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On the domains of indetermination of analytic functions

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Let us consider an analytic function $f(z)$ in the z-plane. A sequence of the elements of the function f is called a *path to point* p if the centres of those elements tend to point p and each of those elements is an immediate continuation of the preceding one. Two paths L_1 and L_2 to the point p are *equivalent* if in every circle whose centre is p there are two elements, $e_1 \in L_1$ and $e_2 \in L_2$, such that one of them is a continuation of the other in that circle. If $f(z)$ is single-valued and p is its isolated singular point, then all the paths to p are equivalent. The same is also true when the dimension of the set of all singularities of the single-valued function $f(z)$ is 0 in a neighborhood of p. On the other hand, if p lies in a continuum consisting of the singularities of $f(z)$ then it is possible that there are $2^{\aleph_0}$ paths to p, no two of which are equivalent (e. g.: for $p = 0$, if the set of the singularities is $0 < r \leqslant 1/2^n$, $\varphi = m\pi/2^n$, where $m, n = 1, 2, \ldots$).

Let K_n denote the circle whose centre is p and radius $1/n$. Let D_n^L denote the set of all values of the elements which are continuations in the circle K_n of the elements belonging to L. The set $D_f(p, L) = \bigcap_{n=1}^{\infty} \overline{D_n^L}$ is called a *domain of indetermination of the function* $f(z)$ *at the point* p *along the path* L. This notion was first introduced in paper [1] by Zoretti.

The domain of indetermination is of course always a continuum. We say that the function $f(z)$ is *determined at the point* p *along the path* L if $D_f(p, L)$ is a single point. It is clear that the domain of indetermination of any single-valued function $f(z)$ at its isolated singular point is a single point (if p is a pole or a removable singular point) or the whole plane (if p is an essential singular point). The same is true if the set of the singularities of $f(z)$ in a neighborhood of p consists of a sequence of points tending to p. If all paths to the point p are equivalent, then the domain of indetermination does not depend on the path L. In this case we shall write $D_f(p)$ instead $D_f(p, L)$.

This note consists of three parts. In the first we consider the single-valued functions only. W. Wolibner has proved[1] that for every plane-continuum C there exist a single-valued function $f(z)$, a point p and the path L to the point p, such that $D_f(p, L) = C$. In this note I shall prove, furthermore, that for any C it is possible to find $f(z)$ such that the set of its singular points is homeomorphic to the Cantor-set and that $f(z)$ is determined at all points except p.

In the second part we consider the domains of indetermination of multiple-valued functions at their isolated singular points p. In the third part we consider the domains of indetermination of analytic functions (single-valued or not), which are univalent in a neighbourhood of p.

This note has been prepared at a seminar conducted by Professor W. Wolibner. It contains answers to his questions and sugestions.

I. THEOREM 1. *For every plane-continuum C there exists a single-valued analytic function $f(z)$ and a point p such that the set of all singularities of $f(z)$ is homeomorphic to the Cantor-set, and $f(z)$ is determined at all points except p and $D_f(p) = C$.*

Proof. Suppose that C is a bounded continuum which contains point 0. We define a function $f(z)$ having the required properties for the point $p = \infty$. As a first step in this construction we define a sequence of polygonal domains C_n such that

(1) $$\operatorname{dist}(C, C_n) \leqslant 1/2^{n-3}$$

for every $n = 0, 1, \ldots$, where $\operatorname{dist}(A, B) = \max[\sup_{x\epsilon A} \varrho(x, B), \sup_{x\epsilon B} \varrho(x, A)]$ denotes the distance in the sense of Hausdorff.

Let $P(z)$ denote the function of Pompeiu [2]. The singularities of P form a set E_0 which is homeomorphic to the Cantor-set and P is determined at all points of E_0. We assume that E_0 lies in the unit circle $|z| < 1$ and $P(\infty) = 0$.

We need the following lemma:

LEMMA. *Let G denote the set of all values $w = P(z)$. There exists a single-valued function $h(w)$ such that $h(G)$ is dense in a given polygonal domain C and $h(0) = 0$.*

Proof. From a theorem of Possel [3] it follows that there exists a conformal mapping φ which represents G on the plane without a certain nondense closed set consisting of parallel segments and single points. Let I be one of those segments. There is a conformal mapping ψ which maps the exterior of I upon the polygonal domain C, and where $\psi(\infty) = 0$.

[1] See Colloquium Mathematicum 2 (1951), p. 304, Comptes Rendus.

The function $h = \psi\varphi$ represents G on C without a certain non-dense closed set consisting of arcs and single points. Furthemore, we have $h(0) = 0$. The above proves the lemma.

Let
$$f_0(z) = h_0 P(z),$$
where $h_0(w)$ is a single-valued function which maps the set of all values of $P(z)$ upon a set which is dense in the polygonal domain C_0, and where $h_0(0) = 0$. Such a function exists according to the lemma. We have $f_0(\infty) = 0$. Therefore, there is an r_0 such that if $|z| > r_0$ then $|f_0(z)| < 1$.

Suppose that the functions $f_0, f_1, \ldots, f_{n-1}$ and the numbers $r_0, r_1, \ldots, r_{n-1}$ are already defined and that they have the following properties:

(2) $$\sum_{j=0}^{n-1} |f_j(z)| < 1/2^{n-1} \quad \text{if} \quad |z| > r_{n-1},$$

(3) $$|f_j(z)| < 1/2^{j-1} \quad \text{for} \quad j = 1, 2, \ldots, n-1, \quad \text{and} \quad |z| < r_{j-1},$$

(4) the singularities of $f_j(z)$, for $j = 1, 2, \ldots, n-1$, lie in the ring $r_{j-1} < |z| < r_j$ only.

Let

(5) $$f_n(z) = h_n P(z - z_n),$$

where $h_n(w)$ is a single-valued function which maps the set of all values of $P(z - z_n)$ upon a set which is dense in the polygonal domain C_n, and where $h_n(0) = 0$. Such a function exists according to the lemma. We have $f_n(\infty) = 0$. The point z_n in (5) is such that

(6) $$|f_n(z)| < 1/2^{n-1} \quad \text{if} \quad |z| < r_{n-1},$$
$$|z_n| \geqslant r_{n-1} + 1.$$

Such a point z_n exists because $f_n(\infty) = 0$. From the same equality and (2) it follows that there exists an r_n such that

(7) $$\sum_{j=0}^{n} |f_j(z)| < 1/2^n \quad \text{if} \quad |z| > r_n,$$
$$|z_n| \leqslant r_n - 1.$$

Therefore, all singularities of $f_n(z)$ lie in the ring $r_{n-1} < |z| < r_n$ only. Thus, the sequence $\{f_n(z)\}$ is defined.

The convergence of the series $\sum_{j=0}^{\infty} f_j(z)$ is uniform in every bounded closed domain, for, according to (6), if $|z| < r_{n-1}$ then $|f_j(z)| < 1/2^{j-1}$ for $j \geqslant n$.

Hence the function $f(z) = \sum_{j=0}^{\infty} f_j(z)$ is regular in the plane except the points of the set $E = \bigcup_{n=0}^{\infty} E_n$, where $E_n = \bigcup_{z \in E_0} (z+z_n)$. It is clear that E is homeomorphic to the Cantor-set.

Since all the paths to the point $p = \infty$ are equivalent the domain of indetermination at $p = \infty$ is of the form $D_f(\infty) = \bigcap_{m=0}^{\infty} \overline{f(Q_m)}$, where $Q_m = \underset{z}{E}\{|z| > r_{m-1}\}$. We shall show that $D_f(\infty) = C$.

Let R_n denote the ring $r_{n-1} < |z| < r_n$. From (6) and (7) it follows that $|f_n(z)| < 1/2^{n-1}$ in the exterior of R_n. Hence

(8) $$\operatorname{dist}[f(R_n), C_n] < 1/2^{n-1}.$$

From (6) it follows that if $z \in R_n$ then $|f_j(z)| < 1/2^{j-1}$ for $j > n$, for in this case $|z| < r_{j-1}$; furthermore from (7) it follows that $\sum_{j=0}^{n-1} |f_j(z)| < 1/2^{n-1}$. Hence, if $z \in R_n$ then $|f(z) - f_n(z)| = |\sum_{j \neq n} f_j(z)| < 1/2^{n-1} + 1/2^{n-1} = 1/2^{n-2}$. This can be written in the form: $\operatorname{dist}[f(R_n), f_n(R_n)] < 1/2^{n-2}$.

Thus, by (8), we obtain $\operatorname{dist}[f(R_n), C_n] < 1/2^{n-3}$ and, by (1), $\operatorname{dist}[f(R_n), C] < 1/2^{n-4}$.

Since $f(Q_m) = \bigcup_{n=m}^{\infty} f(R_n)$, from the above inequality it follows that $\operatorname{dist}[f(Q_m), C] < 1/2^{m-4}$. This proves the theorem.

II. It is easy to see, that the domains of indetermination of single-valued and finitely multiple-valued analytic functions are, at their isolated singular points, single points or the whole plane. This is not true for the infinitely multiple-valued functions.

THEOREM 2. *If p is an isolated singular point of the infinitely multiple-valued analytic function $f(z)$, then the domain of indetermination of $f(z)$ at p is a point or a closed domain (different, or not, from the whole plane).*

Proof. Let $\zeta = F(z)$ be such a function and $p = 0$. It is possible to assume that the function $F(z)$ is of the form $F(z) = f\log z$, where $f(w)$ is single-valued and regular in the half-plane $\operatorname{re} w < \log 1/m$, where m is an integer.

Let K_n be a circle $|z| < 1/n$ whithout the point 0, where $n = m, m+1, \ldots$ The function $F(z)$ is regular in every K_n. The function $w = \log z$ maps those domains onto the half-planes $P_n = \underset{w}{E}\{\operatorname{re} w < \log 1/n\}$. The function $f(w)$ maps every of these half-planes onto domains G_n. We have

(9) $$G_{n+1} \subset G_n \quad \text{for every } n \geqslant m$$

and

(10) $$D_F(0) = \bigcap_{n=m}^{\infty} \overline{G}_n.$$

Let us consider the sequence of functions $f_n(w) = f(w - (\log 1/n - \log 1/m))$, where $n \geqslant m$. All of those functions are regular in the same half-plane P_m, and they map this half-plane onto the domains G_n.

Case 1. There is an integer $n_0 \geqslant m$ such that $\overline{G}_{n_0}$ is different from the whole plane. Hence, by (9), it follows that $\{f_n(z)\}$ is a normal family of analytic functions, and therefore it contains a subsequence $\{f_{n_k}(z)\}$ which is uniformly convergent in P_m to an analytic function. This function maps P_m onto a domain G. Because of the uniform convergence of $\{f_{n_k}(z)\}$ we have $\operatorname{dist}[G, G_{n_k}] \to 0$ as $k \to \infty$. Hence, by (10), $\operatorname{dist}[\overline{G}, D_F(0)] = 0$, and finally $\overline{G} = D_F(0)$.

Case 2. Every $\overline{G}_n$ is the whole plane. Then, by (10), $D_F(0)$ is also the whole plane.

Thus, the theorem is proved.

We now define an analytic function whose domain of indetermination is different both from the single point and from the whole plane.

EXAMPLE. Consider the function $F(z) = e^{i\log\log z}$ and its singular point $p = 0$. The function $\zeta = e^{i\log w}$ maps the half-planes $\operatorname{re} w < \log 1/n$ onto the constant ring

(11) $$e^{-\pi/2} < |\zeta| < e^{\pi/2}.$$

Hence $D_F(0)$ is the closure of this ring.

THEOREM 3. *Every closed domain D in the plane is a domain of indetermination of an (infinitely multiple-valued) analytic functions at one of its isolated singular points.*

Proof. It is easy to see that the function $(\zeta - e^{-\pi/2})^2$ maps the ring (11) onto a simply connected domain. There is a conformal mapping g which maps that domain onto a domain which is dense in D. Therefore, the closed domain D is the domain of indetermination at the point 0 of the function

$$g[(e^{i\log\log z} - e^{-\pi/2})^2].$$

III. THEOREM 4. *If the analytic function $f(z)$ (single-valued or not) is univalent in a neighbourhood of its singular point p, then every domain of indetermination $D_f(p, L)$ is the boundary of a certain plane domain.*

Proof. Let $f(z)$ be univalent in the open circle K the centre of which is p. Let L be a path to p. Let $e \in L$ be an element having its centre in K and such that if $e' \in L$ and e' comes after e in L then the centre of e' is in K.

Let U_L be the set of all the centres of elements e of this kind. Finally, let R_L be a component of the Riemann surface of $f(z)$ over regular points in K such that $e \subset R_L$, and let V_L be the set of all values of $f(z)$ in the domain U_L. Since $f(z)$ is univalent in K, there is a homeomorphism which is induced by $f(z)$ and which maps R_L onto V_L.

Let us note that

(12) $$D_f(p, L) \subset \mathrm{Fr}(V_L).$$

To prove this it is sufficient to remark that $\lim f(z_n) \in \mathrm{Fr}(V_L)$ as $z_n \to p$. This is true since the limit p' of the points z_n' of the Riemann surface corresponding to the points z_n lies in the boundary of R_L, and the homeomorphism, which is induced by $f(z)$ and maps R_L onto V_L at the same time maps the sequence $\{z_n'\}$ into a sequence whose limit points lie in $\mathrm{Fr}(V_L)$.

Let H_L be a component of $E^2 - D_f(p, L)$ such that $V_L \subset H_L$, where E^2 denotes the whole plane. We have $\mathrm{Fr}(E^2 - D_f(p, L)) \subset D_f(p, L)$ and therefore

(13) $$\mathrm{Fr}(H_L) \subset D_f(p, L).$$

Finally, we shall show that

(14) $$D_f(p, L) \subset \mathrm{Fr}(H_L).$$

To prove this, we notice that

(15) $$\mathrm{Fr}(H_L) = \bar{H}_L \cap (E^2 - H_L).$$

Hence, by (12), we have $D_f(p, L) \subset \bar{V}_L$. Therefore, by (15), we obtain (14) since $D_f(p, L) \subset E^2 - H_L$ and $\bar{V}_L \subset \bar{H}_L$. Thus, according to (13) and (14), the theorem is proved.

THEOREM 5. *If the continuum K is the boundary of a plane domain H, then there exist an analytic function F (generally a multiple-valued function), a point p and a path L to point p such that $D_F(p, L) = K$.*

Proof. First we define a function $f(z)$ which has the following properties: 1° it is single-valued and regular in $H - E$, where E is closed and non-dense in H, 2° $f(z)$ vanishes at all points of K and 3° $f(z)$ is determined at all points.

Construction of the function f. Let $Q_n \subset H$, $n = 0, 1, \ldots$, be a sequence of polygonal domains such that $\bar{Q}_n \subset Q_{n+1}$ and $\mathrm{dist}[\mathrm{Fr}(Q_n), K] \to 0$ as $n \to \infty$. Let $K_n = Q_n - Q_{n-1}$, for $n = 1, 2, \ldots$ and let $K_0 = Q_0$.

We define a sequence of functions $\{f_n(z)\}$, $n = 0, 1, \ldots$ Let $f_0(z) \equiv 1/2$.

Let us consider the function of Pompeiu [2] $w = P(z)$ which is bounded, i. e. $|P(z)| < M$, and regular in the unit circle, and which maps one-to-one a certain domain in the unit circle containing 0 onto the circle $|w| < 1$. We assume that $P(0) = 0$. Hence there is a branch $\Phi(z)$ of the inverse function $P^{-1}(w)$ which is regular in $|W| < 1$ and has the properties:

(16) $$\Phi(0) = 0 \quad \text{and} \quad |\Phi(w)| < 1 \quad \text{if} \quad |w| < 1.$$

Suppose that the function $f_j(z)$ for $j \leqslant n-1$ are already defined and have the following properties:

(17) $f_j(z)$ are regular in the exterior of Q_j.

(18) $|f_j(z)| < 1/2^{j+1}$ in the exterior of Q_j,

(19) $|f_j(z) - f_{j-1}(z)| < 1/2^{j+1}$ in Q_{j-1}.

According to the theorem of Runge there exists a rational function $W_n(z)$ having the following properties:

(20) if $z \in Q_{n-1}$ then $|W_n(z)| < A_n$, where A_n is the maximum of $|\Phi(w)|$ in the circle $|w| \leqslant 1/2^{n+1}$,

(21) if $z \notin Q_n$ then $W_n(z) = \Phi(-nf_{n-1}(z)) + \delta_n(z)$, where $|\delta_n(z)| < \delta_n$ and $|P(z+\delta_n) - P(z)| < 1/2^{n+1}$ if $|z| < 1$, and $|W_n(z)| < 1$ (by (16) and (18) it is possible).

The rational function $W_n(z)$ approaches with the error $\delta_n(z)$ the function

$$\Phi_n(z) = \begin{cases} \Phi(-n f_{n-1}(z)) & \text{if} \quad z \notin Q_n, \\ 0 & \text{if} \quad z \in Q_{n-1}, \end{cases}$$

which is regular in $Q_{n-1} \cup (E^2 - Q_n)$.

Let $f_n(z) = f_{n-1}(z) + \frac{1}{n} P W_n(z)$. We shall show that (17)-(19) hold for $j = n$. (17) follows immediately from (21). (18) follows also from (21), as $|f_n(z)| = \left|f_{n-1}(z) + \frac{1}{n} P W_n(z)\right| = \frac{1}{n}|P(W_n(z) - \delta_n(z)) - P W_n(z)| < \frac{1}{2^{n+1}}$. (19) holds too, for if $z \in Q_{n-1}$, then, by (20), we have $|f_n(z) - f_{n-1}(z)| = \frac{1}{n}|P W_n(z)| < \frac{1}{n \cdot 2^{n+1}} \leqslant \frac{1}{2^{n+1}}$.

Thus, the sequence $\{f_n(z)\}$ is defined.

The series $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n} P W_n(z) = \lim_{n \to \infty} f_n(z)$ is convergent for all $z \in H$. This is true, for if $z \in Q_n$ then for $j > n$ we have $|P W_j(z)| < 1/2^{j+1}$. Let us consider the function $f(z) = \lim_{n\to\infty} f_n(z)$. From (18) (if $j = n$) it follows that if $z \in K_{n+1}$ then $|f(z)| \leqslant |f_n(z)| + \sum_{j=n+1}^{\infty} \frac{1}{n+1} |PW_j(z)| < \frac{1}{2^{n+1}} + \frac{1}{n} |PW_{n+1}(z)| + + \frac{1}{2^{n+1}} < \frac{1}{2^n} + \frac{n}{M}$. Thus we conclude that if $z_0 \in K = \mathrm{Fr}(H)$ then $\lim_{\substack{z\to z_0\\ z\in H}} f(z) = 0$. On the other hand, $F(z)$ does not vanish everywhere in the domain H, for if $z \subset Q_0$ then $|f(z)| \geqslant \frac{1}{2} - \sum_{j=1}^{\infty} \frac{1}{j} |PW_j(z)| > \frac{1}{2} - - \sum_{j=1}^{\infty} \frac{1}{j\cdot 2^j} > 0$.

The singularities of $f_n(z)$ form a set $E_n \subset K_n$. It is closed and nondense in K_n. Therefore, the singularities of $f(z)$ form a closed set $E = \bigcup_{n=0}^{\infty} E_n$ which is nondense in H. It is clear that $K \subset \bar{E}$.

Let us now consider the function $F(w) = f^{-1}(w)$, which is univalent (but generally multiple-valued). We shall show that the domain of indetermination of $F(w)$ at point 0 is along a path L the continuum K. To prove this, let L be a path to point 0 such that $D_F(0, L)$ contains the point $p \in K$, and such that the elements of that paths belong to H. Let L' be another path of this kind which contains the point $q \in K$. If $e \in L$ and $e' \in L'$ are two elements such that $e, e' \subset H - Q_n$, then there is an arc which joins the two given values $a \in e$ and $a' \in e'$ in the exterior of Q_n and which does not meet any singular point of $F(w)$. Hence the paths L and L' are equivalent and $K \subset D_F(0, L)$. On the other hand, $D_F(0, L) \subset K$, because the domain of indetermination of any univalent function is always contained in the set of the singularities of the inverse function, and because in our case that set is $K \cup E$.

References

[1] L. Zoretti, *Leçons sur le prolongement analytique*, 1911, p. 59 and 101.

[2] R. de Possel, *Zum Parallelschlitztheorem unendlich zusammenhängender Gebiete*, Nachrichten Ges. Wiss. Göttingen 23 (1931), p. 199-202.

[3] D. Pompeiu, *Sur la continuité des fonctions des variables complexes*, Annales de Toulouse (2), 7 (1905), p. 314.

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Solution of some boundary-value problems by the method of F. Leja

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Introduction. The method of extremal points of F. Leja has been used to solve some problems of the theory of harmonic and analytic functions, for instance Dirichlet's problem and conformal mapping. In this note the above-mentioned method will be applied to the solution of the first boundary-value problem for the differential equation of a more general type than the Laplace equation.

The first boundary-value problem for the equation $\Delta v - c^2 v = 0$. Let D_∞ be a domain in the 3-dimensional Euklidean space containing the point in infinity and let E be the boundary of D_∞. We suppose that the capacity $d(E)$ of the set E is positive. Let $\lambda > 0$ be a real parameter, $f(P)$ a continuous real function defined on E and $\omega_\lambda(P, Q)$ a function of two points P and Q, where P and Q are different in E:

$$\omega_\lambda(P, Q) = \exp\left\{\lambda[f(P)+f(Q)] - \frac{e^{-cPQ}}{PQ}\right\}. \tag{1}$$

PQ denotes the distance between P and Q and $c > 0$ is a constant. Let $Q^{(n)} = \{Q_0, Q_1, \ldots, Q_n\}$ be a system of $n+1$ arbitrary points of E. We denote by

$$P^{(n)} = \{P_0, P_1, \ldots, P_n\} \tag{2}$$

such a system of $n+1$ points of E that

$$\prod_{0\leqslant i<k\leqslant n} \omega_\lambda(Q_i, Q_k) \leqslant \prod_{0\leqslant i<k\leqslant n} \omega_\lambda(P_i, P_k) = v_{n\lambda}, \tag{3}$$

for every system $Q^{(n)} \in E$. System (2) will be called the n-th *extremal system* of points of the set E connected with function (1).

It is known [1] that there exists a limit of the sequence $v_{n\lambda}^{2/n(n+1)}$

$$\lim_{n\to\infty} v_{n\lambda}^{2/n(n+1)} = v_\lambda(E), \quad v_\lambda(E) \geqslant 0 \tag{4}$$