Pour un tenseur antisymétrique il en résulte, en particulier, la non-existence de comitants scalaires ainsi que, pour un espace à nombre impair de dimensions, la non-existence des comitants-densités non triviaux, car dans ce cas nous avons $D = 0$.

Travaux cités

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On some linear functional equations

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Introduction. In the present paper we discuss some particular cases of functional equations of the type

$$A_0(x)\varphi[f^n(x)]+A_1(x)\varphi[f^{n-1}(x)]+\ldots+A_n(x)\varphi(x) = F(x), \tag{1}$$

where $\varphi(x)$ is the required function, and $f(x)$, $F(x)$ and $A_i(x)$ are known functions. $f^k(x)$ denotes here the k-th iteration of the function $f(x)$, i. e.

$$f^0(x) = x,$$

$$\begin{aligned} f^{k+1}(x) &= f[f^k(x)], \\ f^{k-1}(x) &= f^{-1}[f^k(x)], \end{aligned} \qquad k = 0, \pm 1, \pm 2, \ldots$$

We call the number n the order of equation (1).

Equations of type (1) were treated by M. Ghermănescu [1], [2]. However, we consider different problems from those he dealt with.

We shall discuss equation (1) in an interval $\langle a, b\rangle$ where a and b are two consecutive roots of the equation

$$f(x) = x. \tag{2}$$

We shall assume that the function $f(x)$ is a homeomorphism of the interval $\langle a, b\rangle$ onto itself, and the functions $F(x)$ and $A_i(x)$ are complex-valued functions of the real variable, continuous in the interval $\langle a, b\rangle$.

The object of our research will be the number of solutions (1) of equation (1) that are continuous in the intervals (a, b), $(a, b\rangle$, or $\langle a, b)$. It turns out that although equation (1) always possesses infinitely many solutions continuous in the interval (a, b), in some cases there may exist at most one solution continuous in the interval $(a, b\rangle$ or $\langle a, b)$. These results are a continuation of our previous research ([3], [4], [5], [6]).

(1) By a solution of equation (1) we shall understand a complex-valued function $\varphi(x)$ of the real variable, defined in a certain interval and satisfying equation (1) in this interval. In the case of a real-valued function we shall always write "real solution".

I. Equation of the first order. In this section we shall discuss the equation of the first order:

$$A_0(x)\varphi[f(x)]+A_1(x)\varphi(x) = F(x), \qquad x\in\langle a, b\rangle. \tag{3}$$

We assume that $A_0(x)\neq 0$ and $A_1(x)\neq 0$ in the interval $\langle a, b\rangle$. Then equation (3) can be written in the form

$$\varphi[f(x)]-\lambda(x)\varphi(x) = G(x), \tag{4}$$

where $\lambda(x)\overset{\text{df}}{=}-A_1(x)/A_0(x)$, and $G(x)\overset{\text{df}}{=}F(x)/A_0(x)$. Since $A_1(x)\neq 0$, then also $\lambda(x)\neq 0$ in $\langle a, b\rangle$.

One can prove the following

LEMMA I. *If the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation* (2), *and moreover $f(x)>x$ in (a, b), then for every $x\in(a, b)$ the sequences $\{f^n(x)\}$ and $\{f^{-n}(x)\}$ are monotone, and*

$$\lim_{n\to\infty} f^n(x) = b, \qquad \lim_{n\to\infty} f^{-n}(x) = a.$$

A proof of this lemma is to be found in [6].

We shall now prove the following

THEOREM I. *If the function $f(x)$ fulfils the hypotheses of lemma* I, *and the functions $G(x)$ and $\lambda(x)$ are continuous in the interval (a, b), $\lambda(x)\neq 0$ in (a, b), then equation* (4) *possesses infinitely many solutions that are continuous in the open interval (a, b).*

Proof. Let us fix an arbitrary $x_0\in(a, b)$ and let us put $x_n\overset{\text{df}}{=}f^n(x_0)$. Let $\bar\varphi(x)$ be an arbitrary function continuous in the interval $\langle x_0, x_1)$, and fulfilling the condition

$$\lim_{x\to x_1-}\bar\varphi(x) = G(x_0)+\lambda(x_0)\bar\varphi(x_0). \tag{5}$$

Then the formulae

$$\varphi(x) = \begin{cases} \bar\varphi(x) & \text{for} \quad x\in\langle x_0, x_1),\\ G[f^{-1}(x)]+\lambda[f^{-1}(x)]\varphi[f^{-1}(x)] & \text{for} \quad x\in\langle x_n, x_{n+1}),\\ \dfrac{\varphi[f(x)]-G(x)}{\lambda(x)} & \text{for} \quad x\in\langle x_{-n}, x_{-n+1}), \quad n = 1, 2, \ldots \end{cases}$$

define, as we can easily verify on account of lemma I, a function $\varphi(x)$ in the whole interval (a, b) so that it is continuous and satisfies equation (4).

Taking all functions $\bar\varphi(x)$ that are continuous in the interval $\langle x_0, x_1)$, and fulfil condition (5), we get all solutions of equation (4) that are continuous in the interval (a, b). There are infinitely many such solutions.

However, these solutions are not always continuous for $x = b$. We shall show that in some cases at most one of the solutions of equation (4) is continuous for $x = b$.

LEMMA II. *If the function $f(x)$ fulfils the hypotheses of lemma* I, *and the function $\lambda(x)$ is continuous and distinct from zero in the interval $\langle a, b\rangle$ and fulfils the condition*

$$|\lambda(x)|\geqslant 1 \quad \textit{for} \quad x\in(b-\eta, b), \tag{6}$$

resp.

$$|\lambda(x)|\leqslant 1 \quad \textit{for} \quad x\in(a, a+\eta), \tag{7}$$

where η is a positive number, then the unique solution of the equation

$$\varphi[f(x)]-\lambda(x)\varphi(x) = 0 \tag{8}$$

that fulfils the condition

$$\lim_{x\to b}\varphi(x) = 0, \tag{9}$$

resp.

$$\lim_{x\to a}\varphi(x) = 0, \tag{10}$$

is the function $\varphi(x)\equiv 0$ in (a, b).

Proof. Let us assume that condition (6) is fulfilled and that the function $\varphi(x)$ satisfies equation (8) and condition (9). Let us suppose further that there exists $x_0\in(a, b)$ such that $\varphi(x_0)\neq 0$. Let us put $x_n\overset{\text{df}}{=}f^n(x_0)$. We have

$$\varphi(x_{n+1}) = \lambda(x_n)\varphi(x_n), \qquad n = 0, 1, 2, \ldots$$

Then, since $\lambda(x_n)\neq 0$, $\varphi(x_n)\neq 0$ for every n.

On account of lemma I $x_n\to b$ as $n\to\infty$, and thus there exists an N such that $x_n\in(b-\eta, b)$ for $n>N$. We thus have, according to (6)

$$|\lambda(x_n)|\geqslant 1 \quad \text{for} \quad n>N,$$

whence

$$|\varphi(x_{n+1})|\geqslant|\varphi(x_n)|\geqslant|\varphi(x_N)|>0 \quad \text{for} \quad n>N,$$

and thus relation (9) cannot hold.

If we assume condition (7) and relation (10), the proof is analogous.

Now we shall prove

THEOREM II. *Let the function $f(x)$ fulfil the hypotheses of lemma* I, *and let the function $\lambda(x)$ be continuous, $\lambda(x)\neq 0$ in the interval $\langle a, b\rangle$. Moreover, let condition* (6) *with $\lambda(b)\neq 1$, resp. condition* (7) *with $\lambda(a)\neq 1$ be fulfilled. Then there exists at most one solution of equation* (4) *that is continuous in the interval $(a, b\rangle$ resp. $\langle a, b)$.*

Proof. Let us assume that condition (6) is fulfilled and $\lambda(b) \neq 1$.

Let us suppose that there exist two distinct functions $\varphi_1(x)$ and $\varphi_2(x)$, continuous in the interval $(a, b\rangle$ and satisfying equation (4):

$$\varphi_1[f(x)] - \lambda(x)\varphi_1(x) = G(x), \qquad \varphi_2[f(x)] - \lambda(x)\varphi_2(x) = G(x). \tag{11}$$

Then the function $\varphi(x) \overset{\text{df}}{=} \varphi_1(x) - \varphi_2(x)$ is continuous in $(a, b\rangle$ and satisfies equation (8). Putting $x = b$ in (11) we obtain

$$\varphi_1(b) = \varphi_2(b) = \frac{G(b)}{1-\lambda(b)},$$

whence

$$\varphi(b) = 0. \tag{12}$$

Consequently, on account of the continuity of the function $\varphi(x)$ at the point $x = b$ we have

$$\lim_{x\to b} \varphi(x) = 0,$$

and, on account of lemma II and relation (12)

$$\varphi(x) \equiv 0 \quad \text{in} \quad (a, b\rangle,$$

which contradicts the supposition $\varphi_1(x) \not\equiv \varphi_2(x)$.

If we assume that relation (7) is fulfilled and $\lambda(a) \neq 1$, the proof for the interval $\langle a, b)$ is analogous.

THEOREM III. *Let the function $f(x)$ fulfil the hypotheses of lemma* I, *and let the function $\lambda(x)$ be continuous, $\lambda(x) \neq 0$ in the interval $\langle a, b\rangle$. Moreover, let condition* (6) *with $\lambda(b) = 1$, resp. condition* (7) *with $\lambda(a) = 1$ be fulfilled. Then there exists at most one solution of equation* (4) *that is continuous in the interval $(a, b\rangle$ resp. $\langle a, b)$, and assumes a given value for $x = b$ resp. $x = a$.*

The necessary condition of the existence of such a solution is the relation $G(b) = 0$ resp. $G(a) = 0$.

Proof. Let us assume that condition (6) is fulfilled and $\lambda(b) = 1$.

Let us suppose that there exist two distinct functions $\varphi_1(x)$ and $\varphi_2(x)$, continuous in the interval $(a, b\rangle$, satisfying equation (4) and the condition

$$\varphi_1(b) = \varphi_2(b).$$

Then the function $\varphi(x) \overset{\text{df}}{=} \varphi_1(x) - \varphi_2(x)$ is continuous in the interval $(a, b\rangle$, satisfies equation (8) and the condition

$$\varphi(b) = 0. \tag{13}$$

Consequently

$$\lim_{x\to b} \varphi(x) = 0,$$

and on account of lemma II and relation (13)

$$\varphi(x) \equiv 0 \quad \text{in} \quad (a, b\rangle,$$

which contradicts the supposition $\varphi_1(x) \not\equiv \varphi_2(x)$.

Putting $x = b$ in equation (4) we obtain, according to $\lambda(b) = 1$, $G(b) = 0$. Thus the relation $G(b) = 0$ is a necessary condition of the existence of a solution of equation (4) that is continuous in the whole interval $(a, b\rangle$.

If we assume that relation (7) is fulfilled and $\lambda(a) = 1$, the proof for the interval $\langle a, b)$ is analogous.

COROLLARY. *If $\lambda(x) \equiv 1$ in $\langle a, b\rangle$, then the general solution of equation* (4) *continuous in $(a, b\rangle$ resp. $\langle a, b)$ has the form*

$$\varphi(x) = \bar{\varphi}(x) + c,$$

where $\bar{\varphi}(x)$ is a solution of equation (4) *continuous in $(a, b\rangle$ resp. $\langle a, b)$, and c is an arbitrary constant.*

In the above theorems condition (6) resp. (7) is essential. If condition (6) (condition (7)) is not fulfilled, the uniquness of the solutions of equation (4), continuous in the interval $(a, b\rangle$ (in the interval $\langle a, b)$) will not occur. Moreover, if in place of inequality (6) resp. (7) we put the converse inequalities, then every solution of equation (4) that is continuous in the interval (a, b) is also continuous in the interval $(a, b\rangle$ resp. $\langle a, b)$. The corresponding theorem will be preceded by two lemmas.

LEMMA III. *If the function $f(x)$ fulfils the hypotheses of lemma* I, *and the function $\lambda(x)$ is continuous, $\lambda(x) \neq 0$ in the interval (a, b), and fulfils the condition*

$$|\lambda(x)| < L \quad \textit{in} \quad (c, b), \quad c \in (a, b), \quad L \neq 1, \tag{14}$$

resp.

$$|\lambda(x)| > 1/L \quad \textit{in} \quad (a, c), \quad c \in (a, b), \quad L \neq 1, \tag{15}$$

then every function $\varphi(x)$ satisfying equation (4) *fulfils the inequality*

$$|\varphi[f^n(x)]| \leqslant M_n(x)\frac{1-L^n}{1-L} + L^n|\varphi(x)| \quad \textit{for} \quad x \in (c, b), \quad n = 1, 2, \ldots \tag{16}$$

resp.

$$|\varphi[f^{-n}(x)]| \leqslant N_n(x)\frac{L-L^{n+1}}{1-L} + L^n|\varphi(x)| \quad \textit{for} \quad x \in (a, c), \quad n = 1, 2, \ldots \tag{17}$$

where

$$M_n(x) \overset{\text{df}}{=} \sup_{\langle x, f^n(x)\rangle} |G(t)|, \qquad N_n(x) \overset{\text{df}}{=} \sup_{\langle f^{-n}(x), x\rangle} |G(t)|, \qquad x \in (a, b).$$

Proof. Let us assume that relation (14) is fulfilled.

Equation (4) can be written in the form

$$\varphi[f(x)] = G(x)+\lambda(x)\varphi(x).$$

Hence we obtain

$$(18)\qquad |\varphi[f(x)]| \leqslant |G(x)|+|\lambda(x)||\varphi(x)| \leqslant M_1(x)+L|\varphi(x)|, \qquad x\in(c,b).$$

Putting $f(x)$ in place of x in relation (18) we obtain

$$(19)\qquad |\varphi[f^2(x)]| \leqslant M_1[f(x)]+L|\varphi[f(x)]| \leqslant M_1[f(x)]+LM_1(x)+L^2|\varphi(x)|, \qquad x\in(c,b).$$

The functions $M_i(x)$ fulfil the inequalities:

$$M_i[f^j(x)] \leqslant M_{i+j}(x).$$

Hence we have, by (19),

$$|\varphi[f^2(x)]| \leqslant M_2(x)+LM_1(x)+L^2|\varphi(x)|, \qquad x\in(c,b).$$

By induction we obtain from (18)

$$|\varphi[f^n(x)]| \leqslant M_1[f^{n-1}(x)]+L|\varphi[f^{n-1}(x)]| \leqslant M_n(x)+LM_{n-1}(x)+\ldots+L^{n-1}M_1(x)+L^n|\varphi(x)|, \qquad x\in(c,b).$$

For a fixed x the sequence $\{M_n(x)\}$ is increasing:

$$M_i(x) \geqslant M_j(x) \quad \text{for} \quad i>j;$$

consequently

$$|\varphi[f^n(x)]| \leqslant M_n(x)[1+L+\ldots+L^{n-1}]+L^n|\varphi(x)| = M_n(x)\frac{1-L^n}{1-L}+L^n|\varphi(x)|, \qquad x\in(c,b),$$

which was to be proved.

The proof of relation (17) under assumption (15) is analogous.

COROLLARY. *Under the hypotheses of lemma* III *we also have the inequality*

$$(20)\qquad |\varphi[f^n(x)]| \leqslant M(x)\frac{1-L^n}{1-L}+L^n|\varphi(x)| \quad \textit{for} \quad x\in(c,b), \quad n=1,2,\ldots$$

resp.

$$(21)\qquad |\varphi[f^{-n}(x)]| \leqslant N(x)\frac{L-L^{n+1}}{1-L}+L^n|\varphi(x)| \quad \textit{for} \quad x\in(a,c), \quad n=1,2,\ldots$$

where

$$(22)\qquad M(x) \overset{\text{df}}{=} \sup_{\langle x,b)}|G(t)|, \qquad N(x) \overset{\text{df}}{=} \sup_{(a,x\rangle}|G(t)|, \qquad x\in(a,b).$$

Relations (20) and (21) follow from relations (16) and (17) in view of the inequalities

$$M_n(x) \leqslant M(x), \qquad N_n(x) \leqslant N(x), \qquad x\in(a,b), \qquad n=1,2,\ldots$$

LEMMA IV. *Let the function $f(x)$ fulfil the hypotheses of lemma* I, *and let the function $\lambda(x)$ be continuous, $\lambda(x)\neq 0$ in the interval (a,b), and fulfil the condition*

$$(23)\qquad |\lambda(x)| < \vartheta < 1 \quad \textit{in} \quad (b-\eta, b),$$

resp.

$$(24)\qquad |\lambda(x)| > 1/\vartheta > 1 \quad \textit{in} \quad (a, a+\eta),$$

where η is a positive number. Further, let the function $G(x)$ be continuous in the interval $(a,b\rangle$ resp. $\langle a,b)$. Then every function $\varphi(x)$ satisfying equation (4) *and bounded in an interval $\langle x_0, f(x_0)\rangle$, where x_0 is a number from the interval (a,b), is also bounded in an interval (c,b) resp. (a,c) for every $c\in(a,b)$.*

Proof. Let us assume that relation (23) is fulfilled and that

$$(25)\qquad |\varphi(x)| \leqslant C \quad \text{for} \quad x\in\langle x_0, f(x_0)\rangle,$$

and let c be a number from the interval (a,b). We can assume that $c<x_0$. We put

$$x_n \overset{\text{df}}{=} f^n(x_0), \qquad n=0,\pm1,\pm2,\ldots$$

On account of lemma I there exist indices μ and ν such that

$$(26)\qquad x_n\in\big(f(b-\eta),b\big) \quad \text{for} \quad n\geqslant\mu, \qquad x_{-n}\in(a,c) \quad \text{for} \quad n\geqslant\nu.$$

Since the function $\lambda(x)$ is continuous and distinct from zero in the interval (a,b), there exists a number $L>1$ such that

$$1/L < |\lambda(x)| < L \quad \text{for} \quad x\in\langle x_{-\nu}, x_\mu\rangle.$$

Now, let us take an arbitrary $x\in\langle x_{-\nu}, b)$. We shall consider four cases, depending on the position of the point x in the interval $\langle x_{-\nu}, b)$.

1° $x\in\langle x_{-\nu}, x_0)$.

Then there exist $x^*\in\langle x_0, f(x_0)\rangle$ and an index n such that

$$x = f^{-n}(x^*).$$

Evidently $0<n\leqslant\nu$. On account of lemma III

$$|\varphi(x)| = |\varphi[f^{-n}(x^*)]| \leqslant N_n(x^*)\frac{L-L^{n+1}}{1-L}+L^n|\varphi(x^*)|.$$

$N_n(x^*) \leqslant N_\nu(x^*)$, and thus

$$|\varphi(x)| \leqslant N_\nu(x^*)\frac{L-L^{n+1}}{1-L}+L^n|\varphi(x^*)|,$$

whence, putting

$$N \overset{\mathrm{df}}{=} \sup_{\langle x_0, f(x_0)\rangle} N_\nu(x),$$

we have by (25):

$$|\varphi(x)| \leqslant N\frac{L-L^{\nu+1}}{1-L}+L^\nu C,$$

which proves that the function $\varphi(x)$ is bounded in the interval $\langle x_{-\nu}, x_0)$.

2° $x \in \langle x_0, f(x_0)\rangle$.

In the interval $\langle x_0, f(x_0)\rangle$ the function $\varphi(x)$ is assumed to be bounded:

$$|\varphi(x)| \leqslant C.$$

3° $x \in \big(f(x_0), x_\mu\big\rangle$.

Then there exist $x^* \in \langle x_0, f(x_0)\big\rangle$ and an index n such that

$$x = f^n(x^*).$$

Evidently $0 < n \leqslant \mu$. By relation (20)

$$|\varphi(x)| = |\varphi[f^n(x^*)]| \leqslant M(x^*)\frac{1-L^n}{1-L}+L^n|\varphi(x^*)|,$$

whence, putting

$$M \overset{\mathrm{df}}{=} \sup_{\langle x_0, f(x_0)\rangle} M(x),$$

we have by (25):

$$|\varphi(x)| \leqslant M\frac{1-L^\mu}{1-L}+L^\mu C,$$

which proves that the function $\varphi(x)$ is bounded in the interval $\big(f(x_0), x_\mu\big\rangle$.

4° $x \in (x_\mu, b)$.

Since, according to (26), $x_\mu > f(b-\eta)$, $x_{\mu-1} > b-\eta$, and hence

$$|\lambda(y)| < \vartheta < 1 \quad \text{for} \quad y \in \langle x_{\mu-1}, b).$$

There exist $x^* \in \langle x_{\mu-1}, x_\mu\rangle$ and an index $n > 0$ such that

$$x = f^n(x^*).$$

Consequently, according to (20):

$$|\varphi(x)| = |\varphi[f^n(x^*)]| \leqslant M(x^*)\frac{1-\vartheta^n}{1-\vartheta}+\vartheta^n|\varphi(x^*)|.$$

On account of the preceding part of the proof the function $\varphi(y)$ is bounded in the interval $\langle x_{\mu-1}, x_\mu\rangle \subset \big(f(x_0), x_\mu\big\rangle$:

$$|\varphi(y)| \leqslant K \quad \text{for} \quad y \in \langle x_{\mu-1}, x_\mu\rangle.$$

Consequently, putting

$$\overline{M} \overset{\mathrm{df}}{=} \sup_{\langle x_{\mu-1}, x_\mu\rangle} M(x),$$

we have

$$|\varphi(x)| \leqslant \overline{M}\frac{1-\vartheta^n}{1-\vartheta}+\vartheta^n K \leqslant \overline{M}\frac{1}{1-\vartheta}+K,$$

which proves that the function $\varphi(x)$ is bounded in the interval (x_μ, b).

Thus we have proved that the function $\varphi(x)$ is bounded in the intervals $\langle x_{-\nu}, x_0)$, $\langle x_0, f(x_0)\rangle$, $\big(f(x_0), x_\mu\big\rangle$ and (x_μ, b). Then it is bounded in the interval $\langle x_{-\nu}, b)$ and thus also in the interval $(c, b) \subset \langle x_{-\nu}, b)$.

The proof for the interval (a, c) under assumption (24) is analogous.

THEOREM IV. *Let the function $f(x)$ fulfil the hypotheses of lemma* I, *and let the functions $G(x)$ and $\lambda(x)$ be continuous in the interval $(a, b\rangle$, resp. $\langle a, b)$. Moreover, let the function $\lambda(x)$ be distinct from zero in the interval (a, b) and fulfil condition* (23) *resp.* (24). *Then every function $\varphi(x)$ satisfying equation* (4) *and bounded in an interval $\langle x_0, f(x_0)\rangle$, where x_0 is a number from the interval (a, b), is continuous at the point $x = b$ resp. $x = a$.*

Proof. Let us assume that relation (23) is fulfilled and that the function $\varphi(x)$ satisfies equation (4) and is bounded in an interval $\langle x_0, f(x_0)\rangle$. Moreover we shall assume, at first, that $G(b) = 0$.

On account of inequality (20)

$$|\varphi[f^n(x)]| \leqslant M(x)\frac{1-\vartheta^n}{1-\vartheta}+\vartheta^n|\varphi(x)| \quad \text{for} \quad x \in (b-\eta, b).$$

According to lemma IV

$$|\varphi(x)| \leqslant K \quad \text{for} \quad x \in (b-\eta, b);$$

consequently,

$$|\varphi[f^n(x)]| \leqslant M(x)\frac{1-\vartheta^n}{1-\vartheta}+\vartheta^n K \quad \text{for} \quad x \in (b-\eta, b).$$

Hence we have, since $\vartheta < 1$,

$$|\varphi[f^n(x)]| \leqslant M(x)\frac{1}{1-\vartheta}+\vartheta^n K \quad \text{for} \quad x \in (b-\eta, b). \tag{27}$$

Let us take an arbitrary $\varepsilon > 0$. Since the function $G(x)$ is continuous for $x = b$ and $G(b) = 0$, we have by (22)

$$\lim_{x\to b} M(x) = 0.$$

Thus there exists a number δ_1, $0 < \delta_1 < \eta$, such that

$$M(x) < \tfrac{1}{2}(1-\vartheta)\varepsilon \quad \text{for} \quad x \in (b-\delta_1, b).$$

Since $\vartheta < 1$, there exists also an index N such that

$$\vartheta^n < \varepsilon/2K \quad \text{for} \quad n \geqslant N.$$

Consequently, for $x \in (b-\delta_1, b)$ and $n \geqslant N$, we have by (27)

$$|\varphi[f^n(x)]| < \varepsilon,$$

and in particular

$$(28) \qquad |\varphi[f^N(x)]| < \varepsilon \quad \text{for} \quad x \in (b-\delta_1, b).$$

Let us put

$$\delta \overset{\text{df}}{=} b - f^N(b-\delta_1).$$

Since $f^N(x) > x$,

$$0 < \delta = b - f^N(b-\delta_1) < b - b + \delta_1 = \delta_1.$$

And since $f^N((b-\delta_1, b)) = (b-\delta, b)$, for every $x \in (b-\delta, b)$

$$x^* \overset{\text{df}}{=} f^{-N}(x) \in (b-\delta_1, b).$$

Hence, by (28)

$$|\varphi[f^N(x^*)]| < \varepsilon,$$

i. e.

$$|\varphi(x)| < \varepsilon \quad \text{for} \quad x \in (b-\delta, b),$$

which proves that

$$(29) \qquad \lim_{x\to b} \varphi(x) = 0.$$

On the other hand, putting $x = b$ in equation (4), we have, since $G(b) = 0$ and $\lambda(b) \neq 1$:

$$(30) \qquad \varphi(b) = 0.$$

Relations (29) and (30) prove that the function $\varphi(x)$ is continuous for $x = b$.

Now let $G(b)$ be arbitrary. If the function $\varphi(x)$ is an arbitrary solution of equation (4), bounded in the interval $\langle x_0, f(x_0)\rangle$, then the function

$$(31) \qquad \gamma(x) \overset{\text{df}}{=} \varphi(x) - \frac{G(b)}{1-\lambda(b)}$$

satisfies the equation

$$\gamma[f(x)] - \lambda(x)\gamma(x) = G(x) - \frac{1-\lambda(x)}{1-\lambda(b)} G(b),$$

and is also bounded in the interval $\langle x_0, f(x_0)\rangle$. The function

$$H(x) \overset{\text{df}}{=} G(x) - \frac{1-\lambda(x)}{1-\lambda(b)} G(b)$$

is continuous, like the function $G(x)$, and moreover $H(b) = 0$. Consequently, on account of the first part of the proof, the function $\gamma(x)$ is continuous for $x = b$. Hence, by (31), also the function $\varphi(x)$ is continuous for $x = b$.

If we assume condition (24), the proof of the continuity of solutions of equation (4) at the point $x = a$ is analogous.

COROLLARY 1. *Under the hypotheses of theorem* IV *every function* $\varphi(x)$ *satisfying equation* (4) *and continuous in the interval* (a, b) *is also continuous in the interval* $(a, b\rangle$, *resp.* $\langle a, b)$.

COROLLARY 2. *Under the hypotheses of theorem* IV *equation* (4) *possesses infinitely many solutions that are continuous in the interval* $(a, b\rangle$, *resp.* $\langle a, b)$.

II. Equation with constant coefficients. In this section we shall discuss the equation

$$(32) \qquad A_0\varphi[f^n(x)] + A_1\varphi[f^{n-1}(x)] + \ldots + A_n\varphi(x) = F(x), \qquad x \in \langle a, b\rangle,$$

where A_i are constant complex coefficients. We shall assume that $A_0 = 1$ and that $A_n \neq 0$. The latter assumption can be made without loss of generality of our considerations, for if

$$A_{n-i} = 0 \quad \text{for} \quad i = 0, \ldots, k, \quad k < n,$$

and

$$A_{n-k-1} \neq 0,$$

then the transformation

$$y = f^{k+1}(x), \qquad x \in \langle a, b\rangle,$$

changes equation (32) into equation

$$A_0\varphi[f^{n-k-1}(y)] + A_1\varphi[f^{n-k-2}(y)] + \ldots + A_{n-k-1}\varphi(y) = F_1(y), \qquad y \in \langle a, b\rangle,$$

in which $A_{n-k-1} \neq 0$.

We shall now introduce some notions.

DEFINITION. The polynomial

$$(33) \qquad W(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \ldots + A_n$$

will be called a *characteristic polynomial* of equation (32). The roots of polynomial (33) will be called *characteristic roots* of equation (32).

Thus equation (32) always has n characteristic roots (not necessarily distinct). All those roots are distinct from zero.

Let λ_0 be an arbitrary characteristic root of equation (32). There exists a polynomial

$$\overline{W}(\lambda) = B_0\lambda^{n-1}+B_1\lambda^{n-2}+\ldots+B_{n-1}, \quad B_0 = 1, \quad B_{n-1}\neq 0,$$

such that

$$\overline{W}(\lambda)(\lambda-\lambda_0) \equiv W(\lambda).$$

The numbers A_i, B_i, and λ_0 are connected by the formulae:

$$\begin{aligned} &A_0 = B_0 = 1,\\ (34)\qquad &A_i = B_i-\lambda_0 B_{i-1}, \quad i = 1,\ldots,n-1,\\ &A_n = -\lambda_0 B_{n-1}. \end{aligned}$$

We shall prove

LEMMA V. *If the function $\varphi(x)$ satisfies equation* (32), *then the function*

$$(35)\qquad \psi(x) \stackrel{\text{df}}{=} \varphi[f(x)]-\lambda_0\varphi(x)$$

satisfies the equation

$$(36)\qquad B_0\psi[f^{n-1}(x)]+B_1\psi[f^{n-2}(x)]+\ldots+B_{n-1}\psi(x) = F(x),$$

and conversely, if the function $\psi(x)$, defined by (35), *satisfies equation* (36), *then the function $\varphi(x)$ satisfies equation* (32).

Proof. Let us suppose that the function $\varphi(x)$ satisfies equation (32). Making use of formulae (34) we can write equation (32) in the form

$$(37)\qquad B_0\varphi[f^n(x)]+(B_1-\lambda_0 B_0)\varphi[f^{n-1}(x)]+\ldots+ \\ +(B_{n-1}-\lambda_0 B_{n-2})\varphi[f(x)]+(-\lambda_0 B_{n-1})\varphi(x) = F(x),$$

i. e.

$$(38)\qquad B_0\big(\varphi[f^n(x)]-\lambda_0\varphi[f^{n-1}(x)]\big)+B_1\big(\varphi[f^{n-1}(x)]-\lambda_0\varphi[f^{n-2}(x)]\big)+\ldots+ \\ +B_{n-1}\big(\varphi[f(x)]-\lambda_0\varphi(x)\big) = F(x),$$

which means that the function $\psi(x)$ satisfies equation (36).

Now let us suppose that the function $\psi(x)$, defined by relation (35), satisfies equation (36). Then the function $\varphi(x)$ satisfies equation (38), and thus also equation (37). Hence it follows immediately, according to relations (34), that the function $\varphi(x)$ satisfies also equation (32), which was to be proved.

As an immediate consequence of lemma V we obtain

THEOREM V. *Equation* (32) *is equivalent to the system of equations*

$$(39)\qquad \begin{aligned} &\varphi[f(x)]-\lambda_1\varphi(x) = \psi_1(x),\\ &\psi_1[f(x)]-\lambda_2\psi_1(x) = \psi_2(x),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\psi_{n-1}[f(x)]-\lambda_n\psi_{n-1}(x) = F(x), \end{aligned}$$

where $\lambda_1,\ldots,\lambda_n$ are a full sequence of characteristic roots of equation (32).

In the above theorem equivalence is understood in the following manner: If the function $\varphi(x)$ satisfies equation (32), then there exists a system of functions

$$\psi_1(x),\ldots,\psi_{n-1}(x)$$

such that the system of functions

$$(40)\qquad \varphi(x),\psi_1(x),\ldots,\psi_{n-1}(x)$$

satisfies the system of equations (39), and conversely, if the system of functions (40) satisfies the system of equations (39), then the function $\varphi(x)$ satisfies equation (32).

Theorem V enables us to discuss equations of the first order

$$(41)\qquad \varphi[f(x)]-\lambda\varphi(x) = G(x),$$

instead of equation (32). Equations of type (41) have been dealt with in the first part of the present paper. In the sequel we shall deduce from the properties of equation (41) theorems which determine the number of solutions of equation (32) that are continuous in the intervals (a,b), $(a,b\rangle$, or $\langle a,b)$.

THEOREM VI. *If the function $f(x)$ fulfils the hypotheses of lemma* I, *and the function $F(x)$ is continuous in the interval (a,b), then equation* (32) *possesses infinitely many solutions that are continuous in the open interval (a,b).*

If, moreover, the coefficients A_i are real and the function $F(x)$ assumes real values only, then equation (32) *possesses infinitely many real solutions that are continuous in the open interval (a,b).*

The first part of the above theorem follows immediately from theorems I and V. The second part follows from a general theorem on the existence of an infinite number of (real) solutions that are continuous in the interval (a,b) for the equation

$$F\big(x,\varphi(x),\varphi[f_1(x)],\ldots,\varphi[f_n(x)]\big) = 0.$$

This theorem is to be found in [5].

The number of solutions of equation (32) that are continuous in the interval $(a, b\rangle$ resp. $\langle a, b)$ is determined by the following

THEOREM VII. *Let the function $f(x)$ fulfil the hypotheses of lemma* I, *and let λ_i, $i = 1, \ldots, n$, be a full set of characteristic roots of equation* (32).

I. *Let us assume the function $F(x)$ to be continuous in the interval $(a, b\rangle$.*

1° *If*

$$|\lambda_i| < 1, \quad i = 1, \ldots, n,$$

then equation (32) *possesses infinitely many solutions that are continuous in the interval $(a, b\rangle$. More precisely, every solution of equation* (32) *that is continuous in the interval (a, b) is also continuous in the interval $(a, b\rangle$.*

2° *If*

$$|\lambda_i| \geqslant 1, \quad \lambda_i \neq 1, \quad i = 1, \ldots, n,$$

then equation (32) *possesses at most one solution that is continuous in the interval $(a, b\rangle$.*

3° *If*

$$|\lambda_i| \geqslant 1, \quad i = 1, \ldots, n,$$

but some of the characteristic roots are equal to one, then equation (32) *possesses at most one solution, up to an additive constant, that is continuous in the interval $(a, b\rangle$. The necessary condition of existence of such a solution is the relation $F(b) = 0$.*

4° *If there exist indices i_0 and k_0 such that*

$$|\lambda_{i_0}| < 1 \quad \textit{and} \quad |\lambda_{k_0}| \geqslant 1,$$

then equation (32) *either has no solution that is continuous in the interval $(a, b\rangle$, or has infinitely many such solutions.*

II. *Let us assume the function $F(x)$ to be continuous in the interval $\langle a, b)$.*

1° *If*

$$|\lambda_i| > 1, \quad i = 1, \ldots, n,$$

then equation (32) *possesses infinitely many solutions that are continuous in the interval $\langle a, b)$. More precisely, every solution of equation* (32) *that is continuous in the interval (a, b), is also continuous in the interval $\langle a, b)$.*

2° *If*

$$|\lambda_i| \leqslant 1, \quad \lambda_i \neq 1, \quad i = 1, \ldots, n,$$

then equation (32) *possesses at most one solution that is continuous in the interval $\langle a, b)$.*

3° *If*

$$|\lambda_i| \leqslant 1, \quad i = 1, \ldots, n,$$

but some of the characteristic roots are equal to one, then equation (32) *possesses at most one solution, up to an additive constant, that is continuous in the interval $\langle a, b)$. The necessary condition of the existence of such a solution is the relation $F(a) = 0$.*

4° *If there exist indices i_0 and k_0 such that*

$$|\lambda_{i_0}| \leqslant 1 \quad \textit{and} \quad |\lambda_{k_0}| > 1,$$

then equation (32) *either has no solution that is continuous in the interval $\langle a, b)$, or has infinitely many such solutions.*

If the coefficients A_i are real and the function $F(x)$ assumes real values only, then all the assertions of the present theorem remain valid if the word "solution" is replaced by "real solution".

Proof. We shall prove the first part of the theorem. Let us assume that the function $F(x)$ is continuous in the interval $(a, b\rangle$.

1° It is the immediate consequence of theorems IV, V and VI.

2° It is the immediate consequence of theorems II and V.

3° Since the order of roots in system (39) is not essential, we can assume that

$$\lambda_1 = \ldots = \lambda_k = 1,$$

$$|\lambda_i| \geqslant 1, \quad \lambda_i \neq 1, \quad i = k+1, \ldots, n.$$

From theorem II it follows that the system of equations

$$\psi_k[f(x)] - \lambda_{k+1}\psi_k(x) = \psi_{k+1}(x),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\psi_{n-1}[f(x)] - \lambda_n\psi_{n-1}(x) = F(x)$$

has at most one solution such that the function $\psi_k(x)$ is continuous in the interval $(a, b\rangle$. On account of theorem III the equation

$$\psi_{k-1}[f(x)] - \lambda_k\psi_{k-1}(x) = \psi_k(x),$$

i. e. the equation

$$\psi_{k-1}[f(x)] - \psi_{k-1}(x) = \psi_k(x) \tag{42}$$

has at most one solution continuous in the interval $(a, b\rangle$, up to an additive constant. Consequently, equation (42) has at most one solution continuous in the interval $(a, b\rangle$ and fulfilling the condition

$$\psi_{k-1}(b) = 0.$$

Further, the equation

$$\psi_{k-2}[f(x)]-\psi_{k-2}(x) = \psi_{k-1}(x) \tag{43}$$

has no continuous solution in the interval $(a, b\rangle$ at all, when $\psi_{k-1}(b) \neq 0$, and has at most one (up to an additive constant) in the case $\psi_{k-1}(b) = 0$. Thus equation (43) has at most one solution that is continuous in the interval $(a, b\rangle$ and fulfils the condition

$$\psi_{k-2}(b) = 0.$$

Reasoning in this manner, we come to the conclusion that the equation

$$\varphi[f(x)]-\varphi(x) = \psi_1(x)$$

has at most one solution that is continuous in the interval $(a, b\rangle$, up to an additive constant.

Supposing that equation (32) has a solution $\varphi(x)$ continuous at the point $x = b$, and putting $x = b$ in relation (32), we obtain

$$\varphi(b)\sum_{i=1}^{n} A_i = F(b).$$

But, since $\lambda = 1$ is a characteristic root of equation (32),

$$\sum_{i=1}^{n} A_i = 0,$$

whence

$$F(b) = 0. \tag{44}$$

Thus relation (44) is the necessary condition of the existence of a continuous solution of equation (32) in the interval $(a, b\rangle$.

4° As in the preceding case, we can assume that

$$|\lambda_i| < 1 \quad \text{for} \quad i = 1, \ldots, k, \qquad |\lambda_i| \geqslant 1 \quad \text{for} \quad i = k+1, \ldots, n, \qquad 0 < k < n.$$

The system of equations

$$\begin{gathered} \psi_k[f(x)]-\lambda_{k+1}\psi_k(x) = \psi_{k+1}(x), \\ \cdots\cdots\cdots\cdots \\ \psi_{n-1}[f(x)]-\lambda_n\psi_{n-1}(x) = F(x), \end{gathered} \tag{45}$$

either has no continuous solution in the interval $(a, b\rangle$, or has a solution

$$\bar{\psi}_k(x), \ldots, \bar{\psi}_{n-1}(x)$$

such that the function $\bar{\psi}_k(x)$ is continuous in the interval $(a, b\rangle$.

In the first case, on account of theorem V, equation (32) has no solution continuous in the interval $(a, b\rangle$. In the second case the system of equations

$$\begin{gathered} \varphi[f(x)]-\lambda_1\varphi(x) = \psi_1(x), \\ \cdots\cdots\cdots\cdots \\ \psi_{k-1}[f(x)]-\lambda_k\psi_{k-1}(x) = \bar{\psi}_k(x) \end{gathered} \tag{46}$$

possesses, according to theorem IV, infinitely many solutions that are continuous in the interval $(a, b\rangle$. Hence, on account of theorem V, equation (32) possesses infinitely many solutions that are continuous in the interval $(a, b\rangle$.

In this last case a separate proof has to be given for the assertion that if the coefficients A_i are real and the function $F(x)$ is real-valued, then equation (32) has either no real solutions (that are continuous in the interval $(a, b\rangle$), or infinitely many such solutions.

Since the coefficients A_i are real, the characteristic roots λ_i are either real, or form complex-conjugate couples. Conjugate-complex numbers have equal modulus, so if a complex root occurs in system (46), then also its complex-conjugate must occur there (for in system (46) occur those and only those characteristic roots whose absolute values are smaller than 1). Consequently, on account of theorem V, system (46) is equivalent to an equation

$$\varphi[f^k(x)]+C_1\varphi[f^{k-1}(x)]+\ldots+C_k\varphi(x) = \bar{\psi}_k(x) \tag{47}$$

with all coefficients C_i real. Besides, all characteristic roots of equation (47) have absolute values smaller than 1. Thus, if the function $\bar{\psi}_k(x)$ is real-valued, then equation (47) has infinitely many solutions that are continuous in the interval $(a, b\rangle$. And if the function $\bar{\psi}_k(x)$ is not real-valued, then equation (47) cannot have a real solution at all, for otherwise the function $\bar{\psi}_k(x)$, as a linear combination of real functions with real coefficients, would have to be also real-valued. Thus also equation (32) either possesses no real solution (that is continuous in the interval $(a, b\rangle$), or has infinitely many such solutions.

The proof of the second part of the theorem is analogous.

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ANNALES
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On the asymptotic behaviour of harmonic functions in the semi-space

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Let $f(x, y)$ be a continuous and bounded function in the open plane, say $|f(x, y)| \leqslant M$, and let $u(x, y, z)$ be a solution of the Dirichlet problem for the upper semi-space $z > 0$ with $f(x, y)$ as its boundary value. It is known [1] that in the class of functions $u(x, y, z)$ satisfying the condition

$$|u(x, y, z)| \leqslant M_1[|x|^a+|y|^a+|z|^a+1] \quad \text{for} \quad z \geqslant 0, \quad -\infty \leqslant x, y \leqslant \infty,$$

where M_1 and $a < 1$ are positive numbers, this solution is unique and is expressed by the formula

$$u(x, y, z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \frac{dG}{dn} d\xi d\eta$$

where G is Green's function for the upper semi-space $z > 0$ i. e.

$$G = \frac{1}{pq} - \frac{1}{\overline{p}q}, \quad p = p(x, y, z), \quad \overline{p} = \overline{p}(x, y, -z), \quad q = q(\xi, \eta, \zeta)$$

and dG/dn is the inner normal derivative. Next $u(x, y, z)$ will denote the solution of the above-mentioned Dirichlet problem.

Let us consider the semi-line

(1) $$x = x_0+at, \quad y = y_0+bt, \quad z = z_0+ct, \quad t \in \langle 0, \infty),$$

a and b being the arbitrary real, c a positive number and (x_0, y_0, z_0) an arbitrary point in the semi-space $z \geqslant 0$. Then, we shall prove

THEOREM 1. *If* $(x, y, z) \underset{(1)}{\to} \infty$ (¹), *then neither* $\underline{\lim}\, u(x, y, z)$ *nor* $\overline{\lim}\, u(x, y, z)$ *depend on* (x_0, y_0, z_0).

(¹) The symbol $(x, y, z) \underset{(1)}{\to} \infty$ denotes that a point (x, y, z) tends to ∞ along the semi-line (1).