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About the extremal spiral schlicht functions

by J. Zamorski (Wrocław)

Let S denote a class of regular schlicht functions with the expansion

$$f(z) = z + a_2 z^2 + \ldots \tag{1}$$

for $|z| < 1$, and let Σ denote a class of meromorphic schlicht functions with the development

$$F(z) = \frac{1}{z} + b_0 + b_1 z + \ldots \tag{2}$$

for $0 < |z| < 1$. L. Špaček in his paper [3] proves that each function

$$f(z) = z\exp\left\{\frac{1}{1-ai}\int_0^z \frac{p(z)-1}{z}\,dz\right\} \tag{3}$$

where $p(z) = 1 + a_1 z + a_2 z^2 + \ldots$, $|z| < 1$, $\operatorname{re} p(z) \geqslant 0$ and a is any real number, belongs to the class S. Further Špaček proves that for these functions the hypothesis of Bieberbach is true. Of course, similarly

$$F(z) = \frac{1}{z}\exp\left\{-\frac{1}{1-ai}\int_0^z \frac{p(z)-1}{z}\,dz\right\} \tag{4}$$

belongs to the class Σ.

Let S_a denote the class of functions defined by formula (3), and let Σ_a denote the class of functions defined by formula (4). We can easily see that for $a = 0$ class S_a becomes class S^* of all functions starlike for the origin of the system of coordinates. Similarly for the class Σ_a.

Let W_n^a denote the n-th region of the coefficients of functions of the class S_a. We can easily see that it is a $2n-2$ dimensional domain, closed, connected and bounded, including the origin of the system of coordinates. Namely from (3) we have

$$\frac{zf'(z)}{f(z)} = 1 + \varrho\,[p(z)-1] \tag{5}$$

where $\varrho = 1/(1-ai)$ and $\operatorname{re} p(z) \geqslant 0$. From (5) we obtain a connexion of the coefficients of the function with a positive real part and the coefficients of a function belonging to the class S_a,

(6) $$\frac{1-k}{\varrho} a_k + a_1 a_{k-1} + \ldots + a_{k-2} a_2 + a_{k-1} = 0, \qquad k = 2, 3, \ldots$$

Hence of course

(7) $$a_k = \frac{(-1)^k}{(k-1)!} \varrho^{k-1} \begin{vmatrix} a_1 & a_2 & \ldots & a_{k-1} \\ \frac{1-k}{\varrho} & a_1 & \ldots & a_{k-2} \\ 0 & \frac{2-k}{\varrho} & \ldots & a_{k-3} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_1 \end{vmatrix}$$

and

(8) $$a_k = \frac{(-1)^{k-1}}{\varrho} \begin{vmatrix} a_2 & 1 & 0 & \ldots & 0 \\ 2a_3 & a_2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (k-1)a_k & a_{k-1} & a_{k-2} & \ldots & a_2 \end{vmatrix}.$$

Formulae (7) and (8) give homeomorphism of the n-th region of variability of functions belonging to the class S_a and the $n-1$ variability region of the coefficients of functions with a positive real part (see Carathéodory [1]), and this proves the above properties for the region W_n^a. The n-th variability region of coefficients of a function of the class Σ_a, denoted as V_n^a, has analogous properties.

Now we are going to define the functional $H_n(f)$ for functions of the class S_a. Namely let $H_n(x_2, \ldots, x_n; y_2, \ldots, y_n)$ be a real function of $2n-2$ real variables, defined in the region comprising the n-th region of variability W_n^a and such that

(9) $$\sum_{k=2}^{n} \left[\left(\frac{\partial H_n}{\partial x_k} \right)^2 + \left(\frac{\partial H_n}{\partial y_k} \right)^2 \right] \neq 0$$

for all points of the region W_n^a. This function will be considered as a functional defined for the functions of the class S_a, namely

$$H_n(f) = H_n(x_2, \ldots, x_n; y_2, \ldots, y_n), \quad f(z) = z + a_2 z^2 + \ldots, \quad a_k = x_k + i y_k.$$

We are going to denote analogically the functional $E_n(F)$ for the functions of the class Σ_a. Now we shall study the extrema of the functionals $H_n(f)$ and $E_n(F)$.

THEOREM 1. *The function of the class S_a for which the functional $H_n(f)$ obtains its extremal value always has the following form*

$$f(z) = z \Big[\prod_{k=1}^{n-1} (1-\sigma_k z)^{-\beta_k} \Big]^{1/(1-ai)},$$

$$\sum_{k=1}^{n-1} \beta_k = 2, \quad \beta_k > 0, \quad |\sigma_k| = 1.$$

THEOREM 2. *The function of the class Σ_a for which the functional $E_n(F)$ obtains its extremal value always has the following form*

$$F(z) = \frac{1}{z} \Big[\prod_{k=1}^{n+1} (1-\sigma_k z)^{\beta_k} \Big]^{1/(1-ai)},$$

$$\sum_{k=1}^{n+1} \beta_k = 2, \quad \beta_k > 0, \quad |\sigma_k| = 1.$$

In a particular case, when we put $a = 0$, we obtain the known theorems concerning starlike functions (see [4]).

Proof of theorem 1. Formulae (7) and (8) show that each point of the n-th variability region of the functions of the class S_a corresponds continuously to one and only one point of the $n-1$ variability region of the functions with a positive real part, and vice versa. Hence the points of the boundary of one region correspond to the points of the boundary of the other and vice versa. From the assumptions on functional $H_n(f)$ infer this functional has its extremal value on the boundary of the region W_n^a. The extremal function corresponding to the boundary point of the region W_n^a corresponds, according to formula (3), to the function $p(z)$, which is a boundary function of the $n-1$ variability region of functions with a positive real part—on the basis of the above correspondence.

By Carathéodory's theorem [1] this function must be of following form

(10) $$p(z) = 1 + \sum_{k=1}^{n-1} \beta_k \frac{\sigma_k z}{1-\sigma_k z}, \qquad \sum_{k=1}^{n-1} \beta_k = 2, \quad \beta_k > 0, \quad |\sigma_k| = 1$$

or

$$f(z) = z\exp\left\{\frac{1}{1-ai}\int_0^z \sum_{k=1}^{n-1} \frac{\beta_k\sigma_k}{1-\sigma_k z}\,dz\right\} = z\left[\prod_{k=1}^{n-1}(1-\sigma_k z)^{-\beta_k}\right]^{1/(1-ai)},$$

which proves the theorem.

The proof of theorem 2 is analogical.

THEOREM 3. *The parameters defining the function of the class S_a extremal for the functional $H_n(f)$ satisfy the following system of equations:*

$$A_{n-1}\sigma_k^{2(n-1)}+\ldots+A_1\sigma_k^n+\lambda\sigma_k^{n-1}+\bar{A}_1\sigma_k^{n-2}+\ldots+\bar{A}_{n-1}=0,$$

$$2(n-1)A_{n-1}\sigma_k^{2n-3}+\ldots+nA_1\sigma^{n-1}+(n-1)\lambda\sigma_k^{n-2}+$$
$$+(n-2)\bar{A}_1\sigma_k^{n-3}+\ldots+\bar{A}_{n-2}=0,$$

$$A_j=\frac{1}{1-ai}\cdot\frac{1}{j}\cdot\sum_{p=1}^{n-j}a_p\left\{\frac{\partial H_n}{\partial x_{p+j}}-i\,\frac{\partial H_n}{\partial y_{p+j}}\right\},$$

$$\sum_{k=1}^{n-1}\beta_k=2,\quad \beta_k>0,\quad |\sigma_k|=1,\quad k,j=1,\ldots,n-1.$$

THEOREM 4. *The parameters defining the function of the class Σ_a extremal for the functional $E_n(F)$ satisfy the following system of equations:*

$$B_{n+1}\sigma_k^{2(n+1)}+\ldots+B_1\sigma_k^{n+2}+\lambda\sigma_k^{n+1}+\bar{B}_1\sigma_k^n+\ldots+\bar{B}_{n+1}=0,$$

$$2(n+1)B_{n+1}\sigma_k^{2n+1}+\ldots+(n+2)B_1\sigma_k^{n+1}+(n+1)\lambda\sigma_k^n+$$
$$+n\bar{B}_1\sigma_k^{n-1}+\ldots+\bar{B}_n=0,$$

$$B_j=\frac{1}{1-ai}\sum_{p=-1}^{n-j}b_p\left\{\frac{\partial E_n}{\partial x_{p+j}}-i\,\frac{\partial E_n}{\partial y_{p+j}}\right\},$$

$$\sum_{k=1}^{n+1}\beta_k=2,\quad \beta_k>0,\quad |\sigma_k|=1,\quad k,j=1,\ldots,n+1.$$

The proof of theorem 3. The coefficients of function $f(z)$ of class S_a extremal for functional $H_n(f)$ are given by formula (7) where by theorem 1

$$a_j=\sum_{k=1}^{n-1}\beta_k\sigma_k^j,\qquad \sum_{k=1}^{n-1}\beta_k=2,\qquad \beta_k>0,\qquad \sigma_k=e^{i\vartheta_k}.$$

So the coefficients are polynomials of variables β_k and σ_k. Further we have

$$\frac{\partial f}{\partial\beta_k}=-z\left[\prod_{j=1}^{n-1}(1-\sigma_k z)^{-\beta_j}\right]^{1/(1-ai)}\log(1-\sigma_k z)\,\frac{1}{1-ai}$$
$$=\frac{\partial a_2}{\partial\beta_k}z^2+\frac{\partial a_3}{\partial\beta_k}z^3+\ldots+\frac{\partial a_p}{\partial\beta_k}z^p+\ldots$$

since

$$-z\left[\prod_{j=1}^{n-1}(1-\sigma_k z)^{-\beta_j}\right]^{1/(1-ai)}\log(1-\sigma_k z)\,\frac{1}{1-ai}$$
$$=\frac{1}{1-ai}\sum_{p=2}^{\infty}z^p\left\{\frac{\sigma_k^{p-1}}{p-1}+\frac{\sigma_k^{p-2}}{p-2}a_2+\ldots+\sigma_k a_{p-1}\right\},$$

we have

(11) $$\frac{\partial a_p}{\partial\beta_k}=-\frac{1}{1-ai}\left\{\frac{\sigma_k^{p-1}}{p-1}+\frac{\sigma_k^{p-2}}{p-2}a_2+\ldots+\sigma_k a_{p-1}\right\}.$$

Similarly

(12) $$\frac{\partial a_p}{\partial\beta_k}=-\frac{i\beta_k}{1-ai}\cdot\{\sigma_k^{p-1}+\sigma_k^{p-2}a_2+\ldots+\sigma_k a_{p-1}\}.$$

In order to determine the extremum of the functional $H_n(f)$, it is sufficient, to find the extremum of the function $H_n(x_2,\ldots,x_n;\,y_2,\ldots,y_n)$ on the boundary of the region W_n^a. After replacing the variables x_k, y_k by the variables β_k and ϑ_k we have to find the extremum of the function $H_n(\vartheta_1,\ldots,\vartheta_{n-1};\,\beta_1,\ldots,\beta_{n-1})$ with the condition $\sum_{k=1}^{n-1}\beta_k=2$ in the interval of variability $0\leqslant\beta_k\leqslant 2$, $-\infty<\vartheta_k<\infty$. Let λ' be a Lagrange multiplier and let us find the derivatives of the function

$$H_n^*=H_n+\lambda'\left(\sum_{k=1}^{n-1}\beta_k-2\right)$$

at the extremal point. We have

(13) $$\frac{\partial H_n^*}{\partial\beta_k}=\sum_{j=2}^{n}\frac{\partial H_n}{\partial x_j}\cdot\frac{\partial x_j}{\partial\beta_k}+\sum_{j=2}^{n}\frac{\partial H_n}{\partial y_j}\cdot\frac{\partial y_j}{\partial\beta_k}+\lambda'=0,$$
$$\frac{\partial H_n^*}{\partial\vartheta_k}=\sum_{j=2}^{n}\frac{\partial H_n}{\partial x_j}\cdot\frac{\partial x_j}{\partial\vartheta_k}+\sum_{j=2}^{n}\frac{\partial H_n}{\partial y_j}\cdot\frac{\partial y_j}{\partial\vartheta_k}=0.$$

Putting $x_p = \operatorname{re} a_p = \frac{1}{2}(a_p+\bar{a}_p)$, $y_p = \operatorname{im} a_p = \frac{i}{2}(\bar{a}_p - a_p)$ and substituting (11) and (12) into (13) we obtain the following system of equations:

$$\frac{1}{1-ai}\sum_{j=1}^{n-1}\frac{\sigma_k^j}{j}\sum_{p=1}^{n-j} a_p\left\{\frac{\partial H_n}{\partial x_{p+j}} - i\frac{\partial H_n}{\partial y_{p+j}}\right\}+$$

$$+\frac{1}{1+ai}\sum_{j=1}^{n-1}\frac{\bar{\sigma}_k^j}{j}\sum_{p=1}^{n-j}\bar{a}_p\left\{\frac{\partial H_n}{\partial x_{p+j}} + i\frac{\partial H_n}{\partial y_{p+j}}\right\}+\lambda' = 0,$$

(14)
$$\beta_k\left[\frac{1}{1-ai}\sum_{j=1}^{n-1}\sigma_k^j\sum_{p=1}^{n-j}a_p\left\{\frac{\partial H_n}{\partial x_{p+j}} - i\frac{\partial H_n}{\partial y_{p+j}}\right\}-\right.$$

$$\left.-\frac{1}{1+ai}\sum_{j=1}^{n-1}\bar{\sigma}_k^j\sum_{p=1}^{n-j}\bar{a}_p\left\{\frac{\partial H_n}{\partial x_{p+j}} + i\frac{\partial H_n}{\partial y_{p+j}}\right\}\right] = 0,$$

$$\sum_{k=1}^{n-1}\beta_k = 2.$$

Without loss of generality we can assume that no β_k are equal to 0. Further we have $\bar{\sigma}_k = 1/\sigma_k$. To be short let us substitute

$$A_j = \frac{1}{1-ai}\cdot\frac{1}{j}\sum_{p=1}^{n-j}a_p\left\{\frac{\partial H_n}{\partial x_{p+j}} - i\frac{\partial H_n}{\partial y_{p+j}}\right\}.$$

By simple transformations we find from (14) that the parameters of the extremal function must satisfy the system of equations

$$A_{n-1}\sigma_k^{2(n-1)}+\ldots+A_1\sigma_k^n+\lambda\sigma_k^{n-1}+\bar{A}_1\sigma_k^{n-2}+\ldots+\bar{A}_{n-1} = 0,$$

(15)
$$2(n-1)A_{n-1}\sigma_k^{2n-3}+\ldots+nA_1\sigma_k^{n-1}+(n-1)\lambda\sigma_k^{n-2}+$$

$$+(n-2)\bar{A}_1\sigma_k^{n-3}+\ldots+\bar{A}_{n-2} = 0,$$

$$\sum_{k=1}^{n-1}\beta_k = 2, \qquad k = 1,\ldots,n-1.$$

where it is not necessary that all σ_k should be different. The proof of theorem 4 is analogical.

Putting $a = 0$ we obtain from theorems 3 and 4 the wellknown theorems on starlike functions (see [4]).

THEOREM 5. *The coefficients of the class Σ_a satisfy the inequality*

$$|b_n| \leqslant \frac{2}{n+1}\cdot\frac{1}{\sqrt{1+a^2}}.$$

We obtain an equality for the function

$$F(z) = \frac{1}{z}\left[(1+\eta z^{n+1})^{2/(n+1)}\right]^{1/(1-ai)}, \qquad |\eta| = 1.$$

Proof of theorem 5. J. Clunie in his paper [2] proves this for the class of starlike function with a pole. His method may easily be applied to a function of the class Σ_a. Namely

$$F(z) = \frac{1}{z}\exp\left\{-\varrho\int_0^z\frac{p(z)-1}{z}dz\right\}$$

where

$$\varrho = \frac{1}{1-ai}, \qquad \operatorname{re} p(z) \geqslant 0, \qquad p(z) = 1+a_1z+\ldots$$

Hence

(16)
$$\frac{zF'(z)}{F(z)} = -\varrho[p(z)-1]-1.$$

Let

(17)
$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \sum_{k=1}^{\infty}\omega_k z^k.$$

Then $|\omega(z)| \leqslant 1$. From (16) and (17) we have

$$\omega(z)[zF'(z)-(2\varrho-1)F(z)] = zF'(z)+F(z)$$

or

(18)
$$\sum_{k=1}^{\infty}\omega_k z^k\left\{-2\varrho+\sum_{k=0}^{\infty}(k+1-2\varrho)b_k z^{k+1}\right\} = \sum_{k=0}^{\infty}(k+1)b_k z^{k+1}.$$

Hence we have

(19)
$$\sum_{k=1}^{\infty}\omega_k z^k\left\{-2\varrho+\sum_{k=0}^{n-1}(k+1-2\varrho)b_k z^{k+1}\right\}$$

$$= \sum_{k=0}^{n}(k+1)b_k z^{k+1}+\sum_{k=n+2}^{\infty}c_k z^n,$$

c_k being some complex numbers. Since $|\omega(z)| \leqslant 1$, we have

$$\sum_{k=0}^{n}(k+1)^2|b_k|^2r^{2(k+1)}+\sum_{k=n+2}^{\infty}|c_k|^2r^{2k} \leqslant 4|\varrho|^2+\sum_{k=0}^{n-1}|k+1-2\varrho|^2|b_k|^2r^{2k}$$

for $r < 1$. Hence further

$$\sum_{k=0}^{n}(k+1)^2|b_k|^2 \leqslant 4|\varrho|^2+\sum_{k=0}^{n-1}|k+1-2\varrho|^2|b_k|^2$$

or

$$(n+1)^2|b_n|^2 \leqslant 4|\varrho|^2-4|\varrho|^2\sum_{k=1}^{n-1}k|b_k|^2,$$

and thus

$$|b_n| \leqslant \frac{2}{n+1}\cdot\frac{1}{\sqrt{1+a^2}}.$$

We obtain an equality for the coefficients of the function

$$F(z) = \frac{1}{z}[(1+\eta z^{n+1})^{2/(n+1)}]^{1/(1-ai)}, \qquad |\eta| = 1.$$

THEOREM 6. *The coefficients of a function of class S_a satisfy the inequality*

$$|a_n| \leqslant \frac{(1+a^2)^{(1-n)/2}}{(n-1)!}\prod_{k=0}^{n-2}[(2+k)^2+k^2a^2]^{1/2}.$$

The sign of equality occurs only for the coefficients of the function

$$f(z) = z(1+\eta z)^{-2/(1-ai)}, \qquad |\eta| = 1.$$

Proof of theorem 6. Just as in the proof of theorem 5, the function of class S_a

$$f(z) = z\exp\left\{\varrho\int_0^z\frac{p(z)-1}{z}\,dz\right\}$$

satisfies the equation

$$\frac{zf'(z)}{f(z)} = \varrho[p(z)-1]+1. \tag{20}$$

Putting the bounded function $\omega(z)$ we obtain from (20) the equation

$$\omega(z)[zf'(z)+(2\varrho-1)f(z)] = zf'(z)-f(z)$$

or otherwise

$$\sum_{k=1}^{\infty}\omega_k z^k\sum_{k=1}^{\infty}(k+2\varrho-1)a_kz^k = \sum_{k=1}^{\infty}(k-1)a_kz^k. \tag{21}$$

Hence of course

$$\sum_{k=1}^{\infty}\omega_k z^k\sum_{k=1}^{n-1}(k+2\varrho-1)a_kz^k = \sum_{k=1}^{n}(k-1)a_kz^k+\sum_{k=n+1}^{\infty}c_kz^k$$

where c_k are some numbers. Since $|\omega(z)| \leqslant 1$, we have

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|c_k|^2r^{2k} \leqslant \sum_{k=1}^{n-1}|k+2\varrho-1|^2|a_k|^2r^{2k}$$

for $r < 1$. Hence

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2 \leqslant \sum_{k=1}^{n-1}|k+2\varrho-1|^2|a_k|^2 \tag{22}$$

or

$$|a_n|^2 \leqslant \frac{4}{(n-1)^2}\sum_{k=1}^{n-1}\frac{k}{1+a^2}|a_k|^2. \tag{23}$$

For $n = 2$ we have

$$|a_2| \leqslant \frac{2}{\sqrt{1+a^2}}.$$

Hence and from formula (23) by simple induction we conclude that

$$|a_n| \leqslant \frac{(1+a^2)^{(1-n)/2}}{(n-1)!}\prod_{k=0}^{n-2}[(2+k)^2+k^2a^2]^{1/2}.$$

We obtain an equality for the coefficients of the function

$$f(z) = z(1+\eta z)^{-2/(1-ai)}, \qquad |\eta| = 1.$$

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