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PROJECTION METHODS FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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In this note we present a short summary of recent and continuing investigations involving projection methods for retarded functional differential equations. A more detailed account, along with proofs, may be found in [1].

We consider the n-vector nonhomogeneous retarded equation

$$\begin{aligned} \dot{x}(t) &= L(x_t)+f(t), \quad t \in [0, t_1], \\ x_0 &= \xi, \end{aligned} \tag{1}$$

where $x_t \in \mathscr{C} \equiv C([-r, 0], \boldsymbol{R}^n)$ denotes the n-dimensional column vector valued function $\theta \to x(t+\theta)$, $\theta \in [-r, 0]$ and L is a continuous linear functional on $\mathscr{C}$ given by

$$L(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta).$$

The $n \times n$ matrix function η is, by the familiar Riesz theorem, of bounded variation on $[-r, 0]$. We denote the solution of (1) corresponding to initial data $\xi \in \mathscr{C}$ and nonhomogeneous term $f \in L_1([0, t_1], \boldsymbol{R}^n)$ by $x_t(\xi, f)$. Considering the homogeneous form of (1), one can define a family of operators $T(t)\colon \mathscr{C} \to \mathscr{C}$, $t \geqslant 0$,

$$T(t)\xi = x_t(\xi, 0)$$

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and easily verify that $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup with infinitesimal generator $\mathscr{A}: \mathscr{D}(\mathscr{A}) \to \mathscr{C}$ given by

$$(\mathscr{A}\varphi)(\theta) = \begin{cases} \dot{\varphi}(\theta), & -r \leqslant \theta < 0, \\ \int_{-r}^{0} d\eta(s)\varphi(s), & \theta = 0, \end{cases}$$

where

$$\mathscr{D}(\mathscr{A}) \equiv \Big\{\varphi \in \mathscr{C} \mid \varphi \text{ is } C^1 \text{ and } \dot{\varphi}(0) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta)\Big\}.$$

The spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ consists of only point spectrum and one can argue that $\lambda \in \sigma(\mathscr{A}) = \pi(\mathscr{A})$ if and only if $\det \Delta(\lambda) = 0$ where

$$(2) \qquad \Delta(\lambda) \equiv \lambda I - \int_{-r}^{0} d\eta(\theta) e^{\lambda\theta}.$$

One can further argue that given any real γ, there are at most a finite number of $\lambda \in \pi(\mathscr{A})$ having $\operatorname{Re}(\lambda) \geqslant \gamma$ and, if one specializes equation (1) to a differential-difference equation, the eigenvalues are asymptotically distributed in certain curvilinear strips (see [2]).

For $\lambda_j \in \pi(\mathscr{A})$, the operator $\mathscr{A} - \lambda_j I$ has finite ascent and thus the generalized eigenspace $\mathscr{M}_{\lambda_j}(\mathscr{A})$ is given by $\mathscr{M}_{\lambda_j}(\mathscr{A}) = \mathscr{N}_{(\mathscr{A}-\lambda_j I)^k}$ for some finite k, where as usual $\mathscr{N}_B$ denotes the null space of the operator B. The subspace $\mathscr{M}_{\lambda_j}(\mathscr{A})$ is finite dimensional, say of dimension d_j, and is invariant under $\mathscr{A}$ and $T(t)$. We denote by Φ_{λ_j} the $n \times d_j$ matrix function

$$(3) \qquad \Phi_{\lambda_j} = [\varphi^1_{\lambda_j}, \varphi^2_{\lambda_j}, \ldots, \varphi^{d_j}_{\lambda_j}],$$

where $\{\varphi^i_{\lambda_j}\}_{i=1}^{d_j}$ is a basis of generalized eigenfunctions (having form $q_j(\theta)e^{\lambda_j\theta}$, q_j a polynomial) for $\mathscr{M}_{\lambda_j}(\mathscr{A})$. A bilinear form on $C([0,r], \mathbf{R}^{n*}) \times \mathscr{C}$ defined by

$$(4) \qquad \langle\psi, \varphi\rangle \equiv \psi(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(s-\theta)\, d\eta(\theta)\varphi(s)\, ds$$

allows one to define an "adjoint" $\mathscr{A}^*$ to $\mathscr{A}$ via the "usual" definition $\langle\mathscr{A}^*\psi, \varphi\rangle = \langle\psi, \mathscr{A}\varphi\rangle$ and argue that $\pi(\mathscr{A}) = \pi(\mathscr{A}^*)$. Furthermore, $\mathscr{M}_{\lambda_j}(\mathscr{A}^*)$ also has dimension d_j and a "basis" $d_j \times n$ matrix function

$$\Psi_{\lambda_j} = \begin{bmatrix} \psi^1_{\lambda_j} \\ \vdots \\ \psi^{d_j}_{\lambda_j} \end{bmatrix}$$

can be chosen so that $\langle\Psi_{\lambda_j}, \Phi_{\lambda_j}\rangle = I$, the $d_j \times d_j$ identity matrix.

The developments above give rise to (continuous) projection operators $P_{\lambda_j}: \mathscr{C} \to \mathscr{M}_{\lambda_j}(\mathscr{A})$ and Q_{λ_j} defined by

$$P_{\lambda_j}\varphi \equiv \Phi_{\lambda_j}\langle\Psi_{\lambda_j}, \varphi\rangle$$

and

$$Q_{\lambda_j} = I - P_{\lambda_j}.$$

These projections decompose the space $\mathscr{C}$ so that one may write

$$\mathscr{C} = \mathscr{C}_P \oplus \mathscr{C}_Q,$$

where

$$\mathscr{C}_P \equiv \{\varphi \in \mathscr{C} \mid \varphi = \Phi_{\lambda_j} b,\ b \in \mathbf{R}^n\},$$
$$\mathscr{C}_Q \equiv \{\varphi \in \mathscr{C} \mid \langle\Psi_{\lambda_j}, \varphi\rangle = 0\}.$$

On the subspace $\mathscr{M}_{\lambda_j}(\mathscr{A}) = P_{\lambda_j}\mathscr{C} = \mathscr{C}_P$ it can be shown that (1) reduces to an ordinary differential equation (actually a linear d_j-vector system of equations).

Carrying out the above procedure for the first N eigenvalues (ordered by decreasing real parts), one obtains a projection P^N onto the sum of the first N generalized eigenspaces. Defining the $n \times \sum_1^N d_j$ matrix function

$$\Phi^N \equiv [\Phi_{\lambda_1}, \ldots, \Phi_{\lambda_N}]$$

and the corresponding $\sum_1^N d_j \times n$ matrix function Ψ^N, one obtains the projection

$$(5) \qquad P^N\varphi \equiv \Phi^N\langle\Psi^N, \varphi\rangle$$

and again $\mathscr{C}$ is decomposed

$$\mathscr{C} = \mathscr{C}_{\mathscr{P}} \oplus \mathscr{C}_{\mathscr{Q}},$$

when $\mathscr{C}_{\mathscr{P}} = \sum_{j=1}^{N} \mathscr{M}_{\lambda_j}(\mathscr{A})$. On $\mathscr{C}_{\mathscr{P}}$ the equation (1) can be reduced to

$$(6) \qquad \begin{aligned} \dot{y}^N(t) &= B^N y^N(t) + \Psi^N(0) f(t), \\ y^N(0) &= \langle\Psi^N, x_0\rangle, \end{aligned}$$

where $y^N(t) \equiv \langle\Psi^N, x_t\rangle$ and B^N is an appropriately chosen matrix.

If β is a real number such that $\{\lambda \in \sigma(\mathscr{A}) \mid \operatorname{Re}\lambda \geqslant \beta\} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, then there are positive constants K, μ such that

$$(7) \qquad |P^N T(t)\varphi| \leqslant Ke^{(\beta-\mu)t}|P^N\varphi| \quad \text{for} \quad t \leqslant 0,$$

$$(8) \qquad |Q^N T(t)\varphi| \leqslant Ke^{(\beta-\mu)t}|Q^N\varphi| \quad \text{for} \quad t \geqslant 0,$$

where $Q^N = I - P^N$. We remark that the left side of (7) has meaning for $t \leqslant 0$ since on $\mathscr{C}_{\mathscr{P}} \equiv P^N\mathscr{C}$ the equation (1) reduces to an ordinary differential equation with solutions existing on $(-\infty, \infty)$.

All of the above results were established some time ago by Hale [4] and Shimanov [11]. These ideas were subsequently employed to study a number of topics in the qualitative theory of functional differential equations including, among others, bifurcation, asymptotic behavior, existence of periodic solutions, and preservation of saddle point properties under nonlinear perturbations. For an up-to-date account of the above ideas along with some of the applications, one should see the monograph by Hale [5].

A question of great interest from several points of view involves the behavior of the projections P^N on solutions of (1) as we allow $N \to \infty$ (i.e., as we include more and more of the spectrum of $\mathscr{A}$ in the above construction). In particular, for what classes Ξ and $\mathscr{F}$ can one establish $P^N x_t(\xi,f) \to x_t(\xi,f)$, $\xi \in \Xi$, $f \in \mathscr{F}$, as $N \to \infty$? An appealing approach to this question might involve use of the estimates (7), (8) in conjunction with certain representation formulae (for solutions of non-homogeneous equations) which may be found in [5]. However, the constants K, μ in (7), (8) depend very much on N and in fact it can be shown that $K = K(N)$ increases as $N \to \infty$. Examples to show that this approach is fraught with difficulties can be found in [1] and [7].

Results such as $P^N x_t \to x_t$ are in reality expansion theorems (in the $\mathscr{C}$-norm) for solutions of (1) in terms of eigenfunctions $e^{\lambda_j t}$ (actually in terms of generalized eigenfunctions). Expansion theorems of this type were considered by Pitt [9] and Bellman and Cooke [2], among others. We sketch briefly the Bellman–Cooke techniques.

Assuming that $f(t) \equiv 0$ for $t > t_1$ and taking the Laplace transform in (1), one obtains

(9) $$\mathscr{L}[x](s) = \Delta^{-1}(s)q(s)$$

where the analytic function q is defined by

(10) $$q(s) = \xi(0) - \int_{-r}^{0}\int_{0}^{\theta} e^{s(\theta-\tau)}d\eta(\theta)\xi(\tau)d\tau + \int_{0}^{t_1} e^{-s\tau}f(\tau)d\tau.$$

Using the inversion formula one finds

(11) $$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\Delta^{-1}(s)q(s)\,ds$$

where γ is such that all roots of $\det\Delta(s) = 0$ lie in the left half plane $\operatorname{Re}(s) < \gamma$. The expression (11) leads in turn to expansion results

(12) $$x(t) = \lim_{l\to\infty}\sum_{\lambda_j \in C_l} p_j(t)e^{\lambda_j t}$$

where $p_j(t)e^{\lambda_j t} = \operatorname{Res}\{e^{st}\Delta^{-1}(s)q(s)\}_{s=\lambda_j}$ and C_l are the contours described in [2], p. 100.

If the Bellman–Cooke expansion terms were the same as those found via use of the Hale–Shimanov projections described above, one might hope to use results found in [2] to guarantee $P^N x_t \to x_t$ as $N \to \infty$. That these terms are indeed the same is shown in [1]; that is, it is demonstrated under very reasonable assumptions ($\xi \in \mathscr{C}$, $f \in L_1([0,t_1],R^n)$, $f(t) = 0$ for $t > t_1$) that, for $t \geqslant t_1$ and $\theta \in [-r, 0]$, $P_{\lambda_j}x_t(\xi,f)(\theta) = p_j(t+\theta)e^{\lambda_j(t+\theta)}$, where p_j is the polynomial in (12). We are thus in a position to use the ideas and methods in [2] to establish, if possible, convergence of the sequences $P^N x_t$.

Turning to the convergence results, we restrict our considerations here to the special case of (1) given by

(13) $$\dot{x}(t) = \sum_{i=0}^{\nu} A_i x(t-h_i) + f(t), \quad t \in [0, t_1],$$
$$x_0 = \xi,$$

where $0 = h_0 < h_1 < \ldots < h_\nu \leqslant r$, so that $\Delta(s) = sI - \sum_{i=0}^{\nu} A_i e^{-sh_i}$. By making use of fairly intricate arguments similar to those found in [2] (some of the estimates given in [2] are not sufficiently sharp and must be improved) one can prove

THEOREM. *Assume that $\mathscr{F} \subset L_1([0,t_1],\boldsymbol{R}^n)$ is bounded and $\det A_\nu \neq 0$. If $t \to x(t,\xi,f)$ denotes the solution of (13) corresponding to $\xi \in \mathscr{C}$, $f \in \mathscr{F}$, then*

$$x(t,\xi,f) = \lim_{l\to\infty}\sum_{\lambda_j \in C_l} p_j(t)e^{\lambda_j t}$$

obtains for $t > t_1 - h_\nu$. In addition, the convergence is uniform in $f \in \mathscr{F}$ and uniform in t on any finite interval $[a,b]$ such that $t_1 - h_\nu < a$.

An almost immediate corollary of the above is

COROLLARY. *Suppose that $\mathscr{F}_\varepsilon$ is a bounded subset of $L_1([0,t_1],\boldsymbol{R}^n)$ which has the property: $f \in \mathscr{F}_\varepsilon \Rightarrow f(t) = 0$ on $(t_1-\varepsilon, t_1)$, where $\varepsilon > 0$ and $t_1 - h_\nu - \varepsilon \geqslant 0$. Then if $\det A_\nu \neq 0$,*

$$P^N x_t(\xi,f) \to x_t(\xi,f)$$

for each $t > t_1 - \varepsilon$, the convergence being uniform in $f \in \mathscr{F}_\varepsilon$.

The condition $f(t) = 0$ on $(t_1-\varepsilon, t_1)$, $\varepsilon > 0$ arbitrary, in order that $P^N x_{t_1}(\xi,f) \to x_{t_1}(\xi,t)$ is related to the boundary condition $\dot{\varphi}(0) = L(\varphi)$, which each φ in $\mathscr{M}_{\lambda_j}(\mathscr{A}) \subset \mathscr{D}(\mathscr{A})$ must satisfy. Since each $P^N x_{t_1}$ is in $\mathscr{D}(\mathscr{A})$, we observe that it is not surprising that one needs $\dot{x}(t_1) = L(x_{t_1})$ in order to establish the convergence results.

Using simple backward continuation arguments, it is not difficult to obtain expansion results for "initial functions" (i.e., $P^N\xi \to \xi$). (The question of expansion of "initial functions" in uniformly convergent series of exponentials was considered for very special cases of (13) by Markushin in [8], but the conclusions there are based on faulty arguments—see [1].)

If one drops the assumption "$\det A_\nu \neq 0$", one can establish only much weaker results. Roughly speaking, $P^N x_{t_1}(\xi,f) \to x_{t_1}(\xi,f)$ as $N \to \infty$ if $f(\tau) = 0$ for $\tau > t_1 - nh_\nu - \varepsilon$. For the nth order scalar equation (for which, of course, we have $\det A_\nu = 0$) somewhat better results can be given which require $f(\tau) = 0$ for $\tau > t_1 - h_\nu - \varepsilon$. That these are essentially the "best" (in some sense) results possible can be demonstrated by considering (13) with $f \equiv 0$. Using results due to Henry [6] on the "ascent" and "descent" of solution operators and their functional analytic adjoints, it is possible to construct homogeneous systems satisfying (here $\mathscr{R}(T(t))$ denotes the range of $T(t)$ on $\mathscr{C}$)

$$\mathscr{R}(T(t)) \not\subset \overline{\operatorname{span}}\,\{\mathscr{M}_\lambda(\mathscr{A}) |\ \lambda \in \sigma(\mathscr{A})\}$$

for $t < nh_\nu$, while for $t \geqslant nh_\nu$ one has

$$\mathscr{R}(T(t)) \subseteq \overline{\operatorname{span}}\ \{\mathscr{M}_\lambda(\mathscr{A}) |\ \lambda \in \sigma(\mathscr{A})\}.$$

We indicate briefly one possible application of the above-outlined projection ideas and results to the theory of optimal control of systems (1), where we take $f(t) = Du(t)$, D being a given $n \times p$ matrix. The problem we consider is that of minimizing J: $L_2 \to \boldsymbol{R}^1$ over the set $V = \{u \in \mathscr{U} \subset L_2 |\ x_{t_1}(\xi, u) = \zeta\}$. Here the functions ξ (initial) and ζ (terminal) are given (from some specified subset of $\mathscr{C}$) and $\mathscr{U}$ is a specified closed linear subspace (or convex subset, in the case constraints on the controls are desirable) of $L_2([0, t_1], \boldsymbol{R}^p)$.

For a fixed positive integer N, we project the problem onto $\mathscr{C}_{\mathscr{P}} = \sum_{j=1}^{N} \mathscr{M}_{\lambda_j}(\mathscr{A})$ (see (5), (6) above) where the differential equation becomes (6) with $f = Du$. The minimization problem thus obtained is one of minimizing J over

$$V^N \equiv \{u \in \mathscr{U} |\ y^N(t_1; u) = \langle \Psi^N, \zeta \rangle\},$$

where, of course, $y^N(t; u)$ denotes the solution at time t of (6) corresponding to u. Let $\bar{u}^N$ denote the solution of this minimization problem (under reasonable assumptions on J this solution exists uniquely). The question of interest is whether the solutions of these sub-problems tend to the solution of the original problem as one lets $N \to \infty$.

By some manipulations which we shall omit here, the sub-problem can be put in the form of the classical problem of minimizing a functional on a Hilbert space subject to a finite number of equality constraints. That is, one seeks to minimize J over

$$V^N = \Big\{u \in \mathscr{U} |\ \langle g_i, u \rangle_2 = c_i,\ i = 1, 2, \ldots, \sum_{j=1}^{N} d_j\Big\},$$

where c_i, g_i are properly chosen and $\langle\ ,\ \rangle_2$ is the inner product in L_2.

We list some hypotheses of use here:

H1. *J is strictly quasiconvex* ([10], [3]) *and lower semicontinuous on L_2.*

H2. *For any $K \subset L_2$, J bounded on K implies that K is bounded.*

H3. *The system* (1) *and admissible controls $\mathscr{U}$ are chosen such that $P^N x_{t_1}(\xi, u) \to x_{t_1}(\xi, u)$ for ξ fixed in $\mathscr{C}$ and $u \in \mathscr{U}$.*

H4. *J is strongly convex* [10] *on L_2.*

The following theorem is proved in [1].

THEOREM. *Suppose* H1, H2, H3. *Then $J(\bar{u}^N) \nearrow J(u^*)$ and $\bar{u}^N \rightharpoonup u^*$ in L_2 where u^* is the unique solution to the original problem. Under the added hypotheses (a strengthened form of that in* H1) H4 *one actually has $\bar{u}^N \to u^*$ in L_2.*

A number of problems usually studied by control theorists satisfy the hypotheses in this theorem. For example, included are J of the form

$$J(u) = \mathscr{B}(u, u) + \mathscr{L}(u) + \mathscr{K}, \tag{14}$$

where $(u, v) \to \mathscr{B}(u, v)$ is a symmetric continuous bilinear functional satisfying $\mathscr{B}(u, u) \geqslant \delta |u|_{L_2}^2$ for some $\delta > 0$, $u \to \mathscr{L}(u)$ is a continuous linear functional, and $\mathscr{K}$ is a constant.

In particular, if one uses a variation of parameters expression for solutions of (1), the cost functional

$$J(u) = \int_0^{t_1} [x(t)\mathscr{W}x(t) + u(t)\mathscr{V}u(t)]\,dt$$

is seen to have the form (14). Also included, of course, is the minimum norm problem, $J(u) = |u|^2$. In this latter case, the projected problems become those of finding minimal norm elements of V^N. Using well-known projection methods in Hilbert space (and the corresponding alignment condition), one can actually compute a closed form solution for the minimal norm element $\bar{u}^N$ in terms of the λ_j, ξ, ζ, Ψ_{λ_j}, and D.

From a theoretical viewpoint, use of the above projection ideas in optimal control problems appear to have some advantages. We are currently continuing our investigations (see [1]) into the practical feasibility of such methods for computing approximations to optimal controls.

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