vérifient la condition de Hölder. Quant aux autres fonctions qui figurent aux seconds membres des équations (17), il est évident que toutes ces fonctions satisfont à la condition de Hölder soit en vertu des hypothèses faites au début, soit grâce au théorème de Privaloff-Plemelj.

Or, en se basant uniquement sur la continuité des fonctions $\varphi_1(t)$ et $\varphi_2(t)$, on trouve

$$|I_k(t)-I_k(t')| = \left|\int_L \left(\frac{N_k^*(t,\tau)}{|t-\tau|^{1-\mu'}} - \frac{N_k^*(t',\tau)}{|t'-\tau|^{1-\mu'}}\right)\varphi_k(\tau)\,d\tau\right|$$
$$\leqslant \int_L \frac{|N_k^*(t,\tau)-N_k^*(t',\tau)|}{|t-\tau|^{1-\mu'}}|\varphi_k(\tau)|\,d\tau +$$
$$+\int_L |N_k^*(t',\tau)|\left|\frac{1}{|t-\tau|^{1-\mu'}} - \frac{1}{|t'-\tau|^{1-\mu'}}\right||\varphi_k(\tau)|\,d\tau$$
$$< \text{const}\cdot|t-t'|^{\mu^*} + \text{const}\cdot|t-t'|^{\alpha},$$

où α est un nombre positif inférieur à $\frac{1}{2}$ (voir [4], p. 10). Il vient par conséquent

(30) $$|I_k(t)-I_k(t')| < \text{const}\cdot|t-t'|^{\theta\mu^*/2}, \quad \text{où} \quad 0<\theta<1.$$

Par suite, les solutions du système (17) sont aussi des solutions du système (11). Si dans la formule (5) on porte les fonctions trouvées $\Phi^+(t)$ et $\Phi^-(t)$, on obtiendra la fonction $\Phi(z)$ définie à l'intérieur des domaines $S^+, S_0^-, S_1^-, \ldots, S_p^-$, holomorphe séparément dans chacun d'eux, et dont les valeurs limites satisfont à la condition (1). Le problème posé au début se trouve ainsi résolu.

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General solution of a functional equation

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The object of the present paper is the functional equation

(1) $$\varphi[f(x)] = G(x, \varphi(x)),$$

where $\varphi(x)$ denotes the required function and $f(x)$ and $G(x, y)$ are given functions. We assume that the function $f(x)$ satisfies the relation

(2) $$f[f(x)] \equiv x.$$

(Examples of such functions are: $c-x$, c/x, $\sqrt{c^2-x^2}$; S. Łojasiewicz [3] has given a construction of all functions fulfilling (2)).

The linear equation

$$\varphi[f(x)] = A(x)\varphi(x)+B(x)$$

with a function $f(x)$ fulfilling (2) has been treated by N. Gercevanoff [1]. In the present paper I give the general solution of equation (1) with a function $f(x)$ fulfilling (2).

§ 1. We shall discuss equation (1) in a set E such that $f(E) = E$. Using the terminology adopted in [2], we shall call such a set a *modulus-set* for the function $f(x)$. Let us decompose the set E into three sets E_0, E_1, E_2 such that

$$f(x) = x \quad \text{for} \quad x \in E_0,$$
$$f(x) > x \quad \text{for} \quad x \in E_1,$$
$$f(x) < x \quad \text{for} \quad x \in E_2.$$

Lemma I. *If the function $f(x)$ fulfils* (2), *then*

(3) $$f(E_0) = E_0,$$

(4) $$f(E_1) = E_2,$$

(5) $$f(E_2) = E_1.$$

Proof. Relation (3) is evident, relation (5) follows from (4) and (2). Thus it remains only to prove relation (4). To do this, let us take an arbitrary $x \in E_1$. Consequently $f(x) > x$. But according to (2)

$$f[f(x)] = x < f(x),$$

what proves that $f(x) \in E_2$. Thus $f(E_1) \subset E_2$. On the other hand, if $x \in E_2$ then $f(x) \in E_1$ (by an argument similar to the one above). Hence $x = f[f(x)] \in f(E_1)$, which proves that $E_2 \subset f(E_1)$. Thus $E_2 = f(E_1)$, which was to be proved.

Now, let us suppose that equation (1) has a solution $\varphi(x)$, defined in E.

Lemma II. *If a function $y = \varphi(x)$ satisfies the functional equation* (1) *in E, then for every fixed $x \in E$ its value must fulfil the equation*

$$y = G[f(x), G(x, y)]. \tag{6}$$

Moreover, for every fixed $x \in E_0$ its value must fulfil the equation

$$y = G(x, y). \tag{7}$$

Proof. Let us suppose that a function $\varphi(x)$ satisfies equation (1). Putting in (1) $f(x)$ in place of x, we obtain according to (2)

$$\varphi(x) = G\big(f(x), \varphi[f(x)]\big). \tag{8}$$

Inserting (1) to (8) we obtain

$$\varphi(x) = G\big[f(x), G\big(x, \varphi(x)\big)\big],$$

which proves the first part of the assertion of the lemma.

For $x \in E_0$ is

$$f(x) = x. \tag{9}$$

Inserting (9) in (1) we obtain

$$\varphi(x) = G\big(x, \varphi(x)\big),$$

which proves that for $x \in E_0$ the value $y = \varphi(x)$ is a root of equation (7). This completes the proof of the lemma.

Equation (6) can have no roots, as is shown by the following example:

Example I. Let us consider the equation

$$\varphi(1/x) = \varphi(x) + 1, \quad x \in (0, \infty). \tag{10}$$

Here $G(x, y) = y+1$ and equation (6) assumes the form

$$y = y + 2. \tag{11}$$

Equation (11) evidently has no roots. It is also obvious that equation (10) has no solutions.

Nevertheless, if equation (6) has a root for $x = x_0$, then it also has a root for $x = f(x_0)$. Namely, we shall show

Lemma III. *If y is, for a fixed $x \in E$, a root of equation* (6), *then*

$$\bar{y} \stackrel{\text{df}}{=} G(x, y)$$

is a root of the equation

$$y = G\big(x, G[f(x), y]\big), \tag{12}$$

which we obtain replacing x by $f(x)$ in equation (6).

Proof. Let us insert $\bar{y}$ in the right-hand side of equation (12). We have by (6):

$$G\big(x, G[f(x), \bar{y}]\big) = G\big(x, G[f(x), G(x, y)]\big) = G(x, y) = \bar{y},$$

which proves that $\bar{y}$ fulfils equation (12).

Equation (7) can have no roots, although equation (6) has roots (of course, every root of equation (7) is also a root of equation (6)):

Example II. Let the function $G(x, y)$ be defined by the formulae

$$G(x, y) = \begin{cases} y + \frac{1}{2} & \text{for} \quad y \in \langle 0, \frac{1}{2}), \\ y - \frac{1}{2} & \text{for} \quad y \in \langle \frac{1}{2}, 1), \end{cases} \quad x \in E.$$

Equation (6) is identically fulfilled for $x \in E$, $y \in \langle 0, 1)$. On the other hand we always have $G(x, y) \neq y$.

In the above example the function $G(x, y)$ was discontinuous. We shall prove

Lemma IV. *If the function $G(x, y)$ is for $\bar{x} \in E_0$ continuous with respect to y in an interval I, then from the existence of the roots of equation* (6) *for $x = \bar{x}$ follows the existence of the roots of equation* (7) *for $x = \bar{x}$.*

Proof. Let us write

$$g(y) \stackrel{\text{df}}{=} G(\bar{x}, y).$$

Equation (6) can be written for $x = \bar{x}$ in the form

$$g[g(y)] = y. \tag{13}$$

Let us suppose that equation (13) has a root $\bar{y}$, and that $\bar{y}$ does not fulfil equation (7) for $x = \bar{x}$:

$$g(\bar{y}) \neq \bar{y}.$$

Thus for instance let $g(\bar{y}) > \bar{y}$. But then (see lemma I) $g[g(\bar{y})] < g(\bar{y})$. From the continuity of the function $g(y)$ it follows that in the interval $\big(\bar{y}, g(\bar{y})\big)$ there must exist a point η such that $g(\eta) = \eta$, which means that η is a root of equation (7). This completes the proof.

In the sequel we shall assume that equation (6) possesses roots for every $x \in E$, and that equation (7) possesses roots for every $x \in E_0$. If equation (6) has no roots in a set F_1, and equation (7) has no roots in a set $F_2 \subset E_0$, then, on account of lemma II equation (1) has no solution defined in the set

$$F \stackrel{\text{df}}{=} F_1 \cup F_2.$$

From lemma III it follows that the set F is a modulus-set for the function $f(x)$. Then also the set

$$E^* \stackrel{\text{df}}{=} E - F$$

is a modulus-set for the function $f(x)$, but now equation (6) possesses roots for every $x \in E^*$, and equation (7) possesses roots for every $x \in E_0^* \stackrel{\text{df}}{=} E^* \cap E_0$. Thus we can restrict our considerations of equation (1) to the set E^* only.

For an arbitrary $x \in E_1 \cup E_2$ resp. $x \in E_0$ we shall denote by V_x the set of all roots of equations (6) or (7) respectively. Further, let us denote by Ψ the class of all functions $\psi(x)$ defined in the set $E_0 \cup E_1$ and fulfilling the condition

$$\psi(x) \in V_x \quad \text{for} \quad x \in E_0 \cup E_1.$$

With the above notation we shall prove the following

THEOREM I. *The formulae*

$$\varphi(x) = \begin{cases} \psi(x) & \text{for} \quad x \in E_0 \cup E_1, \\ G\big(f(x), \psi[f(x)]\big) & \text{for} \quad x \in E_2, \end{cases} \tag{14}$$

where $\psi(x)$ is an arbitrary function from the class Ψ, determine the general solution of equation (1) *in E.*

Proof. It is evident that the function $\varphi(x)$ is by formulae (14) unambiguously defined in the whole set E. We must show that it satisfies equation (1).

Let us take an arbitrary $x \in E$. We shall consider three cases:

1° $x \in E_0$. Then

$$\varphi(x) = \varphi[f(x)] = \psi(x).$$

On the other hand, since the value $\psi(x)$ fulfils equation (7), we have

$$G\big(x, \varphi(x)\big) = G\big(x, \psi(x)\big) = \psi(x) = \varphi[f(x)],$$

and thus equation (1) is satisfied.

2° $x \in E_1$. Then (on account of lemma I) $f(x) \in E_2$ and according to (14) and (2)

$$\varphi(x) = \psi(x), \quad \varphi[f(x)] = G\big(x, \psi(x)\big) = G\big(x, \varphi(x)\big),$$

and thus equation (1) is satisfied.

3° $x \in E_2$. Then $f(x) \in E_1$ and according to (14) we have

$$\varphi(x) = G\big(f(x), \psi[f(x)]\big), \quad \varphi[f(x)] = \psi[f(x)].$$

Hence

$$G\big(x, \varphi(x)\big) = G\Big(x, G\big(f(x), \psi[f(x)]\big)\Big). \tag{15}$$

But the value $\psi[f(x)]$ fulfils equation (12), then

$$G\Big(x, G\big(f(x), \psi[f(x)]\big)\Big) = \psi[f(x)],$$

whence we have by (15)

$$G\big(x, \varphi(x)\big) = \psi[f(x)] = \varphi[f(x)],$$

which proves that equation (1) is fulfilled.

It remains to show that formulae (14) give the general solution of equation (1), i. e. that every solution of equation (1) can be represented in the form (14). But this follows immediately from the fact that every solution of equation (1) is in the set $E_0 \cup E_1$ a function from the class Ψ, and that every function satisfying equation (1) is unambiguously determined in the set E_2 by its values in the set E_1. This completes the proof of the theorem.

The number of solutions of equation (1) varies considerably. It is illustrated by the following examples:

EXAMPLE III. Let us consider the equation

$$\varphi(\sqrt{1-x^2}\,) = 2\varphi(x), \quad x \in (0, 1). \tag{16}$$

Here $G(x, y) = 2y$ and equation (6) assumes the form

$$y = 4y. \tag{17}$$

The only solution of equation (17) is $y = 0$, and so the only solution of equation (16) is the function $\varphi(x) \equiv 0$.

EXAMPLE IV. Let us consider the equation

$$\varphi(1/x) = x[\varphi(x)]^2, \quad x \in (0, \infty). \tag{18}$$

Here $G(x, y) = xy^2$ and equations (6) and (7) assume the forms

$$y = xy^4, \tag{19}$$

$$y = xy^2 \tag{20}$$

respectively. Equation (19) has (for every $x > 0$) two roots:

$$y_1 = 0, \quad y_2 = \sqrt[3]{1/x},$$

which are for $x = 1$ also the roots of equation (20). Equation (18) has two solutions that are continuous in the interval $(0, \infty)$:

$$\varphi(x) \equiv 0, \quad \varphi(x) = \sqrt[3]{1/x},$$

but besides it possesses infinitely many discontinuous solutions in $(0, \infty)$. The general solution of equation (18) is the function

$$\varphi(x) = \begin{cases} \delta(x)\sqrt[3]{1/x} & \text{for} \quad x \in \langle 1, \infty); \\ \delta(1/x)\sqrt[3]{1/x} & \text{for} \quad x \in (0, 1), \end{cases}$$

where $\delta(x)$ is an arbitrary function defined in the interval $\langle 1, \infty)$ and assuming the values 0 or 1 only.

§ 2. Now let us suppose that the set E is an open interval, and the function $f(x)$ is continuous in E. Let us further suppose that the function $G(x, y)$ is continuous in an open region Ω, normal with respect to the x-axis. For every x we shall denote by Ω_x the set of values y such that $(x, y) \in \Omega$, and by Γ_x we shall denote the set of values assumed by the function $G(x, y)$ for $y \in \Omega_x$. We suppose further that

$$\Omega_x \neq 0, \quad \Gamma_x = \Omega_{f(x)} \quad \text{for} \quad x \in E. \tag{21}$$

As has been proved in [2], equation (1) with a function $f(x)$ continuous, strictly increasing and different from x in an open modulus-interval E possesses in E, under the above assumptions, a continuous solution depending on an arbitrary function. The question arises whether equation (1), with a function $f(x)$ fulfilling (2), has also (under the above assumptions) a continuous solution depending on an arbitrary function. The answer is negative, as is illustrated by the following

Example V. Let us consider the equation

$$\varphi(-x) = (x^4+1)\varphi(x) + x^2, \quad x \in (-\infty, \infty). \tag{22}$$

Here $G(x, y) = (x^4+1)y + x^2$ is a continuous function on the whole plane. Also the function $f(x) = -x$ is continuous in $(-\infty, \infty)$. Relations (21) are fulfiled. Equations (6) and (7) assume the forms

$$y = (x^4+1)^2 y + x^2(x^2+2), \tag{23}$$

and

$$y = (x^4+1)y + x^2 \tag{24}$$

respectively. For $x \neq 0$ the solution of equation (23) is

$$y = -1/x^2.$$

For $x = 0$ equations (23) and (24) are identically fulfilled. Thus the general solution of equation (22) is the function

$$\varphi(x) = -1/x^2 \quad \text{for} \quad x \neq 0,$$
$$\varphi(0) \quad \text{arbitrary}.$$

This solution is evidently discontinuous for $x = 0$.

The considerations of the preceding section imply however the following

Theorem II. *If the function $f(x)$ is continuous in a modulus-interval E and the function $G(x, y)$ is continuous in a region Ω, and if conditions (21) are fulfilled and moreover equation (6) is identically fulfilled in Ω, then equation (1) possesses a continuous solution depending on an arbitrary function.*

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