is completely observable, in the continuous case we must take $D_j = 0$. There is also a difference in the construction for F_j. We may also remark that both in the continuous and in the discrete case we are not obliged to restrict ourselves to the minimal number m of lag terms if that may be useful for other purposes. We would like to point out that one of the main disadvantages of the results described is that we suppose the control to take values in the same space as the state. We do not know any results for the case where the control space has lower dimension. There is one result known for the case where the derivatives are not used: namely we cannot realize our goal with a scalar feedback of the form

$$u(t) = \sum_{j=1}^{m} c_j^* x(t-jh).$$

It is not known if the same is true when we are allowed to use more general delay feedback of the form

$$u(t) = \int_{-h}^{0} d\eta(s)x(t+s).$$

It is also not known whether the same negative answer is obtained by using controls of the form $u(t) = b^*x(t-h)+c^*\dot{x}(t-h)$.

For the discrete-time case, as pointed out also by Charrier, it is not known whether by using feedback of the form $u_n = \sum_{j=1}^{m} B_j x_{n-jk}$ the minimal time l for which we may have $Q^*x_n = 0$, $n \geqslant l$, is just $2k$. For the same discrete case it is not known whether degeneracy is impossible if we use only scalar controls.

It is thus seen that there are still many open questions connected with the simple problem we have considered.

BANACH CENTER PUBLICATIONS
VOLUME 1

ON LOCAL CONTROLLABILITY

HENRY HERMES

Department of Mathematics, University of Colorado, Boulder, USA

Let M^n be an analytic n-dimensional manifold and $\mathscr{D} = \{X^\alpha\colon \alpha \in A\}$ be a collection of analytic tangent vector fields on M. A continuous map $\varphi\colon [0, T] \to M^n$, $T > 0$, is called a *solution* of $\mathscr{D}$ if there exists a partition $0 = t_0 < t_1 < \ldots < t_k = T$ such that on each interval $[t_i, t_{i+1})$, $\dot{\varphi}(t) = X^{\alpha_i}(\varphi(t))$ for some $\alpha_i \in A$. If $M^n = R^n$ and $X^\alpha(x) = f(x, \alpha)$, $\alpha \in A \subset R^m$, then studying solutions of $\mathscr{D}$ is equivalent to studying solutions of $\dot{x} = f(x, \alpha(t))$ where α is a piecewise constant control.

Let φ^* be a fixed (reference) solution of $\mathscr{D}$ with $\varphi^*(0) = p$ and let $\mathscr{A}(t, p, \mathscr{D})$ denote the set of all points attainable at time t by solutions of $\mathscr{D}$ initiating from p at time 0. We define: $\mathscr{D}$ is *locally controllable along* φ^* *at* p if for all $t > 0$, $\varphi^*(t) \in \operatorname{int} \mathscr{A}(t, p, \mathscr{D})$. Clearly, if $\mathscr{D}$ is locally controllable along φ^* at p then $\varphi^*(t) \in \operatorname{int} \mathscr{A}(t, p, \mathscr{D})$ for *all* $t > 0$ and cannot be a time optimal trajectory. Since the Pontriagin maximum principle gives a necessary condition for $\varphi^*(t)$ to belong to the boundary of $\mathscr{A}(t, p, \mathscr{D})$, it is of particular interest to study "singular cases" (i.e., when the maximum principle gives no information; see [2] or [3].) For this reason, and for ease of exposition, we confine our interest, mainly, to systems of the form

$$\mathscr{D} = \{X+\alpha Y\colon\ -1 \leqslant \alpha \leqslant 1\} \tag{1}$$

with X, Y analytic vector fields on M^n_ω, and reference trajectory denoted $T^X(\cdot)p$, the solution of $\dot{x} = X(x)$, $x(0) = p$.

Consider the set, $V(M)$, of all analytic vector fields on M as a real Lie algebra with product the Lie product denoted $[X, Y]$. For $\mathscr{C} \subset V(M)$, let $\tau(\mathscr{C})$ denote the smallest subalgebra containing $\mathscr{C}$, let TM_p denote the tangent space to M at p and let $\mathscr{C}_p = \{X(p) \in TM_p\colon X \in \mathscr{C}\}$. We write $(\operatorname{ad} X, Y) = [X, Y]$, $(\operatorname{ad}^k X, Y) = [X, (\operatorname{ad}^{k-1} X, Y)]$. Now define

$$\mathscr{S}^0 = \{Y, (\operatorname{ad} X, Y), (\operatorname{ad}^2 X, Y), \ldots\}.$$

It is known that rank $\mathscr{S}^0_p = n$ is a sufficient condition for $\mathscr{D}$ to be locally controllable along T^X at p. A necessary condition for local controllability of $\mathscr{D}$ along T^X at p is that $\mathscr{A}(t, p, \mathscr{D})$ have nonempty interior for all $t > 0$. Following [6], we let

$\tau'(\mathcal{D})$ denote the derived algebra of $\tau(\mathcal{D})$ and define $\tau_0(\mathcal{D}) = \tau\{Y, W\colon W \in \tau'(\mathcal{D})\}$. Then Theorem 3.2 of [6] yields a necessary and sufficient condition that $\mathcal{A}(t, p, \mathcal{D})$ have nonempty interior for all $t > 0$ is that $\dim \tau_0(\mathcal{D}) = n$. An easy computation, using the Jacobi identity for the Lie algebra, shows that with $\mathcal{D}$ as in (1), $\tau_0(\mathcal{D}) = \tau(\mathcal{S}^0)$. In summary, a necessary condition for $\mathcal{D}$ to be locally controllable along T^X at p is that $\dim \tau(\mathcal{S}^0)_p = n$; a sufficient condition is that $\operatorname{dim span} \mathcal{S}_p^0 = n$. In [4], for certain two dimensional cases, the gap between necessary and sufficient conditions was closed. Specifically, let

$$\mathcal{S}^1 = \{Y, (\mathrm{ad}^2\, Y, X), (\mathrm{ad}\, X, (\mathrm{ad}^2\, Y, X)), (\mathrm{ad}^2\, X, (\mathrm{ad}^2\, Y, X)), \ldots\},$$
$$\mathcal{S}^2 = \{Y, (\mathrm{ad}^3\, Y, X), (\mathrm{ad}\, X, (\mathrm{ad}^3\, Y, X)), (\mathrm{ad}^2\, X, (\mathrm{ad}^3\, Y, X)), \ldots\}, \qquad \text{etc.}$$

THEOREM 1 [4]. *Let M be a two manifold; assume the necessary condition $\dim \tau(\mathcal{S}^0)_p = 2$ and that $X(p)$, $Y(p)$ are linearly independent. Then there exists an integer $m \geqslant 0$ such that $\operatorname{dim span} \mathcal{S}_p^m = 2$. A necessary and sufficient condition that $\mathcal{D}$ be locally controllable along T^X at p is that the smallest such integer m be even.* (We consider $m = 0$ as even.)

This result may be viewed as a "local, infinite order" version of the maximum principle. One may show that the first order cone of attainability at $p^1 = T^X(t_1)p$ used in the proof of the maximum principle, is $\operatorname{span} \mathcal{S}_p^0$. Thus $T^X(\tau)p$, $0 \leqslant \tau \leqslant t_1$, is a singular arc for $\mathcal{D}$ if $\operatorname{dim span} \mathcal{S}^0_{T^X(\tau)p} < n$ for $0 \leqslant \tau \leqslant t_1$. In the two-dimensional case, Theorem 1 can be used to obtain necessary and sufficient conditions that a singular arc, T^X, of an analytic control system of the form $\dot{x} = X(x) + Y(x)u(t)$, $-1 \leqslant u(t) \leqslant 1$, be time optimal. Examples of the application of Theorem 1, together with a quite complete sketch of its proof, can be found in [4]. We will next discuss extensions of this theorem to higher dimensions.

Suppose X, Y are analytic and involutive (i.e., there exist scalar functions a_1, a_2, a_3 such that $a_1(x)X(x) + a_2(x)Y(x) + a_3(x)[X, Y](x) \equiv 0$) tangent vector fields on M^n. By the Frobenius theorem, all solutions of $\tau(\mathcal{D})$ through an initial point $p \in M^n$ lie on a two manifold M^2, and Theorem 1 can be applied to $\mathcal{D}$ on M^2. Now consider the n-dimensional analytic set of vector fields

$$\mathcal{E} = \Big\{X - \sum_{i=2}^{n} \alpha_i Y^i\colon\ -1 \leqslant \alpha_i \leqslant 1\Big\}$$

where we assume

(a) $X(p), Y^2(p), \ldots, Y^n(p)$ are linearly independent.

(b) For some $\nu \in \{2, \ldots, n\}$, X, Y^ν are involutive.

Let

$$\mathcal{S}^0 = \{Y^2, \ldots, Y^n, (\mathrm{ad} X, Y^2), \ldots, (\mathrm{ad} X, Y^n), (\mathrm{ad}^2 X, Y^2), \ldots\},$$
$$\mathcal{S}^1 = \mathcal{S}^0 \cup \{(\mathrm{ad}^2 Y^\nu, X), (\mathrm{ad} X, (\mathrm{ad}^2 Y^\nu, X)), (\mathrm{ad}^2 X, (\mathrm{ad}^2 Y^\nu, X)), \ldots\},$$
$$\mathcal{S}^2 = \mathcal{S}^1 \cup \{(\mathrm{ad}^3 Y^\nu, X), (\mathrm{ad} X, (\mathrm{ad}^3 Y^\nu, X)), \ldots\}, \qquad \text{etc.}$$

THEOREM 2. *If there exists an integer $m \geqslant 0$ such that $\operatorname{rank} \mathcal{S}_p^m = n$ and if the smallest such integer is even, the system is locally controllable along T^X at p.*

The proof is easy; indeed assumption (a) implies $\operatorname{rank} \mathcal{S}_p^0 \geqslant n-1$. If this rank is n we have local controllability by the "first order" theory. Thus assume $\operatorname{rank} \mathcal{S}_p^0 = n-1$ and that there is an integer $m \geqslant 0$ for which $\operatorname{rank} \mathcal{S}_p^m = n$ and the smallest such m is even. This, together with assumption (b) and Theorem 1, means that the system $\mathcal{D} = \{X + \alpha_\nu Y^\nu\colon -1 \leqslant \alpha_\nu \leqslant 1\}$ is locally controllable along T^X relative to M^2, the two manifold of solutions of $\tau(\mathcal{D})$. Equivalently, for every $\tau > 0$, $\mathcal{A}(\tau, p, \mathcal{D})$ contains a full nbd. of p in M^2. For notational ease, take $\nu = 2$. For $q \in \mathcal{A}(\tau, p, \mathcal{D})$ there exists a piecewise constant function $\alpha_2(\cdot\,; q)\colon [0, \tau] \to [-1, 1]$ such that if φ is the solution of $\dot{x} = X(x) + \alpha_2(t; q)Y^2(x)$, $x(0) = p$, then $\varphi(\tau) = q$. Now let $\alpha_3, \ldots, \alpha_n$ be scalars in a nbd. of zero, $q \in \mathcal{A}(\tau, p, \mathcal{D})$ and $\psi(\tau, \alpha_3, \ldots, \alpha_n, q)$ denote the solution, at time τ, of $\dot{x} = X(x) + \alpha_2(t, q)Y^2(x) + \sum_{j=3}^{n} \alpha_j Y^j(x)$, $x(0) = p$. Then $\psi(\tau, 0, \ldots, 0, q) = q$ while for small τ and $j = 3, \ldots, n$, $\partial\psi/\partial\alpha_j(\tau, 0, \ldots, 0, q) = \tau Y^j(p) + o(\tau)$. Since $Y^3(p), \ldots, Y^n(p)$ are linearly independent and none belong to TM_p^2, the map $\psi(\tau, \cdot)\colon \mathbf{R}^{n-3} \times \mathcal{A}(\tau, p, \mathcal{D}) \to M^n$ covers a full neighbourhood of p, completing the proof.

EXAMPLE 1. Let $M = \mathbf{R}^3$, $p = 0$ and

$$X(x) = \begin{bmatrix} 8 \\ \frac{1}{2}(x_2 - x_3) \\ \frac{1}{2}(x_3 - x_2) \end{bmatrix}, \qquad Y^2(x) = \begin{bmatrix} \frac{1}{4}(x_2 - x_3)^2 \\ 1 \\ -1 \end{bmatrix}, \qquad Y^3(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$[X, Y^2](x) = \begin{bmatrix} -\frac{1}{2}(x_2 - x_3)^2 \\ 1 \\ -1 \end{bmatrix}, \qquad [X, Y^3](x) = 0,$$

$$(\mathrm{ad}^2 X, Y^2)(x) = \begin{bmatrix} -(x_2 - x_3)^2 \\ 0 \\ 0 \end{bmatrix}.$$

One may verify that $\operatorname{rank}\{Y^2(p), Y^3(p), (\mathrm{ad} X, Y^2)(p), (\mathrm{ad} X, Y^3)(p), \ldots\} = 2$ so the "first order theory" gives no information. Also (noting that the last two components of X, Y^2 and $[X, Y^2]$ are negatives of each other) it easily follows that X, Y^2 are involutive. Computation shows that $\operatorname{rank} \mathcal{S}_p^1 = 2$ however $(\mathrm{ad}^3 Y, X)(p)$ and $Y(p)$ are linearly independent (and both in TM_p^2) hence $\operatorname{rank} \mathcal{S}_p^2 = 3$. By Theorem 2, we have local controllability along T^X at p.

The purpose of these "special cases" given in Theorems 1 and 2 is to point the way to a necessary and sufficient condition for local controllability of $\mathcal{D}$ (as given by (1)) along T^X at p. This author has not been successful in extending the method of proof of Theorem 1 to higher dimensional cases. Another possible approach is as follows. Consider $\mathcal{D}$ in the form

$$\dot{x} = X(x) + u(t)Y(x), \qquad x(0) = p, \qquad -1 \leqslant u(t) \leqslant 1. \tag{2}$$

Pick any admissible u; if one attempts to write a solution of (2) in the form $T^X(t) \circ \psi(t, p)$ one finds, [1], that ψ must satisfy the (auxiliary) equation (3)

$$\dot{y} = u(t)Y(y) - tu(t)(\operatorname{ad}X, Y)\big(y(t)\big) + \tfrac{1}{2}t^2u(t)(\operatorname{ad}X, Y)\big(y(t)\big) \ldots$$

with initial data $y(0) = p$. If $\mathscr{B}(t, p)$ denotes the set of points attainable at time t by solutions of (3) initiating from p at time 0, then $\mathscr{A}(t, p, \mathscr{D}) = T^X(t)\mathscr{B}(t, p)$. Since, for fixed $t > 0$, T^X is a homeomorphism, $p \in \operatorname{int}\mathscr{A}(t, p, \mathscr{D})$ if and only if $p \in \operatorname{int}\mathscr{B}(t, p)$. One may thus study the (non-autonomous) equation (3).

We next discuss still another approach to the general problem. Let $t_1 > 0$. One may show that for each integer $i = 0, 1, 2, \ldots$, there is an $\varepsilon_i > 0$ and a C^1 map $\gamma^i\colon (-\varepsilon_i, \varepsilon_i) \to \mathscr{A}(t_1, p, \mathscr{D})$ such that $\gamma^i(0) = p$ and $\dot{\gamma}^i(0) = (\operatorname{ad}^i X, Y)(p)$. We next follow ideas of Krener, [5], and recall that if $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$ and $\dot{\gamma}(0) = 0$ then $\ddot{\gamma}(0) \in TM_p$. Precisely, for $s \geqslant 0$ if $\xi(s) = 2\gamma(\sqrt{s})$ then $\lim_{s \to +0} \dot{\xi}(s) = \ddot{\gamma}(0)$. If $\mathscr{A}(t_1, p, \mathscr{D})$ has interior but p belongs to its boundary, the tangent space to $\mathscr{A}(t, p, \mathscr{D})$ at p is a "half space" hence to obtain it, one must consider curves γ through p with $\dot{\gamma}(0) = 0$. One next must show that, say, if $Y(p)$, $[X, Y](p)$ are linearly dependent, then there exists a map $\gamma\colon (-\varepsilon, \varepsilon) \to \mathscr{A}(t, p, \mathscr{D})$ such that

$$\gamma(0) = p, \quad \dot{\gamma}(0) = 0 = \alpha Y(p) + \beta[X, Y](p), \quad \text{while } \ddot{\gamma}(0) = 0 = [Y, [X, Y]](p).$$

One associates with each term of the form $(\operatorname{ad}^k X, (\operatorname{ad}^m Y, X))$ a coefficient $s^m\tau^{k+1}$, where $(k+1)$ indicates the number of times X appears in the product and m the number of times Y appears. (This parallels the proof of Theorem 1, [4].) It is possible, therefore, that $[Y, [X, Y]](p)$ and $[Y, (\operatorname{ad}^k X, Y)](p)$ (or $[(\operatorname{ad}^m X, Y), (\operatorname{ad}^k X, Y)](p)$) are negatives of each other but this cannot occur with the same coefficient, i.e. the first has coefficient $s^2\tau$, the second $s^2\tau^k$ and if these are negatives of each other $k \neq 1$. Using this approach (but with a full proof not completed at this time) it seems one may obtain the following result for the general n-dimensional system (1).

Let $\mathscr{S}^0 = \{Y, (\operatorname{ad}X, Y), (\operatorname{ad}^2 X, Y), \ldots\}$; define $\mathscr{S}^1$ to be the set of all products of pairs of elements of $\mathscr{S}^0$; $\mathscr{S}^2$ to be the products of all three element subsets of $\mathscr{S}^0$ (repeated usage of an element possible) etc. Assume $\dim \tau(\mathscr{S}^0)_p = n$ so $\mathscr{A}(t, p, \mathscr{D})$ has nonempty interior for all $t > 0$. Then a necessary and sufficient condition that $T^X(t)p \in \operatorname{int}\mathscr{A}(t, p, \mathscr{D})$ for all $t > 0$ is that

$$\dim\operatorname{span}\bigcup_{i=0}^{m}\mathscr{S}_p^i = \dim\operatorname{span}\bigcup_{i=0}^{m+1}\mathscr{S}_p^i \quad \text{for all even } m.$$

References

[1] K. T. Chen, *Decomposition of differential equations*, Math. Annalen 146 (1962), pp. 263–278.

[2] R. Gabasov and F. M. Kirillova, *High order necessary conditions for optimality*, SIAM Control 10 (1972), pp. 27–168.

[3] H. Hermes, *Controllability and the singular problem*, ibid. 2 (1965), pp. 241–260.

[4] —, *On necessary and sufficient conditions for local controllability along a reference trajectory*, in *Geometric methods in system theory*, D. Mayne and R. Brockett eds., Reidel Pub., Dordrecht, The Netherlands, 1974.

[5] A. J. Krener, *The high order maximum principle. Geometric methods on system theory*, ibid. to appear.

[6] H. Sussmann and V. Jurdjevic, *Controllability of nonlinear systems*, J. Diff. Eqs. 12 (1972), pp. 95–116.

BANACH CENTER PUBLICATIONS
VOLUME 1

CONTROLLABLE TWO DIMENSIONAL NEUTRAL SYSTEMS

MARC Q. JACOBS

Department of Mathematics, University of Missouri, Columbia, Mo. 65201, U.S.A.

and

C. E. LANGENHOP

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901, U.S.A.

Consider the following functional differential equation

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) \tag{1}$$

where

$$L(t, \varphi) = \int_{-h}^{0} d_\theta \eta(t, \theta)\varphi(\theta)$$

and $\eta(t, \theta)$ is an $n \times n$ matrix for $t, \theta \in \boldsymbol{R}$ which satisfies the regularity conditions in [3]. In particular, $\theta \to \eta(t, \theta)$, $\theta \in \boldsymbol{R}$ is of bounded variation. The function $t \to B(t)$, $t \in \boldsymbol{R}$ is continuous and $B(t)$ is an $n \times m$ matrix for each $t \in \boldsymbol{R}$. Just as in [1], [2] we use $X = W_2^{(1)}([-h, 0], \boldsymbol{R}^n)$ as our state space, and $\mathscr{U} = L_2([t_0, t_1], \boldsymbol{R}^m)$, $t_1 > t_0 + h$ as our admissible controllers. Given $\varphi \in X$ and $u \in \mathscr{U}$, there is a unique absolutely continuous function $t \to x(t) \equiv x(t, t_0, \varphi, u)$, $t \in [t_0, t_1]$ satisfying (1) a.e. on $[t_0, t_1]$ and the initial condition

$$x_{t_0} = \varphi, \tag{2}$$

where $x_{t_0} \in X$ stands for $x_{t_0}(\theta) = x(t_0 + \theta)$, $-h \leqslant \theta \leqslant 0$. System (1) is *controllable on* $[t_0, t_1]$ means that for each $\varphi, \psi \in X$ there is a $u \in \mathscr{U}$ such that

$$x_{t_1}(\cdot, t_0, \varphi, u) = \psi.$$

Systems (1) which are controllable were characterized in [2] as follows:

In order that (1) *be controllable on* $[t_0, t_1]$, $t_1 > t_0 + h$ *it is necessary and sufficient that the following two conditions be satisfied*:

(i) $\operatorname{Rank} B(t) = n$ *a.e. on* $[t_1 - h, t_1]$,

(ii) $t \to B^*(t)\big(B(t)B^*(t)\big)^{-1}$, $t \in [t_1 - h, t_1]$ *is essentially bounded.*

---

* This research was supported by the National Science Foundation under Grant No. GP-33882.