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The estimation of the third coefficient of the starlike function with a pole

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Let us study the class of functions starlike[1] for the point 0 with expansion

$$F(z) = \frac{1}{z}+b_0+b_1z+\ldots, \qquad 0<|z|<1.$$

For this class we have the following

Theorem. *For the third coefficient of a function of this class the following inequality is true:*

$$|b_3| \leqslant \tfrac{1}{2}.$$

The sign of equation occurs only for the function

$$F(z) = \left(\frac{1}{z^2}+\eta z^2\right)^{1/2}, \qquad |\eta| = 1.$$

Remark. This theorem is known (see [1], [2]) for the class of functions which are starlike for the point 0, and for which $b_0 = 0$. The class here discussed is really broader, because the addition of a constant changes the centre of starlikeness.

Proof. In the paper [4] it has been proved that the functional $\operatorname{re} b_n$ determined for the class of meromorphic starlike functions has its extremal value for functions of the form

$$F(z) = \frac{1}{z}\prod_{k=1}^{n+1}(1-\sigma_k z)^{\beta_k}, \qquad |\sigma_k| = 1, \qquad \beta_k>0, \qquad \sum_{k=1}^{n+1}\beta_k = 2.$$

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(1) The function $F(z)$ is *starlike* in the ring $0<|z|<1$, when the function $f(z) = 1/F(z)$ maps the ring $0<|z|<1$ on a starlike domain.

Moreover the numbers σ_k and β_k satisfy the following system of equations:

$$\sigma_k^{2(n+1)}+\frac{n+1}{n}b_0\sigma_k^{2n+1}+\frac{n+1}{n-1}b_1\sigma_k^{2n}+\ldots+\lambda\sigma_k^{n+1}+\ldots+\frac{n+1}{n}\bar{b}_0\sigma_k+1=0,$$

(1)

$$2(n+1)\sigma_k^{2n+1}+(2n+1)\frac{n+1}{n}b_0\sigma_k^{2n}+\ldots+(n+1)\lambda\sigma_k^{n}+\ldots+\frac{n+1}{n}\bar{b}_0=0,$$

(2) $$b_p=\frac{(-1)^{p+1}}{(p+1)!}\begin{vmatrix} a_1 & a_2 & \ldots & a_p & a_{p+1} \\ p & a_1 & \ldots & a_{p-1} & a_p \\ 0 & p-1 & \ldots & a_{p-2} & a_{p-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a_1 \end{vmatrix},\qquad p=0,1,\ldots,n,$$

(3) $$a_j=\sum_{k=1}^{n+1}\beta_k\sigma_k^j,\qquad j=0,\ldots,n+1,\qquad a_0=2.$$

Now let us put $n=3$ and let us suppose that the extremum $\operatorname{re}b_3$ is obtained for the function

$$F(z)=\frac{1}{z}\prod_{k=1}^{4}(1-\sigma_k z)^{\beta_k}$$

where $\sigma_k\neq\sigma_j$ for $k\neq j$. It follows from this supposition that the equation

(4) $$\sigma^8+\frac{4}{3}b_0\sigma^7+2b_1\sigma^6+4b_2\sigma^5+\lambda\sigma^4+4\bar{b}_2\sigma^3+2\bar{b}_1\sigma^2+\frac{4}{3}\bar{b}_0\sigma+1=0$$

has four different double roots $\sigma_1,\sigma_2,\sigma_3,\sigma_4$. Let us write

(5) $$a_j^{\vee}=\sum_{k=1}^{4}\sigma_k^j,\qquad j=1,2,3.$$

Then using the formulae expressing the coefficients of a polynomial by its roots and from (2), (3), (4), (5) we obtain

(6) $$a_1=\tfrac{3}{2}a_1^{\vee},\qquad a_2=\tfrac{1}{4}a_1^{\vee 2}+a_2^{\vee},$$
$$a_3=-\tfrac{1}{8}a_1^{\vee 3}+\tfrac{3}{2}a_1^{\vee}a_2^{\vee}+\tfrac{1}{2}a_3^{\vee},\qquad \varepsilon=\sigma_1\sigma_2\sigma_3\sigma_4=\pm1.$$

From identity

$$\beta_1+\beta_2+\beta_3+\beta_4=2,\qquad \beta_1\sigma_1^2+\beta_2\sigma_2^2+\beta_3\sigma_3^2+\beta_4\sigma_4^2=a_2,$$
$$\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3+\beta_4\sigma_4=a_1,\qquad \beta_1\sigma_1^3+\beta_2\sigma_2^3+\beta_3\sigma_3^3+\beta_4\sigma_4^3=a_3$$

we get

(7) $$\beta_1=\frac{2\sigma_2\sigma_3\sigma_4-a_1(\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+a_2(\sigma_2+\sigma_3+\sigma_4)-a_3}{(\sigma_2-\sigma_1)(\sigma_3-\sigma_1)(\sigma_4-\sigma_1)};$$

because

$$\beta_k=\bar{\beta}_k\quad\text{and}\quad\bar{\sigma}_k=1/\sigma_k$$

we get

(I) $$2\sigma_2\sigma_3\sigma_4-a_1(\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+a_2(\sigma_2+\sigma_3+\sigma_4)-a_3$$
$$=-2\sigma_1^3+\bar{a}_1\sigma_1^3(\sigma_2+\sigma_3+\sigma_4)-\bar{a}_2\sigma_1^3(\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+\bar{a}_3\sigma_1^2\varepsilon.$$

Then we exchange index 1 for 2, we subtract the resulting equation from equation (I) and divide by $\sigma_2-\sigma_1$. As a result we get

(II) $$2\sigma_3\sigma_4-a_1(\sigma_3+\sigma_4)+a_2$$
$$=2(\sigma_1^2+\sigma_2^2+\sigma_1\sigma_2)-\bar{a}_1(\sigma_1^2\sigma_2+\sigma_1^2\sigma_3+\sigma_1^2\sigma_4+\sigma_1\sigma_2^2+\sigma_2^2\sigma_3+\sigma_2^2\sigma_4+$$
$$+\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4)+\bar{a}_2(\sigma_1^2\sigma_2\sigma_3+\sigma_1^2\sigma_2\sigma_4+\sigma_1^2\sigma_3\sigma_4+\sigma_1\sigma_2^2\sigma_3+\sigma_2^2\sigma_3\sigma_4+\varepsilon)-$$
$$-\bar{a}_3\varepsilon(\sigma_1+\sigma_2).$$

Again we exchange the index 1 for the index 3, the resulting equation is subtracted from equation (II) and divided by $\sigma_3-\sigma_1$. We get as a result

(III) $$a_1=-\varepsilon\bar{a}_3+\bar{a}_2(\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4)-$$
$$-\bar{a}_1(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+2(\sigma_1+\sigma_2+\sigma_3+\sigma_4)$$

and also

(III′) $$\bar{a}_1=\varepsilon[-a_3+a_2(\sigma_1+\sigma_2+\sigma_3+\sigma_4)-a_1(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+$$
$$+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+2(\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4)].$$

Newton's formulae give

(8) $$\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4=\tfrac{1}{6}(a_1^{\vee 3}-3a_1^{\vee}a_2^{\vee}+2a_3^{\vee}),$$
$$\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4=\tfrac{1}{2}(a_1^{\vee 2}-a_2^{\vee}).$$

Putting formulae (6) and (8) into equation (III′) we get

(9) $$a_3^{\vee}=-\tfrac{7}{8}a_1^{\vee 3}.$$

Since

$$\overline{a_1^{\vee}}=\tfrac{1}{6}\varepsilon(a_1^{\vee 3}-3a_1^{\vee}a_2^{\vee}+2a_3^{\vee}),$$

from (9) we have

(10) $$\overline{a_1^{\vee}}=-\tfrac{1}{2}\varepsilon a_1^{\vee}(\tfrac{1}{4}a_1^{\vee 2}+a_2^{\vee})$$

and from formulae (6)

(11) $$\overline{a_1^{\vee}}=-\tfrac{1}{2}\varepsilon a_1^{\vee}a_2.$$

If $\check{a}_1 \neq 0$ then $|a_2| = 2$

$$\beta_1\sigma_1^2+\beta_2\sigma_2^2+\beta_3\sigma_3^2+\beta_4\sigma_4^2 = 2e^{i\theta}$$

and since $\beta_k > 0$, $\sum_{k=1}^{4}\beta_k = 2$, we have

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2,$$

i. e. contrary to the supposition not all σ_k are different.

If $\check{a}_1 = 0$ then from (9) $\check{a}_3 = 0$. In order to denote $\check{a}_2$ we take equation (II). Exchanging its indices successively and summing the equations thus obtained we get the equation

$$\text{(IV)}\quad \begin{aligned} -a_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+2a_2 &= 2(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2)-\bar{a}_1(\sigma_1\sigma_2\sigma_3+\\ &+\sigma_1\sigma_3\sigma_4+\sigma_1\sigma_2\sigma_4+\sigma_2\sigma_3\sigma_4+\sigma_1^2\sigma_2+\sigma_1^2\sigma_3+\sigma_1^2\sigma_4+\sigma_1\sigma_2^2+\sigma_2^2\sigma_3+\\ &+\sigma_2^2\sigma_4+\sigma_1\sigma_3^2+\sigma_2\sigma_3^2+\sigma_3^2\sigma_4+\sigma_1\sigma_4^2+\sigma_2\sigma_4^2+\sigma_3\sigma_4^2)+\\ &+\bar{a}_2(\sigma_1^2\sigma_2\sigma_3+\sigma_1^2\sigma_2\sigma_4+\sigma_1^2\sigma_3\sigma_4+\sigma_1\sigma_2^2\sigma_3+\sigma_1\sigma_2^2\sigma_4+\sigma_2^2\sigma_3\sigma_4+\\ &+\sigma_1\sigma_2\sigma_3^2+\sigma_1\sigma_3^2\sigma_4+\sigma_2\sigma_3^2\sigma_4+\sigma_1\sigma_2\sigma_4^2+\sigma_1\sigma_3\sigma_4^2+\sigma_2\sigma_3\sigma_4^2+2\sigma_1\sigma_2\sigma_3\sigma_4)-\\ &-\bar{a}_3\sigma_1\sigma_2\sigma_3\sigma_4(\sigma_1+\sigma_2+\sigma_3+\sigma_4).\end{aligned}$$

Using formulae (6) and the formulae expressing symmetrical functions by fundamental functions $\check{a}_k$ and ε we get from the equation (IV) the following equation:

$$\text{(12)}\quad -\tfrac{5}{144}\check{a}_1^{6}+\tfrac{1}{16}\check{a}_1^{4}\check{a}_2+\tfrac{1}{8}\check{a}_1^{2}\check{a}_2^{2}-\tfrac{7}{72}\check{a}_1^{3}\check{a}_3-\tfrac{1}{18}\check{a}_3^{2} = \varepsilon(2\check{a}_2-\tfrac{5}{2}\check{a}_1^{2}).$$

If $\check{a}_1 = 0$ and $\check{a}_3 = 0$ then (12) gives $\check{a}_2 = 0$ and from formulae (6) it follows that

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0.$$

Hence and from (2) we infer that the extremal coefficient

$$b_3 = -\tfrac{1}{4}a_4.$$

a_4 is a real quality for which $|a_4| \leqslant 2$. Hence

$$|b_3| \leqslant \tfrac{1}{2}.$$

The equality occurs only when $a_4 = \pm 2$, and thus when $\sigma_k^4 = 1$ or $\sigma_k^4 = -1$. Hence

$$\sigma_1 = 1, \quad \sigma_2 = i, \quad \sigma_3 = -1, \quad \sigma_4 = -i$$

or

$$\sigma_1 = e^{\pi i/4}, \quad \sigma_2 = e^{3\pi i/4}, \quad \sigma_3 = e^{5\pi i/4}, \quad \sigma_4 = e^{7\pi i/4}.$$

In both cases (7) gives $\beta_k = \frac{1}{2}$, and thus the extremal function for the functional $\mathrm{re}\, b_3$ is of the form

$$F(z) = \left(\frac{1}{z^2}\pm z^2\right)^{1/2}.$$

Turning the plane z we find that the function

$$F(z) = \left(\frac{1}{z^2}+\eta z^2\right)^{1/2}, \qquad |\eta| = 1$$

gives the extremum for the functional $|b_3|$.

In order to conclude the proof of our theorem it is sufficient to prove that for all functions of the form

$$\text{(13)}\qquad F_1(z) = \frac{1}{z}\prod_{k=1}^{3}(1-\sigma_k z)^{\beta_k}$$

we have the inequality $|b_3| < \frac{1}{2}$.

Let us study the function

$$\varphi(z) = -z\frac{F_1'(z)}{F_1(z)} = 1+a_1z+\ldots$$

where $F_1(z)$ is the function (13).

This function maps a circle $|z| \leqslant 1$ on the semiplane $\mathrm{re}\varphi \geqslant 0$ once, twice or three times at most. This depends on how many different numbers there are among numbers $\sigma_1, \sigma_2, \sigma_3$. The circumference of the circle is mapped on the line $\mathrm{re}\varphi = 0$. Hence we infer that the function

$$\text{(14)}\qquad f(z) = \frac{1-\varphi(z)}{1+\varphi(z)}$$

is a function bounded in the unit circle. This function transforms the unit circle in a unit circle once, twice or at most three times. From Schurr's known theorem [3] we know that all such functions are either of the form

$$\text{(15')}\qquad f(z) = \frac{1}{h_3}\cdot\frac{h_1z+(h_2+h_1\bar{h}_2h_3)z^2+h_3z^3}{\bar{h}_3+(\bar{h}_2+\bar{h}_1h_2\bar{h}_3)z+\bar{h}_1z^2}$$

where the parameters h_1, h_2, h_3 satisfy the conditions

$$|h_1| \leqslant 1, \quad |h_2| \leqslant 1, \quad |h_3| = 1,$$

or of the form

$$\text{(15'')}\qquad f(z) = \frac{1}{h_2}\cdot\frac{h_1z+h_2z^2}{\bar{h}_2+\bar{h}_1z}$$

where the parameters h_1, h_2 satisfy the conditions

$$|h_1| \leqslant 1, \quad |h_2| = 1$$

or of the form

$$f(z) = h_1 z \tag{15'''}$$

where $|h_1| = 1$.

With the help of the parameters which appear here we can express the coefficients of the expansion of the functions $f(z), \varphi(z)$ and finally $F_1(z)$.

We shall find for the function $F_1(z)$ that either

$$b_3 = \tfrac{1}{6}(1-|h_1|^2)\{h_1^2 h_2 - 2|h_1|^2 h_2^2 - 3\bar{h}_1^2 h_2^3 + (1-|h_2|^2)(2h_1 h_3 + 6\bar{h}_1 h_2 h_3 + 3\bar{h}_2 h_3^2)\}, \tag{16'}$$

or

$$b_3 = \tfrac{1}{6}(1-|h_1|^2)(h_1^2 h_2 - 2|h_1|^2 h_2^2 - 3\bar{h}_1^2 h_2^3), \tag{16''}$$

or

$$b_3 = 0. \tag{16'''}$$

We easily find from the above that always

$$|b_3| \leqslant \tfrac{1}{6}(1-r_1^2)\{r_1^2 r_2 + 2r_1^2 r_2^2 + 3r_1^2 r_2^3 + (1-r_2^2)(2r_1 + 6r_1 r_2 + 3r_2)\} \tag{17}$$

where $0 \leqslant r_1 \leqslant 1$, $0 \leqslant r_2 \leqslant 1$.

Let $0 \leqslant r_1 \leqslant 0{,}5$; then

$$r_1^2(1-r_1^2) \leqslant 0{,}19, \quad r_1(1-r_1^2) \leqslant 0{,}375, \quad 1-r_1^2 \leqslant 1,$$

$$|b_3| \leqslant 0{,}125 + 0{,}907 r_2 - 0{,}062 r_2^2 - 0{,}780 r_2^3, \quad 0{,}59 \leqslant r_2\text{extr} \leqslant 0{,}6,$$

i. e.

$$|b_3| \leqslant 0{,}489 < 0{,}5.$$

Let $0{,}5 \leqslant r_1 \leqslant 1$; then

$$r_1^2(1-r_1^2) \leqslant 0{,}25, \quad r_1(1-r_1^2) \leqslant 0{,}386, \quad 1-r_1^2 \leqslant 0{,}75,$$

$$|b_3| \leqslant 0{,}129 + 0{,}803 r_2 - 0{,}045 r_2^2 - 0{,}636 r_2^3, \quad 0{,}62 \leqslant r_2\text{extr} \leqslant 0{,}63,$$

i. e.

$$|b_3| \leqslant 0{,}47 < 0{,}5.$$

Thus the theorem is completely proved.

References

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