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General solution of the functional equation $\varphi[f(x)] = G(x, \varphi(x))$

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The purpose of the present paper is to give the general solution of the functional equation

$$\varphi[f(x)] = G(x, \varphi(x)), \tag{1}$$

under possibly weak suppositions regarding given functions $f(x)$ and $G(x, y)$. In the particular case, where the function $f(x)$ fulfils the condition $f[f(x)] \equiv x$ the general solution of equation (1) has been given in my previous paper [1].

§ 1. Let E be a set of arbitrary elements, which we shall call points, and let us assume that the function $f(x)$ maps the set E onto itself

$$f(E) = E,$$

in a one-to-one manner.

Definition I. Every set A such that $f(A) = A$ will be called a *modulus-set* for the function $f(x)$.

We denote by $f^n(x)$ the n-th iteration of the function $f(x)$, i. e. we put

$$f^0(x) = x,$$

$$f^{n+1}(x) = f[f^n(x)], \quad f^{n-1}(x) = f^{-1}[f^n(x)].$$

Definition II. For every $x \in E$ the set of the points of the form $f^n(x)$, $n = 0, \pm 1, \pm 2, \ldots$ will be called a *cycle* (¹) determined by x.

Two distinct cycles are always disjoint. If A is a modulus-set for the function $f(x)$ and $x_0 \in A$, then the cycle determined by x_0 is contained in A.

The notion of a cycle is fundamental for our further considerations. Since the cycle C determined by a point x_0 is the smallest modulus-set

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(¹) The notion of a cycle has been introduced by S. Łojasiewicz [2].

containing x_0, we can consider equation (1) on the cycle C independently of the rest of the set E. As we shall see in § 3, a solution of equation (1) is on C uniquely determined by its value at the point x_0.

Now we decompose the set E into the sequence of disjoint sets:

$$E_i = \{x: f^i(x) = x,\ f^j(x) \neq x \text{ for } j = 1, 2, \ldots, i-1\}, \quad i = 1, 2, \ldots$$

$$E_0 = \{x: f^j(x) \neq x,\ j = 1, 2, \ldots\} = E - \bigcup_{i=1}^{\infty} E_i.$$

Lemma I. *Each of the sets E_i, $i = 0, 1, 2, \ldots$ is a modulus-set for the function $f(x)$.*

Proof. Let us fix an arbitrary $i > 0$ and let us take an arbitrary point $x \in E_i$. We have

$$f^i[f(x)] = f^{i+1}(x) = f[f^i(x)] = f(x),$$

and for $0 < j < i$

$$f^j[f(x)] = f^{j+1}(x) = f[f^j(x)] \neq f(x)$$

because $f^j(x) \neq x$, and the function $f(x)$ is single-valued. Thus $f(x) \in E_i$, which proves that $f(E_i) \subset E_i$. Similarly we prove that $f(E_0) \subset E_0$. Thus

(2) $$f(E_i) \subset E_i \quad \text{for} \quad i = 0, 1, 2, \ldots$$

Since the sets E_i are disjoint, $\bigcup_{i=0}^{\infty} E_i = E$ and $f(E) = E$, from relation (2) follows

$$f(E_i) = E_i \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

which was to be proved.

Each of the sets E_i consists of cycles determined by its points. Let us denote (for every i) by E_i^0 a (fixed) set containing exactly one element of every cycle contained in E_i (here we make use of the axiom of choice). We shall now define sets E_i^k (in the sequel we shall denote by $\emptyset$ the empty set):

For $i > 0$:

$$E_i^k \overset{\text{df}}{=} \begin{cases} f^k(E_i^0) & \text{for} \quad k = 0, \ldots, i-1, \\ \emptyset & \text{for the rest of integers } k. \end{cases}$$

For $i = 0$:

$$E_0^k \overset{\text{df}}{=} f^k(E_0^0), \quad k = 0, \pm 1, \pm 2, \ldots$$

The sets E_i^k are disjoint and fulfil the following conditions:

For $i > 0$:

(3) $$f(E_i^k) = E_i^{k+1} \quad \text{for} \quad k \neq -1 \quad \text{and} \quad k \neq i-1, \qquad f(E_i^{i-1}) = E_i^0.$$

For $i = 0$:

(4) $$f(E_0^k) = E_0^{k+1}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Lastly we shall define sets E^n:

$$E^n \overset{\text{df}}{=} \bigcup_{i=0}^{\infty} E_i^n, \quad n = 0, \pm 1, \pm 2, \ldots$$

The sets E^n are disjoint, $\bigcup_{n=-\infty}^{\infty} E^n = E$ (2).

§ 2. Now let us suppose that the set E is a subset of some space X, and let Ξ be some other space (independent of X). Let $G(x, y)$ be a function defined in a subset Ω of the space $X \times \Xi$ (the sign $\times$ denotes here the Cartesian product), and assuming values from Ξ. Let us denote by $G^{-1}(x, y)$ the inverse function of the function $G(x, y)$ with respect to the variable y (provided it exists). We shall now define the sequence of functions $g_n(x, y)$:

$$g_0(x, y) \overset{\text{df}}{=} y,$$

(5) $$g_{n+1}(x, y) \overset{\text{df}}{=} G[f^n(x), g_n(x, y)], \quad n = 0, 1, 2, \ldots,$$

$$g_{n-1}(x, y) \overset{\text{df}}{=} G^{-1}[f^{n-1}(x), g_n(x, y)], \quad n = 0, -1, -2, \ldots$$

Formulae (5) have a formal character for the present. In fact, we do not know whether the function $G(x, y)$ is defined at the point $(f^n(x), g_n(x, y))$ and whether the function $G^{-1}(x, y)$ is defined at the point $(f^{n-1}(x), g_n(x, y))$. If we do not make any assumptions with regard to the function G, then in general they need not be defined.

For every integer n and for every fixed $x \in E$ we denote by Ω_x^n the x-section of the domain of definition of the function $g_n(x, y)$, i. e. the set of values y such that the function $g_n(x, y)$ is defined. More precisely, we make an agreement that if $\Omega_x^N = \emptyset$ for a certain $N > 0$, then $\Omega_x^n = \emptyset$ for every $n > N$, and similarly, if $\Omega_x^N = \emptyset$ for a certain $N < 0$, then $\Omega_x^n = \emptyset$ for every $n < N$. Moreover, we make an agreement that $\Omega_x^n \neq \emptyset$ for $n < 0$ only if the function $G(u, y)$ is invertible with respect to y for each fixed u belonging to the cycle determined by x. Lastly we assume that $\Omega_x^0 = \Xi$ for every $x \in E$.

The following formulae are evident:

(6) $$\Omega_x^{n+1} \subset \Omega_x^n \quad \text{for} \quad n \geqslant 0, \qquad \Omega_x^{n-1} \subset \Omega_x^n \quad \text{for} \quad n \leqslant 0.$$

The reader will easily verify the following

(2) It can be remarked that the upper indices run over the integers from $-\infty$ to $+\infty$, the lower indices run over the integers from 0 to $+\infty$.

Lemma II. *For every $n \geqslant 0$, if one of the functions $g_{n+1}(x, y)$ and $g_n[f(x), G(x, y)]$ is defined, then the other is also defined and they are both equal.*

Similarly, for $n \leqslant 0$, if one of the functions $g_{n-1}(x, y)$ and $g_n[f^{-1}(x), G^{-1}(f^{-1}(x), y)]$ is defined, then the other is also defined and they are both equal.

§ 3. We have undertaken the task of determining all functions $\varphi(x)$ that are defined in the set E and assume values from the space Ξ, and satisfy in E the equation (1).

We shall assume the function $G(x, y)$ to be invertible with respect to y for each $x \in E_0$. Let us take an arbitrary $x_0 \in E$ and let us write

$$x_n = f^n(x_0), \quad n = 0, \pm 1, \pm 2, \ldots$$

Let $\varphi(x)$ be an arbitrary solution of equation (1) in E and let

$$\varphi(x_0) = y_0. \tag{7}$$

From equation (1) we have

$$\varphi(x_{n+1}) = G(x_n, \varphi(x_n)) = G(f^n(x_0), \varphi(x_n)). \tag{8}$$

Comparing (7) and (8) with formulae (5) we see that

$$\varphi(x_n) = g_n(x_0, y_0). \tag{9}$$

Hence it follows that in order that the function $\varphi(x)$ be defined at the point x_n, the function $g_n(x_0, y)$ must be defined for $y = y_0$. If, moreover, the cycle $\{x_n\}$ is finite, i. e. $x_0 \in E_i$ for a certain positive i, then we have from (9)

$$g_i(x_0, y_0) = y_0.$$

Thus we have obtained some conditions for the values which can be assumed by a solution of equation (1) in E.

Definition III. For $x \in E_i$, $i = 1, 2, \ldots$ we shall denote by V_x the set of all $y \in \Omega_x^i$ such that $g_i(x, y) = y$. For $x \in E_0$ we put

$$V_x \stackrel{\mathrm{df}}{=} \bigcap_{k=-\infty}^{+\infty} \Omega_x^k.$$

From the above considerations follows immediately

Lemma III. *Let us assume that the function $f(x)$ maps the set E onto itself in a one-to-one manner, and the function $G(x, y)$ is invertible with respect to y for each $x \in E_0$. If a function $\varphi(x)$ is defined in the whole set E and satisfies equation* (1) *in E, then the value of the function $\varphi(x)$ belongs to the set V_x:*

$$\varphi(x) \in V_x.$$

Thus the relation

$$V_x \neq \emptyset \quad \textit{for} \quad x \in E \tag{10}$$

is a necessary condition that equation (1) *possess a solution in E.*

We shall also prove

Lemma IV. *Under the hypotheses of lemma* III *the set*

$$E^* \stackrel{\mathrm{df}}{=} \{x\colon\ V_x \neq \emptyset\}.$$

is a modulus-set for the function $f(x)$.

Proof. It is enough to prove that if $x \in E^*$, then also $f(x) \in E^*$ and $f^{-1}(x) \in E^*$. We must distinguish two cases:

Case 1. $x \in E_i$, $i > 0$. Let y_0 be an element of the set V_x. Consequently

$$g_i(x, y_0) = y_0. \tag{11}$$

Putting $\bar{y}_0 \stackrel{\mathrm{df}}{=} G(x, y_0)$ and making use of (11) and lemma II we get

$$\bar{y}_0 = g_i(f(x), \bar{y}_0),$$

which (according to lemma I) proves that $\bar{y}_0 \in V_{f(x)}$. Thus $f(x) \in E^*$.

Using the implication

$$x \in E^* \Rightarrow f(x) \in E^*$$

$i-1$ times, we obtain $f^{i-1}(x) \in E^*$, which (since $f^i(x) = x$) is equivalent to the relation $f^{-1}(x) \in E^*$.

Case 2. $x \in E_0$. Let y_0 be an element of the set V_x. It means that for every n the function $g_n(x, y_0)$ is defined. Putting $\bar{y}_0 \stackrel{\mathrm{df}}{=} G(x, y_0)$ we find on account of lemma II and of the invertibility of the function $G(x, y)$ that the function $g_{n-1}[f(x), \bar{y}_0]$ is defined for every n. It means that

$$\bar{y}_0 \in \bigcap_{k=-\infty}^{\infty} \Omega_{f(x)}^k = V_{f(x)}, \quad \text{i. e.} \quad f(x) \in E^*.$$

Similarly, we can prove that $\bar{\bar{y}}_0 \stackrel{\mathrm{df}}{=} G^{-1}[f^{-1}(x), y_0] \in V_{f^{-1}(x)}$, whence $f^{-1}(x) \in E^*$. This completes the proof.

§ 4. In what follows we shall assume that $V_x \neq \emptyset$ for $x \in E$. From lemma III it follows that this assumption is essential if we are to obtain a solution of equation (1) in the whole set E. If this assumption is not fulfilled, i. e. if $V_x = \emptyset$ for a certain $x \in E$, then on account of lemma IV we may restrict ourselves to the consideration of equation (1) in the set E^*, in which $V_x \neq \emptyset$.

We shall denote by Ψ the class of all functions $\psi(x)$ that are defined in E^0 and such that $\psi(x) \in V_x$ for $x \in E^0$.

Now we shall prove the following

THEOREM. *Let us assume that the function $f(x)$ is a one-to-one mapping of the set E onto itself:*

$$f(E) = E,$$

and the function $G(x, y)$ is invertible with respect to y for each $x \epsilon E_0$ and that $V_x \neq \emptyset$ for every $x \epsilon E$. Then the general solution of equation (1) *is given by the formula*

$$\varphi(x) = g_n\big(f^{-n}(x), \psi[f^{-n}(x)]\big) \quad \textit{for} \quad x \epsilon E^n, \tag{12}$$

where $\psi(x)$ is an arbitrary function from the class Ψ.

Proof. First of all we have to prove that the function $\varphi(x)$ given by formula (12) is defined for every $x \epsilon E$, and further that it satisfies equation (1). Let us take an arbitrary $x \epsilon E$. Since

$$E = \bigcup_{j=0}^{\infty} \bigcup_{k=-\infty}^{\infty} E_j^k,$$

there exist indices i and n such that $x \epsilon E_i^n$. We must consider several cases (depending upon the indices i and n).

Case 1. $i > 0$. Then we must have $0 \leqslant n \leqslant i-1$. Evidently $x \epsilon E_i$, and then, according to lemma I, also $f^{-n}(x) \epsilon E_i$. From the definition of the sets V_x it follows that $V_{f^{-n}(x)} \subset \Omega^i_{f^{-n}(x)}$. On account of relations (6) we have

$$\Omega^i_{f^{-n}(x)} \subset \Omega^j_{f^{-n}(x)} \quad \text{for} \quad j = 0, \ldots, i.$$

Hence

$$V_{f^{-n}(x)} \subset \Omega^j_{f^{-n}(x)} \quad \text{for} \quad = 0, \ldots, i.$$

Since $0 \leqslant n < i$, then in particular $V_{f-n(x)} \subset \Omega^n_{f^{-n}(x)}$, whence $\psi[f^{-n}(x)] \epsilon\, \Omega^n_{f^{-n}(x)}$, which proves that the function $g_n\big(f^{-n}(x), y\big)$ is defined for $y = \psi[f^{-n}(x)]$, and thus the function $\varphi(x)$ is defined. To prove that it satisfies equation (1) we shall consider two subcases:

(a) $0 \leqslant n < i-1$. Then, according to (3), $f(E_i^n) = E_i^{n+1}$ and consequently $f(x) \epsilon E^{n+1} \subset E^{n+1}$. Thus we have by (12)

$$\varphi[f(x)] = g_{n+1}\Big(f^{-n-1}\big(f(x)\big), \psi\big[f^{-n-1}\big(f(x)\big)\big]\Big) = g_{n+1}\big(f^{-n}(x), \psi[f^{-n}(x)]\big).$$

But according to relations (5)

$$g_{n+1}\big(f^{-n}(x), \psi[f^{-n}(x)]\big) = G\Big(x, g_n\big(f^{-n}(x), \psi[f^{-n}(x)]\big)\Big) = G\big(x, \varphi(x)\big),$$

hence

$$\varphi[f(x)] = G\big(x, \varphi(x)\big),$$

which was to be proved.

(b) $n = i-1$. Then, according to (3), $f(E_i^{i-1}) = E_i^0$, and consequently $f(x) \epsilon E_i^0 \subset E^0$. Thus we have by (12)

$$\varphi[f(x)] = \psi[f(x)]. \tag{13}$$

On the other hand,

$$G\big(x, \varphi(x)\big) = G\Big(x, g_{i-1}\big(f^{-i+1}(x), \psi[f^{-i+1}(x)]\big)\Big).$$

But since $x \epsilon E_i$, $x = f^i(x)$. Hence

$$G\big(x, \varphi(x)\big) = G\Big(f^i(x), g_{i-1}\big(f(x), \psi[f(x)]\big)\Big) = G\Big(f^{i-1}[f(x)], g_{i-1}\big(f(x), \psi[f(x)]\big)\Big),$$

and according to (5)

$$G\big(x, \varphi(x)\big) = g_i\big(f(x), \psi[f(x)]\big). \tag{14}$$

But $\psi[f(x)] \epsilon V_{f(x)}$ and $f(x) \epsilon E_i$. Hence

$$g_i\big(f(x), \psi[f(x)]\big) = \psi[f(x)],$$

i. e., by (13) and (14)

$$\varphi[f(x)] = G\big(x, \varphi(x)\big),$$

which was to be proved.

Case 2. $i = 0$. Then $x \epsilon E_0$ and $f^{-n}(x) \epsilon E_0$. Consequently $V_{f^{-n}(x)} = \bigcap_{k=-1}^{\infty} \Omega^k_{f^{-n}(x)}$, whence it follows that $\psi[f^{-n}(x)] \epsilon\, \Omega^k_{f^{-n}(x)}$ for each integer k, and thus in particular $\psi[f^{-n}(x)] \epsilon\, \Omega^n_{f^{-n}(x)}$. Hence it follows that the function $g_n\big(f^{-n}(x), y\big)$ is defined for $y = \psi[f^{-n}(x)]$, i. e. the function $\varphi(x)$ is defined. We shall show that it satisfies equation (1).

The relation

$$g_n(x, y) = G^{-1}[f^n(x), g_{n+1}(x, y)]$$

is equivalent to the relation

$$g_{n+1}(x, y) = G[f^n(x), g_n(x, y)], \tag{15}$$

which proves that the relation (15) is valid for every integer n (positive and negative) provided the functions $g_{n+1}(x, y)$ and $g_n(x, y)$ are defined.

According to (4) $f(E_0^n) = E_0^{n+1}$, and consequently $f(x) \epsilon E_0^{n+1} \subset E^{n+1}$. Thus we have by (12)

$$\varphi[f(x)] = g_{n+1}\big(f^{-n}(x), \psi[f^{-n}(x)]\big).$$

On the other hand, we have by (15)

$$g_{n+1}\big(f^{-n}(x), \psi[f^{-n}(x)]\big) = G\Big(x, g_n\big(f^{-n}(x), \psi[f^{-n}(x)]\big)\Big) = G\big(x, \varphi(x)\big),$$

whence

$$\varphi[f(x)] = G(x, \varphi(x)),$$

which was to be proved.

From the considerations of § 3 it follows that formula (12) gives the general solution of equation (1). This completes the proof of the theorem.

COROLLARY. *Under the hypotheses of lemma* III *relation* (10) *is a necessary and sufficient condition that equation* (1) *possess a solution in the set* E.

§ 5. From the hypotheses of the above theorem the hypothesis of the invertibility of the function $G(x, y)$ can be dismissed if we properly extend the notion of an inverse function.

DEFINITION IV. Every function $H(x, y)$ defined in some subset $\overline{\Omega}$ of the set Ω (assuming the values from the space Ξ) and fulfilling for $(x, y) \in \overline{\Omega}$ the relation $G[x, H(x, y)] \equiv y$ will be called an *inverse function of the function* $G(x, y)$.

Let $\{H_\lambda(x, y)\}$ be the family of all inverse functions of the function $G(x, y)$. Let the index λ run over a set Λ. (The set Λ can be, of course, uncountable.) Using any of the functions $H_\lambda(x, y)$ instead of the function $G^{-1}(x, y)$ we can define with the aid of formulae (5) a sequence of functions ${}_\lambda g_n(x, y)$. These functions depend of course upon the choice of the function $H_\lambda(x, y)$, and therefore we add to them an index λ. Similarly, the sets ${}_\lambda\Omega_x^n$ will depend upon λ, and thus also the sets ${}_\lambda V_x$ and the class of functions ${}_\lambda\Psi$.

The following theorem can be proved:

Under the hypothesis that the function $f(x)$ *is a one-to-one mapping of the set* E *onto itself, the general solution of equation* (1) *is given by the formula*

$$\varphi(x) = {}_\lambda g_n\big(f^{-n}(x), {}_\lambda\psi[f^{-n}(x)]\big) \quad \textit{for} \quad x \in E^n,$$

where ${}_\lambda\psi(x)$ *is an arbitrary function from the class* ${}_\lambda\Psi$, $\lambda \in \Lambda$.

The necessary and sufficient condition that equation (1) *possess a solution in the set* E *is that at least for one* $\lambda \in \Lambda$ *the relation*

$${}_\lambda V_x \neq \emptyset \quad \textit{for} \quad x \in E$$

be fulfilled.

§ 6. Now let us replace the condition $f(E) = E$ by the weaker condition $f(E) \subset E$. (But we continue to assume that the function $f(x)$ is invertible in the set E.)

DEFINITION V. The elements of the set $E - f(E)$ will be called *prime elements*(³) of the set E. A cycle determined by a prime element(⁴) will be called *non-full*.

Let us denote by $\overline{\overline{E}}$ the set of all elements of non-full cycles, and let us put $\overline{E} \overset{\mathrm{df}}{=} E - \overline{\overline{E}}$. The set $\overline{E}$ is a modulus-set for the function $f(x)$. In fact, let x be an element of the set $\overline{E}$ and let us suppose that $f(x)$ does not belong to $\overline{E}$. It means that $f(x)$ belongs to a non-full cycle, i. e. there exist an integer N and a prime element x_0 such that $f(x) = f^N(x_0)$. But then $x = f^{N-1}(x_0)$, which means that x is an element of the same non-full cycle, which is impossible since $x \in \overline{E}$. Consequently $f(x) \in \overline{E}$. In the same way it can be shown that $f^{-1}(x) \in \overline{E}$.

Thus we can apply to the set $\overline{E}$ all the considerations of the preceding sections. In particular, we can define sets $\overline{E}^n$, $n = 0, \pm 1, \ldots$ and $\overline{E}_i$, $i = 0, 1, 2, \ldots$, for $x \in \overline{E}$ we can define functions $g_n(x, y)$, sets Ω_x^n and V_x.

Now we shall define sets $\overline{\overline{E}}^n$. As a set $\overline{\overline{E}}^0$ we take the set of prime elements $E - f(E)$ and we put:

$$\overline{\overline{E}}^n \overset{\mathrm{df}}{=} \begin{cases} f^n(\overline{\overline{E}}^0) & \text{for} \quad n > 0, \\ \emptyset & \text{for} \quad n < 0. \end{cases}$$

Evidently

$$\overline{\overline{E}} = \bigcup_{k=-\infty}^{\infty} \overline{\overline{E}}^k.$$

With the aid of formulae (5) we can define also the functions $g_n(x, y)$ for $n \geqslant 0$ and $x \in \overline{\overline{E}}$. Similarly, for $n \geqslant 0$ and $x \in \overline{\overline{E}}$ we can define the sets Ω_x^n. We put

$$V_x \overset{\mathrm{df}}{=} \bigcap_{k=0}^{\infty} \Omega_x^k \quad \text{for} \quad x \in \overline{\overline{E}}.$$

The definition of the class Ψ remains unchanged.

Now, if we put

$$E^n \overset{\mathrm{df}}{=} \overline{E}^n \cup \overline{\overline{E}}^n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

then formula (12) gives the general solution of equation (1) in the set E under the condition that the function $G(x, y)$ is invertible with respect

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(³) This name is due to the fact that these elements have no preceding elements in cycles, because the function $f^{-1}(x)$ is not defined in the set $E - f(E)$.

(⁴) By a cycle determined by a prime element x will be understood the set of points of the form $f^n(x)$, $n = 0, 1, 2, \ldots$

to y for each $x \in \bar{E}_0$ and that $V_x \neq \emptyset$ for every $x \in E$. Also the considerations of § 5 remain valid, because the invertibility of the function $G(x, y)$ occurs only for $x \in \bar{E}_0$.

References

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On the functional equation $F(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]) = 0$

by J. Kordylewski (Kraków)

The present paper contains results regarding the existence of continuous solutions of the functional equation

$$F(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]) = 0, \tag{1}$$

in which the function $\varphi(x)$ is the unknown function, $f(x)$ and $F(x, y_0, \ldots, y_n)$ are given functions, and $f^\nu(x)$ denotes the ν-th iteration of the function $f(x)$, i. e.

$$f^0(x) = x,$$
$$f^\nu(x) = f[f^{\nu-1}(x)], \quad f^{-\nu}(x) = f^{-1}[f^{-\nu+1}(x)], \quad \nu = 1, 2, 3, \ldots$$

Equation (1) is a generalization of the equation

$$F(x, \varphi(x), \varphi[f(x)]) = 0, \tag{2}$$

which was discussed in [1], and is a particular case of the equation

$$F(x, \varphi(x), \varphi[f_1(x)], \varphi[f_2(x)], \ldots, \varphi[f_n(x)]) = 0 \tag{3}$$

(where the functions $f_\nu(x)$ ($\nu = 1, 2, \ldots, n$) are known functions), which was discussed in [2]. Equation (2) is a particular case of equation (3); nevertheless the theorem on the existence of continuous solutions of equation (3) does not, in the case of $n = 1$, pass into the corresponding theorem for equation (2); the hypotheses made with regard to equation (3) are stronger. M. Kuczma has raised the problem of proving the existence of continuous solutions of equation (1) under such assumptions (weaker than the hypotheses on equation (3)) that in the case of $n = 1$ the theorem should pass into the corresponding theorem for equation (2). The present paper is a solution of this problem: we shall show that under suitable assumptions equation (1) possesses infinitely many continuous solutions.

We assume the following hypotheses regarding the function $f(x)$:

(i) The function $f(x)$ is defined, continuous and strictly increasing in an interval $\langle a, b \rangle$; moreover, let $f(a) = a$, $f(b) = b$, $f(x) > x$ for $x \in (a, b)$.