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Some remarks on linear functional sequences

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Let X denote a Banach space with a norm $\| \; \|$, and let X^* be the conjugate space of X with the ordinary norm $\| \; \|^*$. It follows from the Banach-Steinhaus theorem that for any sequence $\{f_n\} \subset X^*$ such that $\overline{\lim\limits_{n\to\infty}} \|f_n\|^* = \infty$, $\overline{\lim\limits_{n\to\infty}} |f_n(x)| = \infty$ on a residual set. It will be shown in the sequel that apart from the trivial case when X is one-dimensional the theorem is not true when "$\overline{\lim}$" and "residual set" are replaced by "lim" and "one point" respectively. Moreover, we shall give certain conditions under which the modified theorem remains true. I wish to express my thanks to Professor W. Orlicz for his valuable suggestions during the preparation of this note.

1. We start from the following instructive example, showing the difference between our problem and the theorem of Banach and Steinhaus. Let $X = C\langle 0, 2\pi\rangle$ denote the space of all periodical continuous functions with period 2π and ordinary norm. For any $x \in X$, let us examine the partial sums of the Fourier series of this function

$$s_n(x, t) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\sin\frac{1}{2}(2n+1)(\tau - t)}{\sin\frac{1}{2}(\tau - t)}\, x(\tau)\, d\tau,$$

where $n = 0, 1, 2, \ldots$ Evidently, $s_n(\cdot, t_0) \in X^*$, for each $t_0 \in \langle 0, 2\pi\rangle$; it is known that

$$\|s_n(\cdot, t_0)\|^* \sim \ln n,$$

whence

$$\lim_{n\to\infty} \|s_n(\cdot, t_0)\|^* = \infty.$$

We shall prove $\underline{\lim\limits_{n\to\infty}} |s_n(x, t)| < \infty$ for each $x \in X$ and arbitrary $t \in \langle 0, 2\pi\rangle$. Suppose that it is not so. Then there exist $x_0 \in X$ and $t_0 \in \langle 0, 2\pi\rangle$, for which

$$\lim_{n\to\infty} |s_n(x_0, t_0)| = \infty,$$

whence

$$\sigma_n(x_0, t_0) = \frac{1}{n}\sum_{k=0}^{n-1} |s_k(x_0, t_0) - x_0(t_0)| \to \infty, \quad \text{for} \quad n \to \infty.$$

But, on the other hand, it is known that $\sigma_n(x_0, t_0) \to 0$ for $n \to \infty$ (see [2], p. 237).

2. *In order that* $\{f_n\} \subset X^*$ *and* $\lim_{n\to\infty} \|f_n\|^* = \infty$ *imply* $\lim_{n\to\infty} |f_n(x_0)| = \infty$, *for a certain* $x_0 \in X$, *it is necessary and sufficient that* X *be one-dimensional.*

Proof. Sufficiency. Let X be a one-dimensional space; it means that there exists $\bar{x} \in X$ such that $\|\bar{x}\| = 1$ and $X = \{\alpha\bar{x}: \alpha$ runs over all real numbers$\}$. Evidently, $f_n(x) = a_n\alpha$ for $x = \alpha\bar{x}$ and $a_n = f_n(\bar{x})$. If

$$|a_n| = \|f_n\|^* \to \infty \quad \text{for} \quad n \to \infty,$$

then

$$\lim_{n\to\infty} |f_n(x)| = \infty \quad \text{for} \quad x \neq 0.$$

Necessity. If X is at least two-dimensional, then there exist $x_0, y_0 \in X$ which are linearly independent and $\|x_0\| = \|y_0\| = 1$. One may find $g, h \in X^*$ such that

$$g(x_0) = 1, \quad g(y_0) = 0; \quad h(x_0) = 0, \quad h(y_0) = 1.$$

Let us consider the set $R = \{(l, m): l = 0, \pm 1, \pm 2, \ldots; m = 1, 2, \ldots\}$ which may be arranged as $\{(a_n, b_n): n = 1, 2, \ldots\}$ so that the sequence $\{(a_n^2 + b_n^2)^{1/2}: n = 1, 2, \ldots\}$ is non-decreasing. Obviously, $\lim_{n\to\infty} (a_n^2 + b_n^2)^{1/2} = \infty$.

Write

$$f_n(x) = a_n g(x) + b_n h(x), \quad n = 1, 2, \ldots$$

Since

$$\|f_n\|^* = \sup_{\|x\| \leq 1} |a_n g(x) + b_n h(x)| \geqslant \max\{|a_n|, |b_n|\} \geqslant \frac{1}{\sqrt{2}}(a_n^2 + b_n^2)^{1/2},$$

we get $\lim_{n\to\infty} \|f_n\|^* = \infty$. We shall show that, for each $x \in X$, $\underline{\lim}_{n\to\infty} |f_n(x)| < \infty$.

Examine the following cases:

α. If $g(x)h(x) = 0$, then either $g(x) = 0$ or $h(x) = 0$; it means that either $f_n(x) = b_n h(x)$ or $f_n(x) = a_n g(x)$. Since $\underline{\lim}_{n\to\infty} |a_n| = 0$ and $\underline{\lim}_{u\to\infty} |b_n| = 1$, we always have in this case $\underline{\lim}_{n\to\infty} |f_n(x)| < \infty$.

β. Let $g(x)h(x) \neq 0$ and $h(x)/g(x) = -p/q$, where $q > 0$ and p are integers; and let $(p_n, q_n) = (np, nq)$, for $n = 1, 2, \ldots$ Notice that

$$\left|\frac{p_n}{q_n} + \frac{h(x)}{g(x)}\right| = \left|-\frac{h(x)}{g(x)} + \frac{h(x)}{g(x)}\right| = 0, \quad n = 1, 2, \ldots$$

Since $\{(p_n, q_n): n = 1, 2, \ldots\} \subset R$ and $(p_n^2 + q_n^2)^{1/2}$ steadily increase as $n \to \infty$, there exists a sequence of positive integers $l_1 < l_2 < \ldots < l_n < \ldots$ such that $(p_n, q_n) = (a_{l_n}, b_{l_n})$; hence $\left|\frac{a_{l_n}}{b_{l_n}} + \frac{h(x)}{g(x)}\right| = 0$, for $n = 1, 2, \ldots$ From $|f_{l_n}(x)| = |a_{l_n} g(x) + b_{l_n} h(x)| = 0$, $n = 1, 2, \ldots$, we see that $\underline{\lim}_{n\to\infty} |f_n(x)| = 0$.

γ. Let $g(x)h(x) \neq 0$ and $h(x)/g(x) = \alpha$ be irrational; then for each positive integer n there exist integers p_n and q_n where $1 \leqslant q_n \leqslant n$ such that $\left|\frac{p_n}{q_n} + \frac{h(x)}{g(x)}\right| \leqslant \frac{1}{q_n n}$ for $n = 1, 2, \ldots$ (see e. g. [1], p. 170). Obviously, there exist sequences of positive integers $k_1 < k_2 < \ldots < k_n < \ldots$, $l_1 < l_2 < \ldots < l_n < \ldots$ such that $(p_{k_n}, q_{k_n}) = (a_{l_n}, b_{l_n})$ for $n = 1, 2, \ldots$ Therefore

$$\left|\frac{a_{l_n}}{b_{l_n}} + \frac{h(x)}{g(x)}\right| \leqslant \frac{1}{b_{l_n} k_n} \quad \text{for} \quad n = 1, 2, \ldots$$

Hence finally

$$|f_{l_n}(x)| = |a_{l_n} g(x) + b_{l_n} h(x)| \leqslant \frac{|g(x)|}{k_n}, \quad n = 1, 2, \ldots,$$

i. e. $\underline{\lim}_{n\to\infty} |f_n(x)| = 0$.

3. Let $S \subset X$ denote a positive cone, i. e. from $x, y \in S$ and $\alpha, \beta \geqslant 0$ it follows that $\alpha x + \beta y \in S$ and S contains more than one element. Let $\bar{S}$ be the closure of S. A linear functional $f \in X^*$ is said to be non-negative on S if $f(x) \geqslant 0$ for $x \in S$.

4. *Let* $f_{mn} \in X^*$, *for* $m, n = 1, 2, \ldots$, *be non-negative on* S. *Put*

$$\mu_{mn} = \|f_{mn}\|_S = \sup_{x \in S, \|x\| = 1} f_{mn}(x).$$

If $\sum_{n=1}^{\infty} 1/\mu_{mn} < \infty$, *for* $m = 1, 2, \ldots$, *then there exists* $x \in \bar{S}$ *such that* $\lim_{n\to\infty} f_{mn}(x) = \infty$, *for* $m = 1, 2, \ldots$

Proof. Take $x_{mn} \in S$ such that $\|x_{mn}\| = 1$ and $f_{mn}(x_{mn}) \geqslant \frac{1}{2}\mu_{mn}$. Obviously there is a double numerical sequence λ_{mn} such that $0 < \lambda_{m1} < \lambda_{m2} < \ldots < \lambda_{mn} < \ldots$, $\lim_{n\to\infty} \lambda_{mn} = \infty$ and $\sum_{n=1}^{\infty} \lambda_{mn}/\mu_{mn} = \nu_m < \infty$, for $m = 1, 2, \ldots$ Put

$$y_{mn} = \frac{1}{2^m \nu_m} \cdot \frac{\lambda_{mn}}{\mu_{mn}} x_{mn}, \quad \text{for} \quad m, n = 1, 2, \ldots$$

The double series $\sum_{m,n=1}^{\infty} y_{mn}$ is absolutely convergent, whence it converges to $x \in \overline{S}$. Evidently,

$$f_{mn}(x) = f_{mn}\left(\sum_{m,n=1}^{\infty} \frac{1}{2^m \nu_m} \cdot \frac{\lambda_{mn}}{\mu_{mn}} x_{mn}\right) \geqslant \frac{1}{2^m \nu_m} \cdot \frac{\lambda_{mn}}{\mu_{mn}} f_{mn}(x_{mn})$$

$$\geqslant \frac{\lambda_{mn}}{2^m \nu_m} \to \infty, \quad \text{as} \quad n \to \infty, \quad \text{for} \quad m = 1, 2, \ldots$$

5. *Let $f_{nh} \in X^*$ for $n = 1, 2, \ldots$; $h \in (0, \delta(n))$ be non-negative on S. Furthermore, suppose that for each pair of positive integers (i, j) there exists an element $x_{ij} \in S$ such that $\|x_{ij}\| \leqslant 1$, $\underline{\lim}_{h\to 0+} f_{ih}(x_{ij}) = \gamma_{ij}$ and $\lim_{j\to\infty} \gamma_{ij} = \infty$, for $i, j = 1, 2, \ldots$ Then there exists an $x \in \overline{S}$ such that $\underline{\lim}_{h\to 0+} f_{nh}(x) = \infty$ for $n = 1, 2, \ldots$*

Proof. Take $\lambda_{ij} \geqslant 0$ such that $\sum_{j=1}^{\infty} \lambda_{ij} \leqslant 1$ and $\sum_{j=1}^{\infty} \lambda_{ij}\gamma_{ij} = \infty$. Series $\sum_{j=1}^{\infty} \lambda_{ij} x_{ij}$ is absolutely convergent, whence it converges to $y_i \in \overline{S}$ for $i = 1, 2, \ldots$ But evidently $\sum_{i=1}^{\infty} \frac{1}{2^i(\|y_i\|+1)} y_i$ is also absolutely convergent, and thus converges to $x \in \overline{S}$. Finally we get

$$\underline{\lim}_{h\to 0+} f_{nh}(x) \geqslant \underline{\lim}_{h\to 0+} f_{nh}\left(\frac{1}{2^n(\|y_n\|+1)} \sum_{j=1}^{p} \lambda_{nj} x_{nj}\right)$$

$$\geqslant \frac{1}{2^n(\|y_n\|+1)} \sum_{j=1}^{p} \lambda_{nj}\gamma_{nj}, \quad \text{for} \quad n, p = 1, 2, \ldots$$

i. e. $\underline{\lim}_{h\to 0+} f_{nh}(x) = \infty$, for $n = 1, 2, \ldots$

6. As an application of 5 we shall show the following example. Let $X = C\langle a, b\rangle$ be the space of all continuous functions on $\langle a, b\rangle$ with ordinary norm, and let $\{t_n\} \subset \langle a, b)$ denote any sequence in $\langle a, b)$. Then there exists an $x \in X$ which is non-decreasing and has infinite right side derivatives at each point t_n, for $n = 1, 2, \ldots$

Proof. Take $\delta(n) = \frac{1}{2}(b - t_n)$ and $S = \{x \in X \colon x(t') \leqslant x(t''),$ when $a \leqslant t' < t'' \leqslant b\}$. Put

$$f_{nh}(x) = \frac{x(t_n+h) - x(t_n)}{h}, \quad \text{for} \quad n = 1, 2, \ldots; \ 0 < h < \delta(n),$$

and

$$x_{ij}(t) \in C\langle a, b\rangle, \qquad x_{ij}(t) = \begin{cases} 0, & a \leqslant t \leqslant t_i, \\ 1, & t_i + \dfrac{b-t_i}{2j} \leqslant t \leqslant b, \quad \text{for} \quad i, j = 1, 2, \ldots \\ \text{linear}, & t_i \leqslant t \leqslant t_i + \dfrac{b-t_i}{2j}. \end{cases}$$

Hence $\gamma_{ij} = \lim_{h\to 0+} f_{ih}(x_{ij}) = 2j/(b-t_i)$, and thus $\lim_{j\to\infty} \gamma_{ij} = \infty$, for $i = 1, 2, \ldots$ It follows at once from 5 that there exists an $x \in \overline{S} = S$ such that

$$\lim_{h\to 0+} \frac{x(t_n+h) - x(t_n)}{h} = \lim_{h\to 0+} f_{nh}(x) = \infty, \quad \text{for} \quad n = 1, 2, \ldots$$

References

[1] L. E. Dickson, *Introduction to the theory of numbers*, Chicago 1929.
[2] A. Zygmund, *Trigonometrical series*, Warszawa-Lwów 1935.

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