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MINIMAL-TIME DELAY FEEDBACK

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The idea of using delay feedback in order to solve minimal time problems was suggested first by V. M. Popov as an application of pointwise degeneracy of delay systems.

In this paper, which represents joint work with B. Asner Jr., we shall show mainly how the minimal time delay feedback can be constructed for discrete systems and point out analogies and differences with respect to the continuous case.

1. The problem

Consider a discrete-time control system

$$x_{n+1} = Ax_n+u_n, \qquad A \text{ invertible,}$$
$$y_n = Q^*x_n, \qquad n \geqslant 0.$$

Here we assume A to be a $d\times d$ matrix (with complex entries) and Q a $d\times p$ matrix. We want to construct a control u_n (by using past information concerning the state) with the property that for all solutions y_n vanishes if $n \geqslant l$; moreover, we want l to be minimal. We assume that in the construction of u_n we are allowed to use only information concerning x_j for $j \leqslant n-(k-1)$, $k \geqslant 1$. This is the delay feedback, and we shall look for linear feedback of the form

$$u_n = \sum_{j=0}^{q} P_j x_{n-(k-1)-j}.$$

By using such feedback control we get the system

$$x_{n+1} = Ax_n+\sum_{j=0}^{q} P_j x_{n-(k-1)-j}.$$

It is very easy to see that for such a system we cannot have all outputs vanishing for $n \geqslant l$ if $l < k$. It is enough to take $x_i = 0$ for $i < 0$ and see that the corresponding solution gives $x_{k-1} = A^{k-1}x_0$; hence $Q^*x_{k-1} = 0$ is not possible for every x_0 (A is invertible). Thus the minimal time is $\geqslant k$.

2. The construction

We shall now propose a construction that gives $Q^*x_n = 0$ for $n \geqslant k$, i.e., a construction for the minimal-time delay feedback.

Let N be a matrix whose columns define a basis for the subspace defined by $Q^*x = 0$.

We shall assume that (A^k, N) is completely controllable and define m as the minimal integer such that $\operatorname{rank}(NA^kN \ldots A^{mk}N) = d$. Assume $k \geqslant 2$; the case $k = 1$ is trivial. Choose matrices $D_j, j = 1, \ldots, m$, such that $Q^*A^iD_j = 0$ for $i = 0, \ldots, k-2$; D_j are $d \times d$ matrices.

Construct matrices $F_1, \ldots, F_{m+1}$ such that

$$NF_{m+1}+A^kNF_m+ \ldots +A^{mk}NF_1 = A^{(m+1)k}+\sum_{j=1}^{m} A^{jk-1}D_{m+1-j};$$

N is $d\times(d-p)$, F_j are $(d-p)\times d$.

The construction is possible because of our rank assumption. Choose

$$C_1 = -A^k+NF_1, \quad C_{j+1} = A^kC_j-A^{k-1}D_j+NF_{j+1}, \quad j = 1, \ldots, m-1,$$
$$B_j = -AC_j+D_j,$$

and consider the system

$$x_{n+1} = Ax_n+\sum_{j=1}^{m}(B_jx_{n-jk}+C_jx_{n+1-jk}).$$

We claim that this system has the property that for all solutions

$$Q^*x_n = 0 \quad \text{for} \quad n \geqslant k.$$

To prove that we consider an arbitrary solution x_n, define $z_n = x_n-\sum_{j=1}^{m} C_jx_{n-jk}$, use the variation of constants formula to compute $z_n-A^kz_{n-k}$ for $n \geqslant$ k, use the construction of matrices D_j to get

$$Q^*z_n-Q^*A^kz_{n-k} = \sum_{j=1}^{m} Q^*A^{k-1}D_jx_{n-k-jk},$$

return to x_n, use the construction for C_j to deduce finally that

$$Q^*x_n = Q^*(A^{k-1}D_m-A^kC_m)x_{n-k-mk}, \quad n \geqslant k,$$

and the way in which matrices F_j have been constructed gives the conclusion.

It can be seen that we have in the construction many free parameters which we may use to satisfy some other possible requirements for the control.

Now we would like to know whether the construction we have described is the only one. This is actually the case if we consider feedback of the form $u_n = \sum B_jx_{n-jk}+ +C_jx_{n+1-jk}$ but we cannot say anything concerning other situations.

3. The continuous case

The corresponding problem in the continuous case is the following. We consider the control system

$$\dot{x} = Ax+u$$

and we want to construct a feedback depending upon delayed values of the state, with a delay h, such that an output of the form $y = Q^*x$ vanishes for $t \geqslant t_1$; moreover we want t_1 to be minimal. The simplest construction is the one proposed by Popov for the case of a scalar output and of a feedback of the form $u(t) = Bx(t-h)$. In this case the minimal time is $2h$ and under some independence conditions the construction of B can actually be performed. The independence conditions may be relaxed if we use controls of the form $u(t) = B_1x(t-h)+B_2x(t-2h)$ but the minimal time is still $2h$. In our joint work with B. Asner, which preceded the construction described above for the discrete case and suggested it, we remarked that we may realize the minimal time h by using a delay feedback containing also the derivative. More precisely, we may describe a construction for matrices B_j, C_j such that all solutions of the system

$$\dot{x}(t) = Ax(t)+\sum B_jx(t-jh)+C_j\dot{x}(t-jh)$$

satisfy

$$Q^*x(t) = 0 \quad \text{for} \quad t \geqslant h.$$

Let N be as above and assume (e^{Ah}, N) to be completely controllable. Choose m as the smallest integer for which $\operatorname{rank}(Ne^{Ah}N \ldots e^{mAh}N) = d$. Construct matrices F_i such that

$$NF_{m+1}+e^{Ah}NF_m+ \ldots +e^{mAh}NF_1 = e^{(m+1)Ah}.$$

Choose matrices D_j such that $Q^*e^{At}D_j \equiv 0$ and then with $C_0 = -I$ construct $C_i = e^{Ah}C_{j-1}+NF_j$ and $B_j = -AC_j+D_j$. Our rank condition makes the construction possible and we may prove that the system thus obtained has the desired property. For the proof one takes again (compare with the discrete-time solution) $z(t) = x(t)-\sum C_jx(t-jh)$, express z using the variation of the constants formula, compute $z(t)-e^{Ah}z(t-h)$, and use the definition of z to get

$$Q^*x(t)-\sum_{j=1}^{m+1} Q^*(C_j-e^{Ah}C_{j-1})x(t-jh) = \sum_{j=1}^{m}\int_{t-h}^{t} Q^*e^{A(t-s)}(B_j+AC_j)x(s-jh)\,ds;$$

recalling that $B_j+AC_j = D_j$ and $Q^*e^{At}D_j = 0$, and taking into account that $Q^*(C_j-e^{Ah}C_{j-1}) = Q^*NF_j = 0$ for $j = 1, \ldots, m$, we deduce that $Q^*x(t) = 0$ for $t \geqslant h$, because, with $C_{m+1} = 0$, the definition of the F_j will give $Q^*e^{Ah}C_m = 0$. A similar computation shows that this structure is necessary for $Q^*x(t) = 0$, $t \geqslant h$.

4. Some comments

It is useful to compare the construction in the continuous case to that in the discrete case. The discrete case allows more freedom in choosing D_j; for instance, if (Q^*, A)

is completely observable, in the continuous case we must take $D_j = 0$. There is also a difference in the construction for F_j. We may also remark that both in the continuous and in the discrete case we are not obliged to restrict ourselves to the minimal number m of lag terms if that may be useful for other purposes. We would like to point out that one of the main disadvantages of the results described is that we suppose the control to take values in the same space as the state. We do not know any results for the case where the control space has lower dimension. There is one result known for the case where the derivatives are not used: namely we cannot realize our goal with a scalar feedback of the form

$$u(t) = \sum_{j=1}^{m} c_j^* x(t-jh).$$

It is not known if the same is true when we are allowed to use more general delay feedback of the form

$$u(t) = \int_{-h}^{0} d\eta(s) x(t+s).$$

It is also not known whether the same negative answer is obtained by using controls of the form $u(t) = b^* x(t-h) + c^* \dot{x}(t-h)$.

For the discrete-time case, as pointed out also by Charrier, it is not known whether by using feedback of the form $u_n = \sum_{j=1}^{m} B_j x_{n-jk}$ the minimal time l for which we may have $Q^* x_n = 0$, $n \geqslant l$, is just $2k$. For the same discrete case it is not known whether degeneracy is impossible if we use only scalar controls.

It is thus seen that there are still many open questions connected with the simple problem we have considered.

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ON LOCAL CONTROLLABILITY

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Let M^n be an analytic n-dimensional manifold and $\mathscr{D} = \{X^\alpha\colon \alpha \in A\}$ be a collection of analytic tangent vector fields on M. A continuous map $\varphi\colon [0, T] \to M^n$, $T > 0$, is called a *solution* of $\mathscr{D}$ if there exists a partition $0 = t_0 < t_1 < \ldots < t_k = T$ such that on each interval $[t_i, t_{i+1})$, $\dot{\varphi}(t) = X^{\alpha_i}(\varphi(t))$ for some $\alpha_i \in A$. If $M^n = R^n$ and $X^\alpha(x) = f(x, \alpha)$, $\alpha \in A \subset R^m$, then studying solutions of $\mathscr{D}$ is equivalent to studying solutions of $\dot{x} = f(x, \alpha(t))$ where α is a piecewise constant control.

Let φ^* be a fixed (reference) solution of $\mathscr{D}$ with $\varphi^*(0) = p$ and let $\mathscr{A}(t, p, \mathscr{D})$ denote the set of all points attainable at time t by solutions of $\mathscr{D}$ initiating from p at time 0. We define: $\mathscr{D}$ is *locally controllable along* φ^* *at* p if for all $t > 0$, $\varphi^*(t) \in \operatorname{int} \mathscr{A}(t, p, \mathscr{D})$. Clearly, if $\mathscr{D}$ is locally controllable along φ^* at p then $\varphi^*(t) \in \operatorname{int} \mathscr{A}(t, p, \mathscr{D})$ for *all* $t > 0$ and cannot be a time optimal trajectory. Since the Pontriagin maximum principle gives a necessary condition for $\varphi^*(t)$ to belong to the boundary of $\mathscr{A}(t, p, \mathscr{D})$, it is of particular interest to study "singular cases" (i.e., when the maximum principle gives no information; see [2] or [3].) For this reason, and for ease of exposition, we confine our interest, mainly, to systems of the form

$$\mathscr{D} = \{X + \alpha Y\colon\ -1 \leqslant \alpha \leqslant 1\} \tag{1}$$

with X, Y analytic vector fields on M^n_ω, and reference trajectory denoted $T^X(\cdot)p$, the solution of $\dot{x} = X(x)$, $x(0) = p$.

Consider the set, $V(M)$, of all analytic vector fields on M as a real Lie algebra with product the Lie product denoted $[X, Y]$. For $\mathscr{C} \subset V(M)$, let $\tau(\mathscr{C})$ denote the smallest subalgebra containing $\mathscr{C}$, let TM_p denote the tangent space to M at p and let $\mathscr{C}_p = \{X(p) \in TM_p\colon X \in \mathscr{C}\}$. We write $(\operatorname{ad} X, Y) = [X, Y]$, $(\operatorname{ad}^k X, Y) = [X, (\operatorname{ad}^{k-1} X, Y)]$. Now define

$$\mathscr{S}^0 = \{Y, (\operatorname{ad} X, Y), (\operatorname{ad}^2 X, Y), \ldots\}.$$

It is known that rank $\mathscr{S}^0_p = n$ is a sufficient condition for $\mathscr{D}$ to be locally controllable along T^X at p. A necessary condition for local controllability of $\mathscr{D}$ along T^X at p is that $\mathscr{A}(t, p, \mathscr{D})$ have nonempty interior for all $t > 0$. Following [6], we let