}(t_1-h)] = e_1^*[\Pi\omega(t_1) - \omega(t_1-h)] = 0, \tag{22}$$

where $e_1^* = [1, 0]$, and

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C_2[-\text{adj } \tilde{A}_{-1}, \tilde{B}].$$

Using (21) the boundary conditions in (22) state that the equations

$$e_1^*\Big[(\Pi e^{Ah} - I_2)A\omega(t_1-h) + \Pi\tilde{\psi}(t_1) - \tilde{\psi}(t_1-h) + \Pi A e^{At_1}\int_{t_1-h}^{t_1} e^{-As}\tilde{\psi}(s)\,ds\Big] \tag{23}$$

$$= e_1^*\Big[(\Pi e^{Ah} - I_2)\omega(t_1-h) + \Pi e^{At_1}\int_{t_1-h}^{t_1} e^{-As}\tilde{\psi}(s)\,ds\Big] = 0$$

must have a solution for arbitrary $\tilde{\psi}$ of the form (20). Consequently, one can show that (18) is controllable on the interval J if and only if the matrix

$$G \equiv \begin{bmatrix} e_1^*(\Pi e^{Ah} - I_2)A \\ e_1^*(\Pi e^{Ah} - I_2) \end{bmatrix} \tag{24}$$

has rank 2. If we compute A in equation (20) we get

$$A = \Pi[(\text{adj } \tilde{A}_0)\tilde{B}, (\text{adj } \tilde{A}_1)\tilde{B}] = \begin{bmatrix} \tilde{\alpha}_0 & \tilde{\alpha}_1 \\ -\tilde{\beta}_0 & -\tilde{\beta}_1 \end{bmatrix}$$

and

$$p(\lambda) = \det(A - \lambda I_2) = (\lambda - \tilde{\alpha}_0)(\lambda + \tilde{\beta}_1) + \tilde{\alpha}_1\tilde{\beta}_0 \tag{25}$$

is the characteristic polynomial of A.

Let λ_i, $i = 1, 2$, denote the characteristic roots of A. Then one readily checks that

$$\text{(a)} \quad e^{Ah} = \frac{1}{\lambda_1 - \lambda_2}[e^{\lambda_1 h}\text{adj}(\lambda_1 I_2 - A) - e^{\lambda_2 h}\text{adj}(\lambda_2 I_2 - A)], \quad \lambda_1 \neq \lambda_2, \tag{26}$$

$$\text{(b)} \quad e^{Ah} = e^{\lambda_1 h}[I_2 + h\,\text{adj}(\lambda_1 I_2 - A)], \quad \lambda_1 = \lambda_2.$$

Now the matrix G in (24) is given by

$$G = -\begin{bmatrix} \tilde{\alpha}_0 & \tilde{\alpha}_1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{\beta}_0 & \tilde{\beta}_1 \\ 0 & -1 \end{bmatrix} e^{Ah}. \tag{27}$$

Thus (18) is controllable on J if and only if

$$\det G \neq 0, \tag{28}$$

where G is given by (27). In order to put these conditions in simpler form it is convenient to consider two cases: (i) $\tilde{\beta}_0 = 0$, (ii) $\tilde{\beta}_0 \neq 0$.

In case (ii) one examines the two possibilities, $\lambda_1 \neq \lambda_2$, $\lambda_1 = \lambda_2$. We shall just state the results of our calculations. In case $\tilde{\beta}_0 = 0$ we get

$$\det G = -[\tilde{\alpha}_1 + (\tilde{\alpha}_0 + \tilde{\beta}_1)e^{-\tilde{\beta}_1 h}]. \tag{29}$$

In the event $\tilde{\beta}_0 \neq 0$ we get that

$$\det G = -E(\lambda_1)E(\lambda_2)/\tilde{\beta}_0, \tag{30}$$

where

$$E(\lambda) \equiv \lambda + \tilde{\beta}_1 + \tilde{\beta}_0 e^{\lambda h}. \tag{31}$$

Hence (18) *is controllable on* $J = [0, t_1]$, $t_1 > 2h$, *if and only if one of the following conditions is satisfied*:

(32) (i) $\tilde{\beta}_0 = 0$ *and* $\tilde{\alpha}_0 + \tilde{\beta}_1 + \tilde{\alpha}_1 e^{\tilde{\beta}_1 h} \neq 0$;

(ii) $\tilde{\beta}_0 \neq 0$ *and* $E(\lambda_1)E(\lambda_2) \neq 0$ *where* λ_i, $i = 1, 2$, *are the roots of* $\lambda^2 + (\tilde{\beta}_1 - \tilde{\alpha}_0)\lambda + \tilde{\alpha}_1\tilde{\beta}_0 - \tilde{\alpha}_0\tilde{\beta}_1 = 0$.

Thus the original system (4) *is controllable on the interval* $[0, t_1]$, $t_1 > 2h$, *if and only if* Rank $C_2[A_{-1}, B] = 2$ *and one of the conditions* (32) (i) *or* (ii) *holds for the corresponding system* (18) *which is connected with* (4) *by means of the transformation* $x = FH^{-1}y$ *where* F, H *are as in* (17).

If we consider the class $\mathscr{S}$ of all matrices $[A_{-1}, A_0, A_1, B]$ where A_i, $i = \pm 1, 0$ and B are respectively 2×2 and 2×1 real matrices with a convenient matrix norm, then it is well known that the subset $\mathscr{S}_{c'}$ of $\mathscr{S}$ which satisfy (15) is open and dense in $\mathscr{S}$. Simple continuity considerations and (32) show that the subset $\mathscr{S}_c$ of all $[A_{-1}, A_0, A_1, B] \in \mathscr{S}_{c'}$, where the corresponding system (18) is controllable is open in $\mathscr{S}_{c'}$. The condition in (1.32) (ii) can be used to establish that $\mathscr{S}_c$ is dense in $\mathscr{S}_{c'}$ and thus in $\mathscr{S}$.

In closing we mention that the same operational methods can be used to characterize two-dimensional null controllable (cf. [8]) retarded systems ($A_{-1} = 0$)

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \tag{33}$$

(i.e., systems where $0 \in \mathscr{A}(0, t_1, \varphi)$ for every $\varphi \in X$, $t_1 > 2h$). Without loss of generality we can assume $B = [0, 1]^*$. Define the 2×2 matrix A by the equation

$$A = \begin{bmatrix} \alpha_0 & \alpha_1 \\ -\beta_0 & -\beta_1 \end{bmatrix}.$$

Then (33) *is null controllable on* $[0, t_1]$, $t_1 > 2h$ *if and only if one of the following conditions is satisfied*:

(34) (i) $\beta_1 = 0$ *and* $\beta_0 \neq 0$,

(ii) $\beta_1 \neq 0$ *and* $\beta_0 + \beta_1 \exp[\det A/\beta_1]h \neq 0$.

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