to $y$ for each $x \in \bar{E}_0$ and that $V_x \neq \emptyset$ for every $x \in E$. Also the considerations of § 5 remain valid, because the invertibility of the function $G(x, y)$ occurs only for $x \in \bar{E}_0$.

## References

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[2] S. Łojasiewicz, *Solution générale de l'équation fonctionnelle $f^n(x) = g(x)$*, Ann. de la Soc. Polon. Math. 24 (1951), p. 88-91.

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# On the functional equation $F(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]) = 0$

by J. Kordylewski (Kraków)

The present paper contains results regarding the existence of continuous solutions of the functional equation

$$F(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]) = 0, \tag{1}$$

in which the function $\varphi(x)$ is the unknown function, $f(x)$ and $F(x, y_0, \ldots, y_n)$ are given functions, and $f^\nu(x)$ denotes the $\nu$-th iteration of the function $f(x)$, i. e.

$$f^0(x) = x,$$
$$f^\nu(x) = f[f^{\nu-1}(x)], \quad f^{-\nu}(x) = f^{-1}[f^{-\nu+1}(x)], \quad \nu = 1, 2, 3, \ldots$$

Equation (1) is a generalization of the equation

$$F(x, \varphi(x), \varphi[f(x)]) = 0, \tag{2}$$

which was discussed in [1], and is a particular case of the equation

$$F(x, \varphi(x), \varphi[f_1(x)], \varphi[f_2(x)], \ldots, \varphi[f_n(x)]) = 0 \tag{3}$$

(where the functions $f_\nu(x)$ ($\nu = 1, 2, \ldots, n$) are known functions), which was discussed in [2]. Equation (2) is a particular case of equation (3); nevertheless the theorem on the existence of continuous solutions of equation (3) does not, in the case of $n = 1$, pass into the corresponding theorem for equation (2); the hypotheses made with regard to equation (3) are stronger. M. Kuczma has raised the problem of proving the existence of continuous solutions of equation (1) under such assumptions (weaker than the hypotheses on equation (3)) that in the case of $n = 1$ the theorem should pass into the corresponding theorem for equation (2). The present paper is a solution of this problem: we shall show that under suitable assumptions equation (1) possesses infinitely many continuous solutions.

We assume the following hypotheses regarding the function $f(x)$:

(i) The function $f(x)$ is defined, continuous and strictly increasing in an interval $\langle a, b\rangle$; moreover, let $f(a) = a$, $f(b) = b$, $f(x) > x$ for $x \in (a, b)$.

Under these assumptions the following lemma (proved in [3]) is valid:

LEMMA I. *If the function $f(x)$ fulfils hypotheses* (i), *then for each $x \in (a, b)$ the sequences $\{f^{\nu}(x)\}$ and $\{f^{-\nu}(x)\}$ are monotone and*

$$\lim_{\nu\to\infty} f^{\nu}(x) = b, \qquad \lim_{\nu\to\infty} f^{-\nu}(x) = a.$$

Let us assume further that:

(ii) The function $F(x, y_0, \dots, y_n)$ is defined in a certain set $\Omega$ of $(n+2)$-dimensional space of the variables $(x, y_0, \dots, y_n)$.

Now we introduce the following notation:

$\mathfrak{F}$ denotes the set of such points $P(x, y_0, \dots, y_n)$ that $P \in \Omega$ and $F(P) = 0$.

$\mathfrak{F}^0$ denotes the projection of the set $\mathfrak{F}$ onto the subspace of the variables $(x, y_1, \dots, y_n)$.

$\mathfrak{F}^n$ denotes the projection of the set $\mathfrak{F}$ onto the subspace of the variables $(x, y_0, \dots, y_{n-1})$.

$\mathfrak{F}_x$ denotes the set of such points $Q(y_0, \dots, y_n)$ that the point $P(x, y_0, \dots, y_n)$ belongs to $\mathfrak{F}$.

$\mathfrak{F}_x^0$ denotes the projection of the set $\mathfrak{F}_x$ onto the subspace of the variables $(y_1, \dots, y_n)$.

$\mathfrak{F}_x^n$ denotes the projection of the set $\mathfrak{F}_x$ onto the subspace of the variables $(y_0, \dots, y_{n-1})$.

We assume further that:

(iii) For each $x \in (a, b)$ the set $\mathfrak{F}_x$ is non-empty.

It is apparent that then also the sets $\mathfrak{F}$, $\mathfrak{F}^0$ and $\mathfrak{F}^n$ and for each $x \in (a, b)$ the sets $\mathfrak{F}_x^0$ and $\mathfrak{F}_x^n$ are non-empty.

Lastly, let us suppose that:

(iv) For each $x \in (a, b)$

$$\mathfrak{F}^n_{f(x)} = \mathfrak{F}^0_x. \tag{4}$$

LEMMA II. *Under the hypotheses* (i)-(iv) *equation* (1) *possesses at least one solution $\varphi(x)$ defined in the interval $(a, b)$.*

Proof. From the definition of the sets $\mathfrak{F}^n$ and $\mathfrak{F}^0$ it follows that there exist at least one function $G(x, y_0, \dots, y_{n-1})$ defined in the set $\mathfrak{F}^n$ such that

$$F\big(x, y_0, \dots, y_{n-1}, G(x, y_0, \dots, y_{n-1})\big) \equiv 0 \quad \text{in} \quad \mathfrak{F}^n, \tag{5}$$

and at least one function $H(x, y_1, \dots, y_n)$ defined in the set $\mathfrak{F}^0$ such that

$$F\big(x, H(x, y_1, \dots, y_n), y_1, \dots, y_n\big) \equiv 0 \quad \text{in} \quad \mathfrak{F}^0. \tag{6}$$

Let $x_0$ be an arbitrary point of the interval $(a, b)$. We put

$$x_\nu = f^{\nu}(x_0), \qquad \nu = \pm 1, \pm 2, \pm 3, \dots$$



From hypothesis (iii) it follows that there exist functions $y_0(x), \dots, y_{n-1}(x)$ defined in the interval $\langle x_0, f(x_0)\big)$ such that

$$\big(x, y_0(x), \dots, y_{n-1}(x)\big) \in \mathfrak{F}^n \quad \text{for} \quad x \in \langle x_0, f(x_0)\big). \tag{7}$$

We shall show that the formulae

$$\varphi(x) = \begin{cases} y_\nu[f^{-\nu}(x)] & \text{for} \quad x \in \langle x_\nu, x_{\nu+1}), \quad \nu = 0, 1, \dots, n-1, \\ G\big(f^{-n}(x), \varphi[f^{-n}(x)], \varphi[f^{-n+1}(x)], \dots, \varphi[f^{-1}(x)]\big) & \\ \qquad \text{for} \quad x \in \langle x_\nu, x_{\nu+1}), \quad \nu = n, n+1, \dots, & \\ H\big(x, \varphi[f(x)], \varphi[f^2(x)], \dots, \varphi[f^n(x)]\big) & \\ \qquad \text{for} \quad x \in \langle x_\nu, x_{\nu+1}), \quad \nu = -1, -2, \dots & \end{cases} \tag{8}$$

define the function $\varphi(x)$ for every $x \in (a, b)$ and that the function $\varphi(x)$ thus defined satisfies equation (1) in the interval $(a, b)$.

On account of lemma I we have

$$(a, b) = \bigcup_{-\infty}^{+\infty} \langle x_\nu, x_{\nu+1}) = (a, x_0) + \langle x_0, x_n) + \langle x_n, b). \tag{9}$$

The proof of the existence of the function $\varphi(x)$ will be carried out independently for each of the intervals $(a, x_0)$, $\langle x_0, x_n)$ and $\langle x_n, b)$. Proving that the function $\varphi(x)$ is defined in the interval $\langle x_n, b)$ we shall show that it satisfies the equation in the interval $\langle x_0, b)$, and proving that the function $\varphi(x)$ is defined in the interval $(a, x_0)$ we shall show that it satisfies the equation in the interval $(a, x_0)$.

I. Let us take an arbitrary $x \in \langle x_0, x_n)$. There exists a $\nu$ $(0 \leqslant \nu < n)$ such that $x \in \langle x_\nu, x_{\nu+1})$. Since $f^{-\nu}(x) \in \langle x_0, x_1)$, $y_\nu[f^{-\nu}(x)]$ has a meaning and consequently $\varphi(x)$ has a meaning. Since $x$ has been an arbitrary point of the interval $\langle x_0, x_n)$, formulae (8) define the function $\varphi(x)$ in the interval $\langle x_0, x_n)$.

II. For the interval $\langle x_n, b)$ the proof will be by induction. Let us take an arbitrary $x \in \langle x_n, x_{n+1})$. Since for $\nu = 1, 2, \dots, n$

$$f^{-\nu}(x) \in \langle x_{n-\nu}, x_{n-\nu+1}) \subset \langle x_0, x_n),$$

$\varphi[f^{-\nu}(x)]$ $(\nu = 1, 2, \dots, n)$ has, on account of the first part of the proof, a meaning and we have by (8)

$$\varphi[f^{-\nu}(x)] = y_{n-\nu}\big[f^{-n+\nu}\big(f^{-\nu}(x)\big)\big] = y_{n-\nu}[f^{-n}(x)] \quad \text{for} \quad \nu = 1, 2, \dots, n.$$

Hence we have according to (7), since $f^{-n}(x) \in \langle x_0, x_1)$,

$$\big(f^{-n}(x), \varphi[f^{-n}(x)], \varphi[f^{-n+1}(x)], \dots, \varphi[f^{-1}(x)]\big)$$
$$= \big(f^{-n}(x), y_0[f^{-n}(x)], y_1[f^{-n}(x)], \dots, y_{n-1}[f^{-n}(x)]\big) \in \mathfrak{F}^n.$$

Since the function $G$ is defined in the set $\mathfrak{F}^n$, $G(f^{-n}(x), \varphi[f^{-n}(x)], \varphi[f^{-n+1}(x)], \ldots, \varphi[f^{-1}(x)])$ has a meaning, and by (5)

$$F\Big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)], G\big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big)\Big) = 0.$$

Consequently, $\varphi(x)$ has a meaning and equation (1) is satisfied at the point $f^{-n}(x)$.

Since $x$ has been an arbitrary point of the interval $\langle x_n, x_{n+1})$ and

$$f^{-n}\big(\langle x_n, x_{n+1})\big) = \langle x_0, x_1),$$

formulae (8) define the function $\varphi(x)$ in the interval $\langle x_n, x_{n+1})$ and equation (1) is satisfied in the interval $\langle x_0, x_1)$.

Now let us suppose that formulae (8) define the function $\varphi(x)$ in an interval $\langle x_n, x_{n+k})$ $(k \geqslant 1)$ and that it satisfies equation (1) in the interval $\langle x_0, x_k)$. We shall show that formulae (8) define the function $\varphi(x)$ in the interval $\langle x_n, x_{n+k+1})$ and that it satisfies equation (1) in the interval $\langle x_0, x_{k+1})$.

Let us take an arbitrary $x \in \langle x_{n+k}, x_{n+k+1})$. Since for $\nu = 1, 2, \ldots, n+1$

$$f^{-\nu}(x) \in \langle x_{n+k-\nu}, x_{n+k+1-\nu}) \subset \langle x_0, x_{n+k}),$$

$\varphi[f^{-\nu}(x)]$ $(\nu = 1, 2, \ldots, n+1)$ has a meaning in view of the first part of the proof and of the inductive hypothesis. Since $f^{-n-1}(x) \in \langle x_{k-1}, x_k) \subset \langle x_0, x_k)$ in virtue of the inductive hypothesis equation (1) is satisfied at the point $f^{-n-1}(x)$ and we have

$$F\big(f^{-n-1}(x), \varphi[f^{-n-1}(x)], \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big) = 0.$$

Consequently,

$$\big(f^{-n-1}(x), \varphi[f^{-n-1}(x)], \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big) \in \mathfrak{F},$$

whence

$$\big(\varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big) \in \mathfrak{F}^0_{f^{-n-1}(x)}.$$

On account of hypothesis (iv)

$$\big(\varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big) \in \mathfrak{F}^n_{f^{-n}(x)},$$

i. e.

$$\big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big) \in \mathfrak{F}^n.$$

Consequently, $G(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)])$ has a meaning and by (5)

$$F\Big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)], G\big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big)\Big) = 0.$$

Thus $\varphi(x)$ has a meaning and equation (1) is satisfied at the point $f^{-n}(x)$.

Since $x$ has been an arbitrary point of the interval $\langle x_{n+k}, x_{n+k+1})$ and

$$f^{-n}\big(\langle x_{n+k}, x_{n+k+1})\big) = \langle x_k, x_{k+1}),$$



formulae (8), on account of the inductive hypothesis, define the function $\varphi(x)$ in the interval $\langle x_n, x_{n+k+1})$, and it satisfies equation (1) in the interval $\langle x_0, x_{k+1})$. Thus, by way of induction, according to (9), we conclude that formulae (8) define the function $\varphi(x)$ in the interval $\langle x_n, b)$ and that $\varphi(x)$ satisfies equation (1) in the interval $\langle x_0, b)$.

III. Modifying slightly the second part of the proof we obtain the proof for the interval $(a, x_0)$.

Remark. If in the statement of lemma II we require the existence of a solution of equation (1) in an interval $(c, b)$, where $c \in (a, b)$, then it would be sufficient to postulate in hypothesis (iv) instead of relation (4) the inclusion

$$\mathfrak{F}^n_{f(x)} \supset \mathfrak{F}^0_x \quad \text{for} \quad x \in (c, b).$$

Requiring the existence of a solution in an interval $(a, c)$ it is enough to postulate that

$$\mathfrak{F}^n_{f(x)} \subset \mathfrak{F}^0_x \quad \text{for} \quad x \in (a, c).$$

Now let us assume additionally that:

(v) There exist at least one function $G(x, y_0, \ldots, y_{n-1})$ defined and continuous in the set $\mathfrak{F}^n$, fulfilling relation (5), and at least one function $H(x, y_1, \ldots, y_n)$, defined and continuous in the set $\mathfrak{F}^0$, fulfilling relation (6), and, moreover, that at least one of the conditions

$$(10)\quad y_0 \equiv H\big(x, y_1, \ldots, y_{n-1}, G(x, y_0, \ldots, y_{n-1})\big) \quad \text{for every} \quad (x, y_0, \ldots, y_{n-1}) \in \mathfrak{F}^n,$$

$$(11)\quad y_n \equiv G\big(x, H(x, y_1, \ldots, y_n), y_1, \ldots, y_{n-1}\big) \quad \text{for every} \quad (x, y_1, \ldots, y_n) \in \mathfrak{F}^0$$

is fulfilled.

(vi) There exists an $x_0 \in (a, b)$ such that each two points of the set $\mathfrak{F}^n$

$$P_1(x_0, c_0, \ldots, c_{n-1}), \qquad P_2(f(x_0), c_1, \ldots, c_n)$$

can be joined by a continuous curve, passing within the set $\mathfrak{F}^n$, of the form

$$\begin{cases} x = t, \\ y_0 = y_0(t), \qquad t \in \langle x_0, f(x_0)\rangle, \\ \cdots\cdots\cdots \\ y_{n-1} = y_{n-1}(t), \end{cases}$$

i. e. the functions $y_\nu(t)$ $(\nu = 0, 1, \ldots, n-1)$ are defined and continuous in the interval $\langle x_0, f(x_0)\rangle$, $y_\nu(x_0) = c_\nu$, $y_\nu[f(x_0)] = c_{\nu+1}$ $(\nu = 0, 1, \ldots, n-1)$ and

$$P\big(t, y_0(t), \ldots, y_{n-1}(t)\big) \in \mathfrak{F}^n \quad \text{for} \quad t \in \big(x_0, f(x_0)\big).$$

Remark. Concerning the beginning of the formulation of hypothesis (vi) it can be shown (which is done in the proof of lemma III) that if a point $P_1(x_0, c_0, \ldots, c_{n-1})$ belongs to $\mathfrak{F}^n$ then there exists a $c_n$ such that the point $P_2(f(x_0), c_1, \ldots, c_n)$ belongs to $\mathfrak{F}^n$.

Remark. In hypothesis (vi) the set $\mathfrak{F}^n$ can be replaced by the set $\mathfrak{F}^0$. Both forms of hypothesis (vi) are equivalent: from the fulfilment of hypothesis (vi) for the set $\mathfrak{F}^n$ follows the fulfilment for the set $\mathfrak{F}^0$, and conversely.

Lemma III. *Under hypotheses* (i)-(vi) *equation* (1) *possesses at least one solution* $\varphi(x)$ *defined and continuous in the interval* $(a, b)$.

Proof. Suppose that in formulae (8) the functions $G$ and $H$ are those for which hypothesis (v) with condition (10) is fulfilled (if we supposed the fulfilment of relation (11), the proof would run analogically). Let hypothesis (vi) be fulfilled for the point $x_0$. We shall show that in the proof of lemma II the functions $y_\nu(x)$ $(\nu = 0, 1, \ldots, n-1)$ can be chosen in such a manner that the function $\varphi(x)$ defined by formulae (8) (and satisfying equation (1)) be a continuous function.

From hypothesis (iii) it follows that there exists a point $P_1(x_0, c_0, \ldots, c_{n-1}) \in \mathfrak{F}^n$. Since the function $G$ is defined in the set $\mathfrak{F}^n$, $G(x_0, c_0, \ldots, c_{n-1})$ has a meaning, and (according to (5))

$$F\big(x_0, c_0, \ldots, c_{n-1}, G(x_0, c_0, \ldots, c_{n-1})\big) = 0.$$

Consequently

$$\big(x_0, c_0, \ldots, c_{n-1}, G(x_0, c_0, \ldots, c_{n-1})\big) \in \mathfrak{F},$$

whence

$$\big(c_1, \ldots, c_{n-1}, G(x_0, c_0, \ldots, c_{n-1})\big) \in \mathfrak{F}^0_{x_0}.$$

But on account of hypothesis (iv)

$$\big(c_1, \ldots, c_{n-1}, G(x_0, c_0, \ldots, c_{n-1})\big) \in \mathfrak{F}^n_{f(x_0)},$$

i. e. the point

$$P_2\big(f(x_0), c_1, \ldots, c_{n-1}, G(x_0, c_0, \ldots, c_{n-1})\big) \in \mathfrak{F}^n.$$

According to hypothesis (vi) there exists a sequence of functions $\{y_\nu(x)\}$ $(\nu = 0, 1, \ldots, n-1)$ defined and continuous in the interval $\langle x_0, f(x_0)\rangle$ such that

$$P\big(x, y_0(x), \ldots, y_{n-1}(x)\big) \in \mathfrak{F}^n \quad \text{for} \quad x \in \big(x_0, f(x_0)\big)$$

and

$$\begin{aligned} y_\nu(x_0) &= c_\nu \quad &&\text{for} \quad \nu = 0, 1, \ldots, n-1, \\ y_\nu[f(x_0)] &= c_{\nu+1} \quad &&\text{for} \quad \nu = 0, 1, \ldots, n-2, \\ y_{n-1}[f(x_0)] &= G(x_0, c_0, \ldots, c_{n-1}). \end{aligned} \tag{12}$$



It follows by (8), on account of the continuity of the functions $y_\nu(x)$, $G$ and $H$, that the function $\varphi(x)$ is continuous for $x \in \langle x_\nu, x_{\nu+1})$ $(\nu = 0, \pm 1, \pm 2, \ldots)$. It is thus enough to prove the continuity of the function $\varphi(x)$ at the points $x_\nu$ $(\nu = 0, \pm 1, \pm 2, \ldots)$.

Continuity for $x_1, x_2, \ldots, x_{n-1}$ follows from relations (12). For $x_n, x_{n+1}, x_{n+2}, \ldots$ and $x_0, x_{-1}, x_{-2}, \ldots$ the proof will be by induction.

For $x = x_n$ we have by (8) and (12), on account of the continuity of the functions $G$ and $y_{n-1}$ and of the function $\varphi(x)$ for $x \in \langle x_n, x_{n+1})$,

$$\lim_{x \to x_n -} \varphi(x) = \lim_{x \to x_n -} y_{n-1}[f^{1-n}(x)] = y_{n-1}(x_1) = G(x_0, c_0, \ldots, c_{n-1}),$$

$$\lim_{x \to x_n +} \varphi(x) = \varphi(x_n) = G\big(f^{-n}(x_n), \varphi[f^{-n}(x_n)], \varphi[f^{-n+1}(x_n)], \ldots, \varphi[f^{-1}(x_n)]\big)$$

$$= G\big(x_0, \varphi(x_0), \ldots, \varphi(x_{n-1})\big) = G\big(x_0, y_0(x_0), \ldots, y_{n-1}(x_0)\big) = G(x_0, c_0, \ldots, c_{n-1});$$

thus the function $\varphi(x)$ is continuous for $x = x_n$.

Now let us suppose that the function $\varphi(x)$ is continuous for $x_n, x_{n+1}, \ldots, x_{n+k}$ $(k \geqslant 0)$. We have by (8), on account of the continuity of the function $G$, and of the function $\varphi(x)$ for $x \in \langle x_{n+k+1}, x_{n+k+2})$,

$$\lim_{x \to x_{n+k+1} -} \varphi(x) = \lim_{x \to x_{n+k+1} -} G\big(f^{-n}(x), \varphi[f^{-n}(x)], \ldots, \varphi[f^{-1}(x)]\big)$$

$$= G\big(x_{k+1}, \varphi(x_{k+1}), \ldots, \varphi(x_{k+n})\big),$$

$$\lim_{x \to x_{n+k+1} +} \varphi(x) = \varphi(x_{n+k+1}) = G\big(f^{-n}(x_{n+k+1}), \varphi[f^{-n}(x_{n+k+1})], \ldots, \varphi[f^{-1}(x_{n+k+1})]\big)$$

$$= G\big(x_{k+1}, \varphi(x_{k+1}), \ldots, \varphi(x_{k+n})\big);$$

thus the function $\varphi(x)$ is continuous for $x = x_{n+k+1}$. By way of induction it follows that the function $\varphi(x)$ is continuous for $x_n, x_{n+1}, x_{n+2}, \ldots$

Further, we have for $x = x_0$, on account of (8), (10), (12), of the continuity of the functions $H$ and $y_0$ and of the function $\varphi(x)$ for $x_1, x_2, \ldots, x_n$ and for $x \in \langle x_0, x_1)$,

$$\lim_{x \to x_0 +} \varphi(x) = \varphi(x_0) = y_0(x_0) = c_0,$$

$$\lim_{x \to x_0 -} \varphi(x) = \lim_{x \to x_0 -} H\big(x, \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]\big)$$

$$= H\big(x_0, c_1, \ldots, c_{n-1}, G(x_0, c_1, \ldots, c_{n-1})\big) = c_0;$$

thus the function $\varphi(x)$ is continuous for $x = x_0$.

Now let us suppose that the function $\varphi(x)$ is continuous for $x_0, x_{-1}, \ldots, x_{-k}$ $(k \geqslant 0)$. We have by (8), on account of the continuity of the function $H$ and of the function $\varphi(x)$ for $x_{-k}, x_{-k+1}, \ldots, x_{-k+n-1}$ and for $x \in \langle x_{-k-1}, x_{-k})$,

$$\lim_{x \to x_{-k-1}+} \varphi(x) = \varphi(x_{-k-1}) = H\big(x_{-k-1}, \varphi[f(x_{-k-1})], \varphi[f^2(x_{-k-1})], \ldots, \varphi[f^n(x_{-k-1})]\big)$$

$$= H\big(x_{-k-1}, \varphi(x_{-k}), \varphi(x_{-k+1}), \ldots, \varphi(x_{-k+n-1})\big),$$

$$\lim_{x \to x_{-k-1}-} \varphi(x) = \lim_{x \to x_{-k-1}-} H\big(x, \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]\big)$$

$$= H\big(x_{-k-1}, \varphi(x_{-k}), \varphi(x_{-k+1}), \ldots, \varphi(x_{-k+n-1})\big);$$

thus the function $\varphi(x)$ is continuous for $x = x_{-k-1}$. By way of induction, it follows that the function $\varphi(x)$ is continuous for $x_0, x_{-1}, x_{-2}, \ldots$ This completes the proof of the lemma.

Remark. If in hypothesis (v) we required the existence of the function $G$ or of the function $H$ only, then we should obtain the existence of a continuous solution of equation (1) in the interval $\langle x_0, b)$ or $(a, f^n(x_0))$ respectively. In that case we should not need to postulate relations (10) and (11).

Now let us suppose that:

(ii′) The function $F(x, y_0, \ldots, y_n)$ is defined and continuous in a region $\Omega$ and possesses in it continuous derivatives $\partial F/\partial y_0$ and $\partial F/\partial y_n$, neither of which vanishes in the region $\Omega$.

(ii″) One of the two sets $\mathfrak{F}^n$ and $\mathfrak{F}^0$ is a convex region.

We shall prove the following

THEOREM. *Under the hypotheses* (i), (ii′), (ii″), (iii) *and* (iv) *equation* (1) *possesses infinitely many solutions* $\varphi(x)$ *defined and continuous in the interval* $(a, b)$.

Proof. In view of lemma III it is enough to show that the hypotheses (ii), (v) and (vi) are fulfilled and that there exist infinitely many sequences of functions $\{y_n(x)\}$ $(\nu = 0, 1, \ldots, n-1)$ for which hypothesis (vi) is fulfilled.

The fulfilment of hypothesis (ii) follows from hypothesis (ii′).

Moreover, from hypothesis (ii′) follows the existence of exactly one function $G(x, y_0, \ldots, y_{n-1})$, defined and continuous in the set $\mathfrak{F}^n$ and fulfilling relation (5), and exactly one function $H(x, y_1, \ldots, y_n)$, defined and continuous in the set $\mathfrak{F}^0$ and fulfilling relation (6). Since in this case the equations

$$F(x, y_0, \ldots, y_n) = 0, \quad y_0 = H(x, y_1, \ldots, y_n), \quad y_n = G(x, y_0, \ldots, y_{n-1})$$

represent the same set $\mathfrak{F}$, relations (10) and (11) are also fulfilled, and thus hypothesis (v) is fulfilled.



Now let us suppose that, for instance, the set $\mathfrak{F}^n$ is a convex region (if the set $\mathfrak{F}^0$ is convex, we reason in a similar way). Let us take an arbitrary $x_0 \in (a, b)$ and two points $P_1(x_0, c_0, \ldots, c_{n-1})$ and $P_2(f(x_0), c_1, \ldots, c_n)$ belonging to the set $\mathfrak{F}^n$. Since $\mathfrak{F}^n$ is convex, there exists a continuous curve of the form (12), passing within the region $\mathfrak{F}^n$ and joining the points $P_1$ and $P_2$, and thus hypothesis (vi) is fulfilled.

Since $\mathfrak{F}^n$ is an open set, there exist infinitely many such curves of the form (12), and taking in formulae (8) all sequences $\{y_\nu(x)\}$ $(\nu = 0, 1, \ldots, n-1)$ determinig these curves we obtain infinitely many continuous solutions of equation (1). This completes the proof.

## References

[1] J. Kordylewski and M. Kuczma, *On the functional equation* $F(x, \varphi(x), \varphi[f(x)]) = 0$, Ann. Polon. Math. 7 (1959), p. 21-32.

[2] — *On the functional equation* $F(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) = 0$, Ann. Polon. Math. 8 (1960), p. 55-60.

[3] M. Kuczma, *On the functional equation* $\varphi(x) + \varphi[f(x)] = F(x)$, Ann. Polon. Math. 6 (1959), p. 281-287.

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