ANNALES
POLONICI MATHEMATICI
IX (1960)

Differential inequalities with unbounded operators in Banach spaces

by W. Mlak (Kraków)

The present paper attempts to give an abstract formulation of theorems concerning differential inequalities. We apply the Hille-Yosida theory of semi-groups of linear bounded operators (see for instance [7]). More precisely, we deal with positivity-preserving semi-groups. Such semi-groups have been considered in connection with temporally homogeneous stochastic processes of Markoff type. For details in this field we refer to [3], [4], [5], [6], [17]. It must be remarked that infinitesimal generators of positivity-preserving semi-groups in concrete functional spaces of continuous or summable functions are in a certain sense second order elliptic operators possessing the maximum property (see [4], [15] and [18]). Thus the theory developed in the present paper combined with the results of the papers mentioned above gives immediately an operator-theoretical treatment of parabolic differential inequalities. Our theorems make it possible to consider countable systems of differential inequalities in the space (l). It is sufficient in this case to combine our theorems with the theory of integration of Kolmogoroff equations of Markoff processes with a countable number of possible states (see [8]). However, a more detailed study in that field may be conducted by the application of methods similar to that used in [6] and [12].

In our investigations we make use of some theorems similar to the generalized mean value theorem (see [2], [11] and [16]). The relation of inequality is defined by means of a positive cone (see [7]).

For the terminology and some simple properties of semi-groups of operators we refer to [7].

We discuss linear differential inequalities in sections 2, 3 and 4. Section 5 concerns some simple almost linear differential inequalities.

1. Let E be a Banach space. The norm of $x \in E$ is denoted by $|x|$. By $\xi, \eta, \ldots$ we denote continuous and linear functionals over E. The totality of such functionals, i. e. the adjoint space of E is denoted by E^*. The real-valued functions of the real variable t are denoted by small Greek letters $\varphi, \psi, \sigma, \ldots$ etc. The operator A is linear if it is additive and homogeneous. The domain of A is denoted by $D[A]$. We deal only with operators whose domain and range are included in the same space E.

S is a cone if the following conditions hold:

(α) if $x \in S$ and $y \in S$ then $x+y \in S$;

(β) if $\lambda \geqslant 0$ and $x \in S$ then $\lambda x \in S$;

(γ) S is closed.

We define the relation "$\leqslant$" as follows:

$$x \leqslant y \overset{\text{def}}{\equiv} y-x \in S. \tag{δ}$$

Let S^* be the set of all functionals $\xi \in E^*$ such that $\xi x \geqslant 0$ for each $x \in S$. The lemma below follows immediately from the well known results of Mazur [10]:

LEMMA 1. *Suppose that for every $\xi \in S^*$ the inequality $\xi x \geqslant 0$ holds. Then $x \in S$.*

It is thus seen that S may be identified with the set of x-es satisfying $\xi x \geqslant 0$ with suitable ξ.

A linear operator is called *positive* if $Bx \geqslant \theta$ for $x \in S \cap D[B]$. We say that a certain condition holds *nearly everywhere* [19] if it holds everywhere except an at most denumerable set of points. For the sake of clarity we use in the following only the right-hand upper Dini derivative $\overline{D}_+$. However, the properties discussed remain true if $\overline{D}_+$ is replaced by any other Dini derivative.

We now establish some notational conventions. The symbol w-lim denotes a weak limit, s-lim a strong one. Let $x(t)$ be defined in a neighborhood of t_0. We define

$$D_+^w x(t_0) = \operatorname*{w-lim}_{h\to 0+} \frac{x(t_0+h)-x(t_0)}{h},$$

$$D_+^s x(t_0) = \operatorname*{s-lim}_{h\to 0+} \frac{x(t_0+h)-x(t_0)}{h},$$

$$x'(t_0) = \operatorname*{s-lim}_{h\to 0} \frac{x(t_0+h)-x(t_0)}{h}.$$

The right-hand weak (strong) partial differentiation is denoted by $\partial_+^w/\partial s$ $(\partial_+^s/\partial s)$. The bilateral strong derivative is denoted by $\partial^s/\partial s$. The Bochner integral of $x(t)$ is shortly written as $\int_a^b x(\tau)d\tau$; the Pettis integral is denoted by $(\mathrm{P})\int_a^b x(\tau)d\tau$.

It follows from Zygmund's lemma that if $\varphi(t)$, $\psi(t)$ are continuous and satisfy nearly everywhere the inequality $\overline{D}_+\varphi(t) \leqslant \psi(t)$, then $\varphi(\tau_1)-\varphi(\tau_2) \leqslant \int_{\tau_2}^{\tau_1} \psi(\tau)d\tau$. For $\varphi(t)$ absolutely continuous and $\psi(t)$ summable, the inequality $\varphi'(t) \leqslant \psi(t)$ satisfied almost everywhere implies that $\varphi(\tau_1)-\varphi(\tau_2) \leqslant \int_{\tau_1}^{\tau_2} \psi(\tau)d\tau$. It is thus seen that the lemma below follows from lemma 1 combined with the definition of the Pettis integral (see for instance [7]).

LEMMA 2. *Suppose that $x(t)$ and $y(t)$ satisfy one of the following conditions:*

(1) *The function $x(t)$ is weakly continuous and for every $\xi \in S^*$ there exists an at most denumerable set $Z_\xi \subset (0, a)$ such that $\overline{D}_+\xi x(t) \leqslant \xi y(t)$ for $t \in (0, a)-Z_\xi$. The function $y(t)$ is weakly continuous and Pettis integrable.*

(2) *For every $\xi \in S^*$ the function $\xi x(t)$ is absolutely continuous and $\frac{d}{dt}\xi x(t) \leqslant \xi y(t)$ for $t \in (0, a)-Z_\xi$, $\operatorname{mes} Z_\xi = 0$. The function $y(t)$ is Pettis integrable.*

Then

$$x(t_2)-x(t_1) \leqslant (\mathrm{P})\int_{t_1}^{t_2} y(s)ds \quad (0 < t_1 < t_2). \tag{3}$$

2. Suppose that we are given a one-parameter family $A(t)$ $(0 < t < a)$ of linear operators. Let the family $\{U(t, s)\}$ of linear bounded operators possess the following properties:

(4) $U(t, s)$ is positive for $0 < s \leqslant t$.

(5) The strong derivative $\frac{\partial_+^s U(t, s)x}{\partial s}$ $(s < t)$ exists for $x \in D[A]$ and $\frac{\partial_+^s U(t, s)x}{\partial s} = -U(t, s)A(s)x$.

(6) For every $x \in E$ the function $u(s) = U(t, s)x$ is strongly continuous in s and $U(t, t) = I$.

THEOREM 1. *Let $U(t, s)$ satisfy (4), (5) and (6). Suppose that $x(t)$, $y(t)$ are strongly continuous in $(0, a)$. We assume that $x(t) \in D[A(t)]$ $(0 < t < a)$. Let the inequality $D_+^s x(t) \leqslant A(t)x(t)+y(t)$ be satisfied nearly everywhere in $(0, a)$. Then*

$$x(t) \leqslant U(t, s)x(s)+\int_s^t U(t, \tau)y(\tau)d\tau \quad \textit{for} \quad 0 < s < t. \tag{7}$$

Proof. It follows from (4) that for a fixed t the inequality

$$U(t, s)D_+^s x(s) \leqslant U(t, s)A(s)x(s)+U(t, s)y(s) \tag{8}$$

holds nearly everywhere in $(0, t)$. Using (6) we conclude that $U(t, s)x(s)$ is continuous in s. Suppose that $D^s_+ x(s_0)$ exists. It follows from the formula

$$\frac{1}{h}[U(t, s_0+h)x(s_0+h)-U(t, s_0)x(s_0)]$$
$$=\frac{1}{h}[U(t, s_0+h)x(s_0)-U(t, s_0)x(s_0)]+U(t, s_0+h)D^s_+ x(s_0)+$$
$$+U(t, s_0+h)\frac{o(h)}{h}$$

and from (6) that

$$\left(\frac{\partial^s_+ U(t, s)x(s)}{\partial s}\right)_{s=s_0}=\left(\frac{\partial^s_+ U(t, s)x(s_0)}{\partial s}\right)_{s=s_0}+U(t, s_0)D^s_+ x(s_0). \tag{9}$$

By (5), (8) and (9) we get

$$\frac{\partial^s_+ \big(U(t, s)x(s)\big)}{\partial s}\leqslant U(t, s)y(s) \tag{10}$$

nearly everywhere in $(0, t)$. Using (10) and lemma 2 we get

$$U(t, s_2)x(s_2)\leqslant U(t, s_1)x(s_1)+\int_{s_1}^{s_2} U(t, \tau)y(\tau)\,d\tau \tag{11}$$

for $s_1 < s_2 < t$. Let $s = s_1$ and $s_2 \to t$. (7) now follows from (11).

Some obvious consequences of theorem 1 must now be mentioned.

THEOREM 2. *Let $U(t, s)$ satisfy* (4), (5) *and* (6). *Let $x_i(t)$, $y_i(t)$ $(i = 1, 2)$ be strongly continuous in $(0, a)$. Let the inequalities*

$$D^s_+ x_1(t)\leqslant A(t)x_1(t)+y_1(t),$$
$$D^s_+ x_2(t)\geqslant A(t)x_2(t)+y_2(t)$$

be satisfied nearly everywhere in $(0, a)$. Then

$$x_1(t)-x_2(t)\leqslant U(t, s)[x_1(s)-x_2(s)]+\int_s^t U(t, \tau)[y_1(\tau)-y_2(\tau)]\,d\tau \qquad (s\leqslant t).$$

Moreover, if $x_1(0)\leqslant x_2(0)$ and $y_1(t)\leqslant y_2(t)$ then $x_1(t)\leqslant x_2(t)$.

Assumptions (4), (5) and (6) hold for $U(t, s) = T(t-s)$ where $\{T(\tau)\}$ is a positivity-preserving semi-group of class (C_0): $A(t) = A$ and A is the infinitesimal generator of $\{T(\tau)\}$. Other conditions which ensure the existence of $U(t, s)$ satisfying (5) and (6) for time-dependent $A(t)$ are given in [9], [14]. In order to ensure (4) one must assume that the semi-groups generated by $A(t)$ are positive.

3. Suppose now that A is the infinitesimal generator of positivity-preserving semi-group $\{T(u)\}$ of class $(0, A)$. We shall prove the following theorem:

THEOREM 3. *Let $x(t)$ be weakly continuous in $(0, a)$, and let the inequality*

$$D^w_+ x(t)\leqslant Ax(t) \tag{12}$$

be satisfied nearly everywhere in $(0, a)$. Then $x(t)\leqslant T(t-s)x(s)$ for $s<t$.

Proof. Define $x_\lambda(t)$ as follows: $x_\lambda(t) = \lambda R(\lambda, A)x(t)$. Suppose that $D^w_+ x(t_0)$ exists and let $\xi\in E^*$. We have

$$\xi\frac{x_\lambda(t_0+h)-x_\lambda(t_0)}{h}=\xi\lambda R(\lambda, A)\frac{x(t_0+h)-x(t_0)}{h}.$$

But $x(t_0-h)-x(t_0)/h$ tends weakly to $D^w_+ x(t_0)$ and $\xi\lambda R(\lambda, A)\in E^*$. Hence $D^w_+ x_\lambda(t_0) = \lambda R(\lambda, A)D^w_+ x(t_0)$. The resolvent $R(\lambda, A)$ is positive. We thus infer by (12) that

$$D^w_+ x_\lambda(t)\leqslant Ax_\lambda(t) \tag{13}$$

nearly everywhere and consequently

$$T(t-s)D^w_+ x_\lambda(s)-T(t-s)Ax_\lambda(s)\leqslant\theta \tag{14}$$

nearly everywhere in $(0, t)$. Observe now that

$$[T(\tau_2)-T(\tau_1)]\lambda R(\lambda, A)x=\int_{\tau_1}^{\tau_2} T(\tau)\lambda AR(\lambda, A)x\,d\tau, \qquad x\in E \tag{15}$$

$(0<\tau_1<\tau_2)$. This implies that

$$|[T(\tau_2)-T(\tau_1)]\lambda R(\lambda, A)x|\leqslant\sup_\tau\lambda|T(\tau)|\,|AR(\lambda, A)|\,|x|\,|\tau_2-\tau_1|, \tag{16}$$

where sup is taken over an arbitrary compact interval of $(0, a)$ which includes τ_1 and τ_2. Thus, the operator-valued function $T(t)\lambda R(\lambda, A)$ is continuous in t $(t>0)$ in the uniform operator topology. We will prove now that the derivative $\dfrac{\partial^w_+ T(t-s)x_\lambda(s)}{\partial s}$ exists nearly everywhere and is equal to the difference appearing in the left-hand member of (14). Let $\xi\in E^*$ and let $D^w_+ x(t)$ exist at $t = t_0$. We have

$$\xi\frac{T(t-t_0-h)x_\lambda(t_0+h)-T(t-t_0)x_\lambda(t_0)}{h}$$
$$=-\xi\frac{T(t-t_0-h)-T(t-t_0)}{-h}x_\lambda(t_0)+\xi T(t-t_0-h)\frac{x_\lambda(t_0+h)-x_\lambda(t_0)}{h}.$$

The first member in the right-hand sum tends to $-\xi AT(t-t_0)x_\lambda(t_0)$. The second member is equal to

$$(17)\quad \xi T(t-t_0-h)\lambda R(\lambda, A)\frac{x(t_0+h)-x(t_0)}{h}$$

$$= \xi[T(t-t_0-h)-T(t-t_0)]\lambda R(\lambda, A)\frac{x(t_0+h)-x(t_0)}{h}+$$

$$+\xi T(t-t_0)\frac{x_\lambda(t_0+h)-x_\lambda(t_0)}{h}.$$

Suppose that $h_n \to 0+$. The second member of the right-hand sum of (17) tends to $T(t-t_0)D_+^w x_\lambda(t_0)$. It follows from the weak convergence of $\frac{x(t_0+h_n)-x(t_0)}{h_n}$ that there is an $M>0$ such that $\left|\frac{x(t_0+h_n)-x(t_0)}{h_n}\right| \leqslant M$. By (16) we thus infer that

$$(18)\quad \left|\xi[T(t-t_0-h_n)-T(t-t_0)]\lambda R(\lambda, A)\frac{x(t_0+h_n)-x(t_0)}{h_n}\right| \leqslant MNh_n$$

with suitable N. From the previous part of the proof and from (18) it follows that $\frac{\partial_+^w(T(t-s)x_\lambda(s))}{\partial s}$ exists and is equal to the left-hand difference of (14). Hence the inequality

$$\frac{\partial_+^w(T(t-s)x_\lambda(s))}{\partial s} \leqslant \theta$$

holds nearly everywhere in $(0, t)$. Applying (16) one easily proves that $T(t-s)x_\lambda(s)$ is weakly continuous in s. We thus infer that $T(t-s_2)x_\lambda(s_2) \leqslant T(t-s_1)x_\lambda(s_1)$ for $s_1<s_2$. But $\operatorname{s-lim}_{\lambda\to\infty} x_\lambda(s) = x(s)$. It follows from the last inequality that $T(t-s_2)x(s_2) \leqslant T(t-s_1)x(s_1)$. Now introduce $\tau = t-s_2$. This leads to the following inequality: $T(\tau)x(s_2) \leqslant T(\tau)T(s_2-s_1)x(s_1)$. Thus

$$(19)\quad \lambda R(\lambda, A)x(s_2) = \lambda\int_0^\infty e^{-\lambda\tau}T(\tau)x(s_2)\,d\tau$$

$$\leqslant \lambda\int_0^\infty e^{-\lambda\tau}T(\tau)T(s_2-s_1)x(s_1)\,d\tau$$

$$= \lambda R(\lambda, A)T(s_2-s_1)x(s_1).$$

On the other hand, $\operatorname{s-lim}_{\lambda\to\infty}\lambda R(\lambda, A)x = x$ for $x \in E$. The assertion of our theorem follows from (19).

Remark 1. Suppose that $\text{w-lim}\, x(t) = x(0) \leqslant \theta$. From (16) and from the assertion of theorem 3 it follows that $x_\lambda(t) \leqslant T(t-s)x_\lambda(s)$, and $\text{w-lim}_{s\to 0+} T(t-s)x_\lambda(s) = T(t)x_\lambda(0) \leqslant \theta$. Hence $x(t) \leqslant \theta$.

Remark 2. Suppose that (12) is replaced by the following inequality:

$$D_+^w x(t) \leqslant Ax(t)+y(t).$$

We assume that $y(t)$ is weakly continuous and $T(t-s)y(s)$ is supposed to be Pettis integrable in s. Then the assertion of theorem 3 may be replaced by

$$x(t) \leqslant T(t-s)x(s)+(\mathrm{P})\int_s^t T(t-\tau)y(\tau)\,d\tau \qquad (s<t).$$

We have the following generalization of a theorem of Reuter [13]:

THEOREM 4. *Let A_1 and A_2 be infinitesimal generators of positivity-preserving semi-groups $T_1(t), T_2(t)$ of class $(0, A)$. Suppose that $D[A_1] = D[A_2] = D$ and $A_1 \leqslant A_2$. Then $T_1(t) \leqslant T_2(t)$ for $t>0$.*

Proof. Define $x_1(u) = T_1(u)x$ and let $x \in S\cap D$. It follows from the inequality $A_1 \leqslant A_2$ that

$$x_1'(u) = A_1T_1(u)x \leqslant A_2T_1(u)x = A_2x_1(u).$$

By theorem 3 we get

$$e^{-\lambda s}T_1(\tau)T_1(s)x \leqslant e^{-\lambda s}T_2(\tau)T_1(s)x$$

for $0<\tau<\infty$, $s>0$. We thus obtain

$$(20)\quad T_1(\tau)\lambda R(\lambda, A_1)x \leqslant T_2(\tau)\lambda R(\lambda, A_1)x.$$

The inequality $T_1(\tau) \leqslant T_2(\tau)$ is now obtained from the following relations: $\operatorname{s-lim}_{\lambda\to\infty}\lambda R(\lambda, A_1)x = x$ for $x\in E$, $\overline{S\cap D} = S$.

It should be remarked that for semi-groups of class $(0, A)$ the condition

$$(21)\quad \theta \leqslant R(\lambda, A_1) \leqslant R(\lambda, A_2)$$

implies that $A_1 \leqslant A_2$. If $R(\lambda, A_1)$ and $R(\lambda, A_2)$ satisfy (21), then the inequality $T_1(u) \leqslant T_2(u)$ follows from the Hille inversion formula:

$$T_i(u)x = \operatorname{s-lim}_{n\to\infty}\left\{\frac{n}{u}R\left(\frac{n}{u}, A_i\right)\right\}^n x, \qquad i=1,2.$$

4. The aim of this section is to discuss the case where a differential inequality holds almost everywhere. In what follows we assume that the function $x(t)$ possesses the strong derivative $x'(t)$ almost everywhere [1].

[1] The strong differentiability is not more restrictive than the weak one: see [1], th. 4.

For the sake of clarity it is supposed that $x(t)$ can be expressed as the indefinited Bochner integral of $x'(t)$, i. e. $x(\tau_2)-x(\tau_1)=\int_{\tau_1}^{\tau_2}x'(\tau)d\tau$ for $\tau_1,\tau_2\in(0,a)$. Observe that $x(t)$ becomes now strongly absolutely continuous.

THEOREM 5. *Let A be an infinitesimal generator of a positivity-preserving semi-group $\{T(u)\}$ of class $(0,A)$. Suppose that the inequality*

$$x'(t)\leqslant Ax(t) \tag{22}$$

holds almost everywhere in $(0,a)$. Then

$$x(t)\leqslant T(t-s)x(s)\quad \textit{for}\quad 0<s<t. \tag{23}$$

Proof. Just as in the proof of theorem 3 we introduce the function $x_\lambda(t)=\lambda R(\lambda,A)x(t)$. It follows from (16) that the function $T(t-s)x_\lambda(s)$ is strongly absolutely continuous in s in an arbitrary compact interval of $(0,a)$. Using arguments similar to that used in the proof of theorem 3 we derive from (22) the inequality

$$\frac{\partial^s\big(T(t-s)x_\lambda(s)\big)}{\partial s}\leqslant\theta.$$

This inequality holds almost everywhere and $T(t-s)x_\lambda(s)$ is absolutely continuous in s. By lemma 2 we thus obtain

$$T(t-s_2)x_\lambda(s_2)\leqslant T(t-s_1)x_\lambda(s_1)\quad (s_1<s_2).$$

In order to prove (23) we can now apply a procedure used in the proof of theorem 3.

We will prove the following theorem:

THEOREM 6. *Suppose that A is an infinitesimal generator of a positivity-preserving semi-group $\{T(u)\}$ of class $(0,A)$. Let the function $x(t)$ be weakly absolutely continuous and Bochner integrable in any subinterval of $(0,a)$. Assume that $\int_{\tau_1}^{\tau_2}x(\tau)d\tau\in D[A]$ for $\tau_1,\tau_2\in(0,a)$. Let $Ax(t)$ be Bochner integrable in any subinterval of $(0,a)$. It is supposed that for $\xi\in S^*$ the inequality*

$$\frac{d}{dt}\xi x(t)\leqslant\xi Ax(t) \tag{24}$$

holds for $t\in(0,a)-Z_\xi$, $\operatorname{mes}Z_\xi=0$.

Then $x(t)$ satisfies (23).

Proof. Define $x_h(t)=\int_t^{t+h}x(\tau)d\tau$. The integration of (24) shows that

$$\xi x\big((t+h)-x(t)\big)\leqslant\xi\int_t^{t+h}Ax(\tau)d\tau:\qquad \xi\in S^*. \tag{25}$$

The operator A is closed. It follows from our assumptions and from th. 3.7.12 of [7] that

$$\xi\int_t^{t+h}Ax(\tau)d\tau=\xi Ax_h(t). \tag{26}$$

On the other hand, $\frac{d}{dt}x_h(t)=x(t+h)-x(t)$ for almost all t. Using (25) and (26) we infer that the inequality

$$\frac{d}{dt}x_h(t)\leqslant Ax_h(t)$$

holds almost everywhere. Obviously $x_h(t)$ satisfies the regularity assumptions needed in theorem 5. Hence $x_h(t)\leqslant T(t-s)x_h(s)$ and consequently

$$\frac{1}{h}\xi x_h(t)\leqslant\frac{1}{h}\xi T(t-s)x_h(s). \tag{27}$$

The function $x(t)$ is weakly continuous. We infer therefore that $\frac{1}{h}\xi x_h(t)\to\xi x(t)$, $\frac{1}{h}\xi T(t-s)x_h(s)\to\xi T(t-s)x(s)$. The assertion of the theorem follows now from (27).

Remark 3. Suppose that $x(t)$ is merely weakly continuous. Let $Ax(t)$ be weakly continuous. Then (23) holds if (24) is replaced by the following condition: for each $\xi\in S^*$ the inequality $\overline{D}_+\xi x(t)\leqslant\xi Ax(t)$ holds nearly everywhere in $(0,a)$.

5. The purpose of this section is to discuss almost linear differential inequalities. For the sake of simplicity we restrict ourselves to the case of time-independent A. Throughout our investigations we assume that A is an infinitesimal generator of a positivity-preserving semi-group $\{T(u)\}$ of class (C_0). We say that $f(t,x)$ defined in $\langle 0,a\rangle\times E$ increases in x if the condition $x_1\leqslant x_2$ implies that $f(t,x_1)\leqslant f(t,x_2)$. In what follows we suppose that $f(t,x)$ is bounded, say $|f(t,x)|<M$.

THEOREM 7. *Let the function $x(t)$ be strongly continuous and let the inequality*

$$D_+^w x(t)\leqslant Ax(t)+f\big(t,x(t)\big) \tag{28}$$

be satisfied nearly everywhere in $(0, a)$. *Suppose that the transformation* F *defined by*

$$(29)\qquad z(t) \to \int_0^t T(t-\tau) f\big(\tau, z(\tau)\big)\, d\tau$$

is completely continuous when considered in the space $C_E\langle 0, a\rangle$. *Let the function* $f(t, x)$ *increase in* x. *Then there exists a solution* $y(t)$ *of the equation*

$$(30)\qquad y(t) = T(t)x(0) + \int_0^t T(t-\tau) f\big(\tau, y(\tau)\big)\, d\tau$$

such that

$$(31)\qquad x(t) \leqslant y(t) \quad \text{for} \quad t \in \langle 0, a\rangle.$$

Proof. (28) implies that

$$(32)\qquad x(t) \leqslant T(t)x(0) + \int_0^t T(t-\tau) f\big(\tau, x(\tau)\big)\, d\tau.$$

Suppose that $M = \sup|f|$ and $N = \sup_{\langle 0, a\rangle} |T(t)|$. Let V be defined as follows:

$$z(\cdot) \in V \equiv z \in C_E\langle 0, a\rangle,\ x(t) \leqslant z(t) \text{ in } \langle 0, a\rangle \text{ and } |z(t)| \leqslant N|x(0)| + M.$$

V is closed and bounded in $C_E\langle 0, a\rangle$. Using (32) and the monotonicity of $f(t, x)$ one easily verifies that $F(V) \subset V$. By Schauder's fixed point theorem we infer that there is a solution $y(t)$ of (30) and $y(\cdot) \in V$. Therefore $x(t) \leqslant y(t)$, q. e. d.

Using the method of successive approximations one easily proves the following theorem:

THEOREM 8. *Let* $f(t, x)$ *be continuous and let it satisfy the condition*

$$|f(t, x_1) - f(t, x_2)| \leqslant K|x_1 - x_2|.$$

Suppose that $f(t, x)$ *increases in* x. *Let* $x(t)$ *be strongly continuous and let* $x(t)$ *satisfy* (28). *Then* $x(t) \leqslant y(t)$, *where* $y(t)$ *is the unique solution of* (30).

References

[1] A. Alexiewicz, *On differentiation of vector-valued functions*, Studia Math. 11 (1949), p. 185-196.

[2] — *On a theorem of Ważewski*, Ann. Soc. Polon. Math. 24 (1951), p. 129-131.

[3] W. Feller, *The general diffusion operator and positivity preserving semi-groups in one dimension*, Ann. of Math. 66 (1954) No. 3, p. 417-436.

[4] — *On positivity preserving semi-groups of transformations on* $C[r_1, r_2]$, Ann. Soc. Polon. Math. 25 (1952), p. 85-94.

[5] E. Hille, *The abstract Cauchy problem and Cauchy's problem for parabolic equations*, Journ. d'Analyse Math. 3 (1953/54), p. 81-196.

[6] — *Perturbation methods in the study of Kolmogoroff's equations*, Proc. Int. Congres. Math. 1954, Vol. III, Amsterdam 1956, p. 365-376.

[7] — and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Soc. Col. Publ. 1957.

[8] T. Kato, *On the semi-groups generated by Kolmogoroff's differential equations*, Journ. Math. Soc. Japan 6 (1954) No. 1, p. 1-15.

[9] — *Integration of equation of evolution in Banach space*, Journ. Math. Soc. Japan 5 (1953), p. 208-234.

[10] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), p. 70-84.

[11] W. Mlak, *Differential inequalities in linear spaces*, Ann. Pol. Math. 5 (1958), p. 95-101.

[12] G. E. H. Reuter and W. Ledermann, *On the differential equations for the transition probabilities of Markov processes with enumerably many states*, Proc. Cambridge Phil. Soc. 49 (1953), p. 247-262.

[13] G. E. H. Reuter, *A note on contraction semi-groups*, Math. Scand. 3.2 (1955), p. 275-280.

[14] П. Е. Соболевский, *Обобщенные решения дифференциальных уравнений первого порядка в гильбертовом пространстве*, Доклады А. Н. СССР 122 (1958), p. 994-996.

[15] А. Д. Вентцель, *О граничных условиях для многомерных диффузионных процессов*, Теория вер. и её прим. 4 (1959), p. 172-185.

[16] T. Ważewski, *Une généralisation des théorèmes sur les accroissements finis au cas des espaces de Banach et application à la généralisation du théorème de l'Hôpital*, Ann. Soc. Polon. Math. 24 (1951), p. 132-147.

[17] K. Yosida, *An operator-theoretical treatment of temporaly homogeneous Markoff process*, Journ. Math. Soc. Japan 1 (1949), p. 244-253.

[18] — *On Brownian motion in a homogeneous Riemannian space*, Pacific Journ. of Math. 2 (1952), p. 263-270.

[19] I. S. Gál, *On the fundamental theorems of the calculus*, Trans. Amer. Math. Soc. 86.2 (1957), p. 309-320.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 18. 10. 1959