sider the function $f(z) = \lim_{n\to\infty} f_n(z)$. From (18) (if $j = n$) it follows that if $z \in K_{n+1}$ then $|f(z)| \leqslant |f_n(z)| + \sum_{j=n+1}^{\infty} \frac{1}{n+1}|PW_j(z)| < \frac{1}{2^{n+1}} + \frac{1}{n}|PW_{n+1}(z)| + + \frac{1}{2^{n+1}} < \frac{1}{2^n} + \frac{n}{M}$. Thus we conclude that if $z_0 \in K = \mathrm{Fr}(H)$ then $\lim_{\substack{z\to z_0\\ z\in H}} f(z) = 0$. On the other hand, $F(z)$ does not vanish everywhere in the domain H, for if $z \subset Q_0$ then $|f(z)| \geqslant \frac{1}{2} - \sum_{j=1}^{\infty}\frac{1}{j}|PW_j(z)| > \frac{1}{2} - - \sum_{j=1}^{\infty}\frac{1}{j\cdot 2^j} > 0$.

The singularities of $f_n(z)$ form a set $E_n \subset K_n$. It is closed and nondense in K_n. Therefore, the singularities of $f(z)$ form a closed set $E = \bigcup_{n=0}^{\infty} E_n$ which is nondense in H. It is clear that $K \subset \bar{E}$.

Let us now consider the function $F(w) = f^{-1}(w)$, which is univalent (but generally multiple-valued). We shall show that the domain of indetermination of $F(w)$ at point 0 is along a path L the continuum K. To prove this, let L be a path to point 0 such that $D_F(0, L)$ contains the point $p \in K$, and such that the elements of that paths belong to H. Let L' be another path of this kind which contains the point $q \in K$. If $e \in L$ and $e' \in L'$ are two elements such that $e, e' \subset H - Q_n$, then there is an arc which joins the two given values $a \in e$ and $a' \in e'$ in the exterior of Q_n and which does not meet any singular point of $F(w)$. Hence the paths L and L' are equivalent and $K \subset D_F(0, L)$. On the other hand, $D_F(0, L) \subset K$, because the domain of indetermination of any univalent function is always contained in the set of the singularities of the inverse function, and because in our case that set is $K \cup E$.

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Solution of some boundary-value problems by the method of F. Leja

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Introduction. The method of extremal points of F. Leja has been used to solve some problems of the theory of harmonic and analytic functions, for instance Dirichlet's problem and conformal mapping. In this note the above-mentioned method will be applied to the solution of the first boundary-value problem for the differential equation of a more general type than the Laplace equation.

The first boundary-value problem for the equation $\Delta v - c^2 v = 0$. Let D_∞ be a domain in the 3-dimensional Euklidean space containing the point in infinity and let E be the boundary of D_∞. We suppose that the capacity $d(E)$ of the set E is positive. Let $\lambda > 0$ be a real parameter, $f(P)$ a continuous real function defined on E and $\omega_\lambda(P, Q)$ a function of two points P and Q, where P and Q are different in E:

$$\omega_\lambda(P, Q) = \exp\left\{\lambda[f(P)+f(Q)] - \frac{e^{-cPQ}}{PQ}\right\}. \tag{1}$$

PQ denotes the distance between P and Q and $c > 0$ is a constant. Let $Q^{(n)} = \{Q_0, Q_1, \ldots, Q_n\}$ be a system of $n+1$ arbitrary points of E. We denote by

$$P^{(n)} = \{P_0, P_1, \ldots, P_n\} \tag{2}$$

such a system of $n+1$ points of E that

$$\prod_{0\leqslant i<k\leqslant n} \omega_\lambda(Q_i, Q_k) \leqslant \prod_{0\leqslant i<k\leqslant n} \omega_\lambda(P_i, P_k) = v_{n\lambda}, \tag{3}$$

for every system $Q^{(n)} \in E$. System (2) will be called the n-th *extremal system* of points of the set E connected with function (1).

It is known [1] that there exists a limit of the sequence $v_{n\lambda}^{2/n(n+1)}$

$$\lim_{n\to\infty} v_{n\lambda}^{2/n(n+1)} = v_\lambda(E), \quad v_\lambda(E) \geqslant 0 \tag{4}$$

called the *span* of the set E. From (4), (3) and (1) follows the existence of the limit of the sequence

$$\sigma_{n\lambda}(P^{(n)}) = \frac{2\lambda}{n(n+1)} \sum_{0 \leqslant i < k \leqslant n} [f(P_i)+f(P_k)] - \frac{2}{n(n+1)} \sum_{0 \leqslant i < k \leqslant n} \frac{e^{-cP_iP_k}}{P_iP_k}$$

and the inequality $\lim_{n\to\infty} \sigma_{n\lambda} = \sigma_\lambda > -\infty$. In fact, let m be the lower bound of the function $f(P)$ on E and let $R^{(n)} = \{R_0, R_1, \ldots, R_n\}$ be such a system of $n+1$ points of E that

$$\sum_{0 \leqslant i < k \leqslant n} \frac{1}{R_iR_k} = \min_{Q^{(n)} \epsilon E} \sum_{0 \leqslant i < k \leqslant n} \frac{1}{Q_iQ_k}.$$

Then

$$\sigma_{n\lambda}(P^{(n)}) \geqslant \sigma_{n\lambda}(R^{(n)}) \geqslant 2m\lambda - \frac{2}{n(n+1)} \sum_{0 \leqslant i < k \leqslant n} \frac{1}{R_iR_k}.$$

It is known [2] that

$$\frac{2}{n(n+1)} \sum_{0 \leqslant i < k \leqslant n} \frac{1}{R_iR_k} \to \frac{1}{d(E)} \quad \text{when} \quad n \to \infty.$$

We supposed that $d(E) > 0$, and therefore $\sigma_\lambda \geqslant 2m\lambda - 1/d(E) > -\infty$. It is obvious that $\sigma_\lambda < 2M\lambda$ where $M = \sup_{P \epsilon E} f(P)$.

We denote by Δ a Borelian set. Let $\mu_{n\lambda}(\Delta)$ be a function of a set Δ defined by the following formula;

$$\mu_{n\lambda}(\Delta) = \begin{cases} 0 & \text{if } \Delta \text{ does not contain any point of the system (2),} \\ \dfrac{k}{n+1} & \text{if } \Delta \text{ contains } k \text{ points of the system (2).} \end{cases}$$

We have $0 \leqslant \mu_{n\lambda}(\Delta) \leqslant 1$. Each of the functions $\mu_{n\lambda}(\Delta)$ gives a certain distribution of the unit mass on the set E defined by the extremal system (2). The sequence $\{\mu_{n\lambda}(\Delta)\}$ is a uniformly bounded sequence of non-negative functions. Let $\mu_\lambda(\Delta) = \mu_\lambda$ be the limit of a convergent subsequence chosen from $\{\mu_{n\lambda}(\Delta)\}$. Using a similar proof to that used in the paper [3] we can prove the formula

$$\sigma_\lambda = 2\lambda \int f(Q)\,d\mu_\lambda - \int\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda d\mu_\lambda \tag{5}$$

$$= \sup_\tau \left[2\lambda \int f(Q)\,d\tau - \int\int \frac{e^{-cPQ}}{PQ}\,d\tau\,d\tau\right],$$

where τ denotes an arbitrary distribution of the unit mass on E.

Let E_λ be the kernel of a mass corresponding to the distribution μ_λ. We denote by $u_\lambda(P)$ the function

$$u_\lambda(P) = \int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda - \lambda f(P) \tag{6}$$

defined for $P \epsilon E$.

THEOREM 1. *The function* (6) *is constant on the set* E_λ *except for a set of capacity* 0 *contained in* E_λ. *We denote this constant by* c_λ (1). *On* E *is* $u_\lambda(P) \geqslant c_\lambda$.

The proof of theorem 1 is similar to that given in [3]. The constant c_λ is equal to the upper bound of function (6) in E_λ.

Let D_λ be a domain whose boundary is the set E_λ. It is easy to prove that $E_\lambda \to E$ as $\lambda \to 0$ (cf. [3]). We suppose that every point of the boundary E is regular for Dirichlet's problem.

THEOREM 2. *The function*

$$v_\lambda(P) = \frac{1}{\lambda}\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda \left(= \frac{1}{\lambda} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{e^{-cPP_i}}{PP_i} \text{ for } P \epsilon E_\lambda \right) \tag{7}$$

satisfies outside the set E_λ *the differential equation*

$$\Delta v_\lambda - c^2 v_\lambda = 0. \tag{8}$$

The function $v_\lambda(P)$ *is continuous in* $D_\lambda + E_\lambda$ *and* $v_\lambda(P) = f(P) + c_\lambda/\lambda$ *for* $P \epsilon E_\lambda$.

Proof. If we take the derivatives of function (7) it is obvious that equation (8) is satisfied outside E_λ. For $P_0 \epsilon E_\lambda$ we have (see (1)) $v_\lambda(P_0) = f(P_0) + c_\lambda/\lambda$. Since function (7) is lower semi-continuous, we have

$$\lim_{\substack{P\to P_0 \\ P \epsilon E_\lambda}} \frac{1}{\lambda}\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda \geqslant f(P_0) + \frac{c_\lambda}{\lambda}. \tag{9}$$

On the other hand (see [5], p. 69), we have

$$\overline{\lim_{\substack{P\to P_0 \\ P \epsilon E_\lambda}}} \frac{1}{\lambda}\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda \geqslant \overline{\lim_{\substack{P\to P_0 \\ P \epsilon D_\lambda}}} \frac{1}{\lambda}\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda. \tag{10}$$

But

$$\overline{\lim_{\substack{P\to P_0 \\ P \epsilon E_\lambda}}} \frac{1}{\lambda}\int \frac{e^{-cPQ}}{PQ}\,d\mu_\lambda = f(P_0) + \frac{c_\lambda}{\lambda}. \tag{11}$$

(1) At every regular point $P_0 \epsilon E_\lambda$ we have $u_\lambda(P_0) = c_\lambda$ (cf. [5]).

From (9), (10) and (11) follows

$$\lim_{P\to P_0\epsilon E_\lambda} v_\lambda(P) = f(P_0)+\frac{c_\lambda}{\lambda}.$$

THEOREM 3. *There exists only one function, μ_λ, which satisfies formula* (5).

Proof. Conversely, assume that there exist two functions, μ_λ and $\sigma_\lambda \not\equiv \mu_\lambda$, which satisfy (5). Then

$$0 = 2\lambda\int f(Q)[d\mu_\lambda - d\sigma_\lambda] - \iint \frac{e^{-cPQ}}{PQ}[d\mu_\lambda d\mu_\lambda - d\sigma_\lambda d\sigma_\lambda].$$

Denoting by τ_λ the difference $\sigma_\lambda - \mu_\lambda$ we obtain

$$0 = 2\int\Big[-\lambda f(Q)+\int\frac{e^{-cPQ}}{PQ}d\mu_\lambda\Big]d\tau_\lambda + \iint\frac{e^{-cPQ}}{PQ}d\tau_\lambda d\tau_\lambda.$$

But

$$-\lambda f(Q)+\int\frac{e^{-cPQ}}{PQ}d\mu_\lambda \begin{cases}\geqslant c_\lambda & \text{for} \quad Q\epsilon E,\\ = c_\lambda & \text{for} \quad Q\epsilon E_\lambda,\end{cases}$$

hence

$$\int\Big[-\lambda f(Q)+\int\frac{e^{-cPQ}}{PQ}d\mu_\lambda\Big]d\tau_\lambda$$
$$\geqslant c_\lambda\tau_\lambda(E_\mu E_\sigma)+c_\lambda\tau_\lambda(E_\mu - E_\mu E_\sigma)+c_\lambda\tau_\lambda(E_\sigma - E_\mu E_\sigma) = c_\lambda \tau(E) = 0.$$

Therefore

$$0 \geqslant \iint\frac{e^{-cPQ}}{PQ}d\tau_\lambda d\tau_\lambda.$$

It is known [4] that

$$\iint\frac{e^{-cPQ}}{PQ}d\tau_\lambda d\tau_\lambda \geqslant 0;$$

the equality holds only when $\tau_\lambda \equiv 0$. Then $\sigma_\lambda \equiv \mu_\lambda$.

THEOREM 4. *Let E be the common boundary of two domains D and D_∞. Suppose that E is a Liapounoff surface and $f(P)$ satisfies the Lipschitz condition. Then $v_\lambda(P) = f(P)+c_\lambda/\lambda$ on E for a sufficiently small value of $\lambda > 0$.*

Proof. Denote by $w_\lambda(P)$ the following function: $w_\lambda(P)$ = the solution of Dirichlet's problem for the domain D with the boundary value $f(P)+c_\lambda/\lambda$ if $P\epsilon D$ and $w_\lambda(P)$ = the same solution for the domain D_∞ if $P\epsilon D_\infty$ $(w_\lambda(\infty) = 0)$.

If E and $f(P)$ satisfy the above-mentioned conditions the function $w_\lambda(P)$ can be expressed for a sufficiently small value of $\lambda > 0$ as the Newtonian potential of a simple layer (see [6]). Then $w_\lambda(P)$ is a superharmonic function in the whole space and $w_\lambda(P) = f(P)+c_\lambda/\lambda$ for $P\epsilon E$. We have

(12) $$v_\lambda(P) \geqslant f(P)+c_\lambda/\lambda \quad \text{for} \quad P\epsilon E, \quad v_\lambda(\infty) = 0$$

and $v_\lambda(P) = f(P)+c_\lambda/\lambda$ for $P\epsilon E_\lambda$. Since $v_\lambda(P)$ is subharmonic outside (2) E_λ, we have

(13) $$w_\lambda(P) \geqslant v_\lambda(P) \quad \text{outside} \quad E_\lambda.$$

For $P\epsilon E - E_\lambda$ we have $w_\lambda(P) = f(P)+c_\lambda/\lambda$; therefore from (12) and (13) follows $v_\lambda(P) = f(P)+c_\lambda/\lambda$ on E.

The parabolic equation $v''_{xx} = v'_t$. Let E be a curve

$$x = \chi(t), \quad t\epsilon\langle 0, a\rangle,$$

where the function $\chi(t)$ satisfies the Lipschitz condition, and $f(x, t)$ is a real, continuous function defined on E. Let $(\chi(t_i), t_i)$, $i = 0, 1, \ldots, n$, be a system of $n+1$ arbitrary points of E. Consider the following expression:

$$\sigma_n(t_0, \ldots, t_n) = \frac{2}{n(n+1)}\Big\{\sum_{0\leqslant i<k\leqslant n}[f(\chi(t_i), t_i)+f(\chi(t_k), t_k)] - \sum_{0\leqslant i<k\leqslant n}\frac{\exp\{-[\chi(t_i)-\chi(t_k)]^2/4|t_i-t_k|\}}{2\sqrt{\pi|t_i-t_k|}}\Big\}.$$

A system of $n+1$ points $(\chi(\bar t_i), \bar t_i)$, $i = 0, 1, \ldots, n$, of E will be called an *extremal system* of points of E if

$$\sigma_n(\bar t_0, \bar t_1, \ldots, \bar t_n) \geqslant \sigma_n(t_1, \ldots, t_n)$$

for every other system of points $(\chi(t_i), t_i)$, $i = 0, 1, \ldots, n$, of E.

Let $\nu_n(\Delta)$ be a function of a Borelian set Δ which gives the distribution of the unit mass defined by the extreme system of points $\{\chi(\bar t_i), \bar t_i\}$ and let $\nu = \nu(\Delta)$ be the limit of a convergent subsequence chosen from $\{\nu_n(\Delta)\}$. As before (cf. [3]), it can easily be proved that

$$\sigma(\nu) = 2\int f d\nu - \iint\frac{\exp\{-[\chi(t)-\chi(\eta)]^2/4|t-\eta|\}}{2\sqrt{\pi|t-\eta|}}d\nu d\nu \geqslant \sigma(\tau)$$

for every other distribution of the 1 mass on E defined by $\tau = \tau(\Delta)$.

(2) because $\Delta v_\lambda = c^2 v_\lambda > 0$.

We have $\sigma(\nu) > -\infty$. In fact, let $t_k = ak/(n+1)$, $x_k = \chi(t_k)$, $k = 0, 1, \ldots, n$; then $\sigma_n(\bar{t}_0, \ldots, \bar{t}_n) \geqslant \sigma_n(t_0, \ldots, t_n)$ and therefore

$$\sigma(\nu) \geqslant 2\int_0^a f\,d\eta - \int_0^a\int_0^a \frac{\exp\{-[\chi(t)-\chi(\eta)]^2/4|t-\eta|\}}{2\sqrt{\pi|t-\eta|}}\,dt\,d\eta > -\infty.$$

Put $\dfrac{\exp\{-[x-\chi(\eta)]^2/4|t-\eta|\}}{2\sqrt{\pi|t-\eta|}} = U(x, t; \chi(\eta), \eta)$.

THEOREM 5. *Let E_ν be the kernel of a mass corresponding to the distribution ν. The function*

$$u(x, t) = \int U(x, t; \chi(\eta), \eta)\,d\nu - f(x, t), \qquad (x, t) \in E \tag{14}$$

is constant almost everywhere on E_ν.

Proof. The function (14) is lower semicontinuous on E_ν. In fact let (x_0, t_0) be a point on E_ν. If [3]

$$\int U(x_0, t_0; \chi(\eta), \eta)\,d\nu < \infty$$

we can choose a radius r of a circle K with the centre in (x_0, t_0) so small that

$$\int_{KE} U(x_0, t_0; \chi(\eta), \eta)\,d\nu < \varepsilon, \qquad \varepsilon > 0.$$

On the other hand, we have

$$\int U(x, t; \chi(\eta), \eta)\,d\nu = \int_{E-EK} + \int_{EK} \geqslant \int_{E-EK}.$$

Therefore

$$\underline{\lim}_{(x,t)\to(x_0,t_0)} u(x, t) \geqslant \int_{E-EK} U(x_0, t_0; \chi(\eta), \eta)\,d\nu - f(x_0, t_0)\,u(x_0, t_0) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, it follows that

$$\underline{\lim}_{(x,t)\to(x_0,t_0)} u(x, t) \geqslant u(x_0, t_0).$$

Let

$$\int u(x, t)\,d\nu = c_\nu.$$

We cannot have $u(x, t) < c_\nu - \varepsilon$ for every point $(x, t) \in E_\nu$ because it would contradict the definition of c_ν. Therefore there exists such a point $(x_0, t_0) \in E_\nu$ that $u(x, t) > c_\nu - \varepsilon$. Hence $u(x, t)$ is semicontinuous on E_ν

(3) In the case of $\int U(x_0, t_0; \chi(\eta), \eta)\,d\nu = \infty$ we have $\lim\limits_{r\to 0}\int_{E-EK} U(x_0, t_0; \chi(\eta), \eta)\,d\nu = \infty$ and therefore $\underline{\lim}\, u(x, t) = \infty$ as $(x, t) \to (x_0, t_0)$.

and we have $u(x, t) > c_\nu - \varepsilon$ in a neighbourhood $O(x_0, t_0)$ of (x_0, t_0). We shall prove that the inequality $u(x, t) \geqslant c_\nu$ holds almost everywhere on E. In fact, suppose that there exists a subset $F \subset E$ of positive measure such that $u(x, t) \leqslant c_\nu - 2\varepsilon$ on F. Let $\tau = \tau(\Delta)$ be a distribution of mass defined by the formula

$$\tau = -\nu \quad \text{on } O(x_0, t_0), \qquad \tau(F) = \nu(O(x_0, t_0)),$$

$$\tau \equiv 0 \quad \text{outside} \quad O(x_0, t_0) + F,$$

$$A = \int_E\int_E U(\chi(t), t; \chi(\eta), \eta)\,d\tau\,d\tau < \infty,$$

and let $0 < h \leqslant 1$. We have

$$\begin{aligned}\sigma(\nu + h\tau) - \sigma(\nu) &= 2h\int_E f\,d\tau - h^2 A - 2h\int_E\int_E U\,d\nu\,d\tau \\ &\geqslant -h^2A - 2h[(c_\nu - \varepsilon)\tau(O(x_0, t_0)) + (c_\nu - 2\varepsilon)\tau(F)] \\ &= h[-hA + 2\varepsilon\tau(F)] > 0\end{aligned}$$

for a sufficiently small value of $h > O$. But this contradicts the inequality $\sigma(\nu) \geqslant \sigma(\nu + h\tau)$. Since $\varepsilon > 0$ is arbitrarily small, we have $u(x, t) \geqslant c_\nu$ almost everywhere on E. We cannot have $u(x, t) > c_\nu$ at a point P on E_ν because it follows from the lower semicontinuity of $u(x, t)$ that that inequality holds in a neighbourhood of P and $\int u\,d\nu$ would be $> c_\nu$. Therefore

$$u(x, t)\begin{cases} = c_\nu & \text{almost everywhere on } E_\nu, \\ \geqslant c_\nu & \text{on } E.\end{cases} \tag{15}$$

The function $\nu(\Delta)$ possesses almost everywhere a finite derivative $\lim\limits_{|\Delta|\to 0}\dfrac{\nu(\Delta)}{|\Delta|}$. In the remaining points the degree of infinity of $\dfrac{\nu(\Delta)}{\Delta}$ as $|\Delta| \to 0$ cannot be $> -\frac{1}{2}$ because in the opposite case we should have $\int U\,d\nu = \infty$, which contradicts (15).

Let

$$v(x, t) = \int_0^t U(x, t; \chi(\eta), \eta)\,d\nu. \tag{16}$$

Suppose that the point (x, t), $-\infty < x < \infty$, $t \in \langle 0, a\rangle$ is outside the set E_ν; then

$$v''_{xx}(x, t) = \int_0^t \frac{\partial^2 U}{\partial x^2}\,d\nu,$$

$$v'_t(x, t) = \int_0^t \frac{\partial U}{\partial t}\,d\nu - \lim_{\Delta t\to 0}\int_t^{t+\Delta t} \frac{U(x, t+\Delta t; \chi(\eta), \eta)\,d\nu}{\Delta t};$$

but

$$\lim_{\Delta t\to 0}\int_{t}^{t+\Delta t}\frac{U\big(x,t+\Delta t;\chi(\eta),\eta\big)d\nu}{\Delta t}=0.$$

Therefore we have

THEOREM 6. *The function* (16) *satisfies outside the set* E_ν *the equation* $v''_{xx}=v'_t$ *and* $v(x,0)=0$, $v(\pm\infty,t)=0$.

The function $\int_E U d\nu$ is the sum of two functions: $v(x,t)=\int_0^t U\big(x,t;\chi(\eta),\eta\big)d\nu$ and $\tilde{v}(x,t)=\int_t^a U\big(x,t;\chi(\eta),\eta\big)d\nu$, where $v(x,t)$ satisfies outside E_ν the equation $v''_{xx}=v'_t$ and $\tilde{v}(x,t)$ the equation $\tilde{v}''_{xx}=-\tilde{v}'_t$. For $x\to\infty$ or $x\to-\infty$ the functions $v(x,t)$ and $\tilde{v}(x,t)$ tend to 0.

Suppose now that $f(x,t)\equiv 0$ and E is a segment $x=x_0$, $t\in\langle 0,a\rangle$. In that case the function $\int_E U d\nu$ is equal to a constant c_ν everywhere on E. In fact, we have

$$\int_E U d\nu\leqslant\frac{1}{2\sqrt{\pi}}\int_E\frac{d\nu}{\sqrt{|t-\eta|}}\quad\text{everywhere}$$

and

$$\int_E U d\nu=\frac{1}{2\sqrt{\pi}}\int_E\frac{d\nu}{\sqrt{|t-\eta|}}\quad\text{on}\quad E.$$

But the function

$$\frac{1}{2\sqrt{\pi}}\int_E\frac{d\nu}{\sqrt{|t-\eta|}}$$

is subharmonic outside E_ν and from the maximum principle results the inequality

$$\frac{1}{2\sqrt{\pi}}\int_E\frac{d\nu}{\sqrt{|t-\eta|}}\leqslant c_\nu\quad\text{everywhere.}\tag{17}$$

From (15) and (17) it follows that

$$\int_E U\big(x,t;\chi(\eta),\eta\big)d\nu=c_\nu\quad\text{almost everywhere on } E.$$

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