ANNALES
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Certain results concerning $\sigma_k(n)$ and $\varphi_k(n)$

by M. Sugunamma (Tirupati, South India)

Let $\sigma_k(n)$ be the function representing the sum of k-th powers of positive divisors of n, and let $\varphi_k(n)$ be the Jordan function defined as $\varphi_k(n) = n^k\prod(1-p_i^{-k})$, the product on the right being taken over all distinct prime factors of n.

In this we will prove the following:

(A) $$\lim_{n\to\infty}\frac{\sigma_n(n+a)}{\sigma_n(n)} = e^a,$$

(B) $$\lim_{n\to\infty}\frac{\varphi_n(n+a)}{\varphi_n(n)} = e^a,$$

(C) $$\lim_{n\to\infty}\frac{\sigma_n(n+a)\varphi_n(n+a)}{\sigma_n(n)\varphi_n(n)} = e^{2a},$$

for all positive integral n and $n+a$.

To prove these three results we will use the following inequalities:

(A′) $$\left(1+\frac{a}{n}\right)^k\zeta(k) > \frac{\sigma_k(n+a)}{\sigma_k(n)} > \left(1+\frac{a}{n}\right)^k\frac{1}{\zeta(k)},$$

(B′) $$\left(1+\frac{a}{n}\right)^k\zeta(k) > \frac{\varphi_k(n+a)}{\varphi_k(n)} > \left(1+\frac{a}{n}\right)^k\frac{1}{\zeta(k)},$$

(C′) $$\left(1+\frac{a}{n}\right)^{2k}\zeta(2k) > \frac{\sigma_k(n+a)\varphi_k(n+a)}{\sigma_k(n)\varphi_k(n)} > \left(1+\frac{a}{n}\right)^{2k}\frac{1}{\zeta(2k)},$$

for all $k > 1$, and for all positive integral n and $n+a$, where, $\zeta(k)$ is the well known Riemann ζ-function.

In this connection it may be said that the inequality (A′) for special values of a, $a = 1$ and, $a = -1$ implies and improves S. Swetharanyam's [1] inequalities, namely

$$\left(\frac{3}{2}\right)^k\zeta(k)\zeta(2k)\big(\zeta(2)\big)^2 > \frac{\sigma_k(n+1)}{\sigma_k(n)} > \frac{1}{\zeta(k)\zeta(2k)\big(\zeta(2)\big)^2}$$

and

$$\zeta(k)\zeta(2k)(\zeta(2))^2 > \frac{\sigma_k(n-1)}{\sigma_k(n)} > \left(\frac{1}{2}\right)^k \frac{1}{\zeta(k)\zeta(2k)(\zeta(2))^2}.$$

Proof of (A′). We know that if $n = \prod p_i^{a_i}$, then

$$\sigma_k(n) = n^k \prod \frac{(1-p_i^{-k(a_i+1)})}{(1-p_i^{-k})},$$

the product on the right being taken over all distinct prime factors of n. Hence we have

$$\frac{1}{\prod(1-p_i^{-k})} > \frac{\sigma_k(n)}{n^k} > 1,$$

the product on the left being taken over all distinct prime factors of n.

Hence

$$\zeta(k) = \frac{1}{\prod(1-p^{-k})} > \frac{1}{\prod(1-p_i^{-k})} > \frac{\sigma_k(n)}{n^k} > 1,$$

the product on the left end being taken over all primes. From this it follows that

$$\left(1+\frac{a}{n}\right)^k \zeta(k) > \frac{\sigma_k(n+a)}{\sigma_k(n)} > \left(1+\frac{a}{n}\right)^k \frac{1}{\zeta(k)}.$$

Proof of (A). If we put $k = n$ in (A′) and take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\sigma_n(n+a)}{\sigma_n(n)} = e^a,$$

using the result $\zeta(n) \to 1$ as $n \to \infty$, for all positive integral n and $n+a$.

Proof of (B′).

$$\varphi_k(n) = n^k \prod (1-p_i^{-k}).$$

Hence we have

$$1 > \frac{\varphi_k(n)}{n^k} > \prod(1-p^{-k}) = \frac{1}{\zeta(k)},$$

the product on the right being taken over all primes. From this it follows that

$$\left(1+\frac{a}{n}\right)^k \zeta(k) > \frac{\varphi_k(n+a)}{\varphi_k(n)} > \left(1+\frac{a}{n}\right)^k \frac{1}{\zeta(k)}$$

for all positive integral n and $n+a$.

Proof of (B). If we put $k = n$ in (B′) and take the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{\varphi_n(n+a)}{\varphi_n(n)} = e^a,$$

as above, for all positive integral n and $n+a$.

Here we can note that from (A′) and (B′) we will get the inequality

$$\left(1+\frac{a}{n}\right)^{2k} (\zeta(k))^2 > \frac{\sigma_k(n+a)\varphi_k(n+a)}{\sigma_k(n)\varphi_k(n)} > \left(1+\frac{a}{n}\right)^{2k} \frac{1}{(\zeta(k))^2},$$

which is less sharp than the inequality (C′) for $1 < \zeta(2k) < \zeta(k)$.

Proof of (C′). If $n = \prod p_i^{a_i}$,

$$\frac{\sigma_k(n)\varphi_k(n)}{n^{2k}} = \prod (1-p_i^{-k(a_i+1)}),$$

the product on the right being taken over all distinct prime factors of n. Hence,

$$1 > \frac{\sigma_k(n)\varphi_k(n)}{n^{2k}} > \prod(1-p_i^{-2k}) > \prod(1-p^{-2k}) = \frac{1}{\zeta(2k)}$$

for $a_i \geqslant 1$, the product on the right end being taking over all primes. From this it follows that

$$\left(1+\frac{a}{n}\right)^{2k} \zeta(2k) > \frac{\sigma_k(n+a)\varphi_k(n+a)}{\sigma_k(n)\varphi_k(n)} > \left(1+\frac{a}{n}\right)^{2k} \frac{1}{\zeta(2k)},$$

for all positive integral n and $n+a$.

Proof of (C). The result (C) can be got directly from (A) and (B); or if we put $k = n$ in (C′) and take the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{\sigma_n(n+a)\varphi_n(n+a)}{\sigma_n(n)\,\varphi_n(n)} = e^{2a},$$

for all positive integral n and $n+a$.

In conclusion it can be seen that it is always possible to make the upper and lower limits of the concerned functions in the inequalities (A′), (B′), and (C′) independent of n. For example, if $a = 1$

$$\left(\frac{3}{2}\right)^k \zeta(k) > \frac{\sigma_k(n+1)}{\sigma_k(n)} > \frac{1}{\zeta(k)}$$

since $\frac{3}{2} \geqslant 1+1/n > 1$ for $n \geqslant 2$. Similarly

$$\left(\frac{3}{2}\right)^k \zeta(k) > \frac{\varphi_k(n+1)}{\varphi_k(n)} > \frac{1}{\zeta(k)},$$

and

$$\left(\frac{3}{2}\right)^k \zeta(2k) > \frac{\sigma_k(n+1)\varphi_k(n+1)}{\sigma_k(n)\varphi_k(n)} > \frac{1}{\zeta(2k)}.$$

In a similar manner if $a = -1$

$$\zeta(k) > \frac{\sigma_k(n-1)}{\sigma_k(n)} > \left(\frac{1}{2}\right)^k \frac{1}{\zeta(k)},$$

$$\zeta(k) > \frac{\varphi_k(n-1)}{\varphi_k(n)} > \left(\frac{1}{2}\right)^k \frac{1}{\zeta(k)},$$

and

$$\zeta(2k) > \frac{\sigma_k(n-1)\varphi_k(n-1)}{\sigma_k(n)\varphi_k(n)} > \left(\frac{1}{2}\right)^x \frac{1}{\zeta(2k)}.$$

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Reference

[1] S. Swetharanyam, *On the function* $\sigma_k(n)$, Ann. Polon. Math. 4 (1958), p. 340-343.

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Über die korrekte Definition des Ranges eines nomographischen Polynoms und über die Stetigkeit und die Differenzierbarkeit der verallgemeinerten nomographischen Polynome

von J. Wojtowicz (Warszawa)

Der Begriff des Ranges eines nomographischen Polynoms, obwohl in jedem nomographischen Lehrbuch angewandt, ist bisher nicht korrekt definiert worden. Die von verschiedenen Verfassern aufgestellten Definitionen sind ungenau und sogar falsch. In der vorliegenden Arbeit wird eine korrekte Definition des Ranges eines nomographischen Polynoms aufgestellt.

Außerdem enthält die Arbeit Beweise der Sätze, welche in einem gewissen Sinne umgekehrt sind zu bekannten Sätzen über die Summe stetiger Funktionen und über die Differenzierbarkeit der Summe differenzierbarer Funktionen.

Definition. Als ein *nomographisches Polynom* wird eine Funktion von der Form

$$F(x_1, \ldots, x_k) \equiv \sum_{i=1}^{n} f_{i1}(x_1)\ldots f_{ik}(x_k) \tag{1}$$

bezeichnet.

Definition. Unter einem *Summanden* des nomographischen Polynoms versteht man den Ausdruck $f_{i1}(x_1)\ldots f_{ik}(x_k)$.

Definition. Ein nomographisches Polynom $\sum_{i=1}^{n} f_{i1}(x_1)\ldots f_{ik}(x_k)$ *ist von reduzierter Form in Bezug auf die Veränderliche* x_1, wenn die Funktionen $f_{i2}(x_2)\ldots f_{ik}(x_k)$ $(i = 1, 2, \ldots, n)$ linear unabhängig sind.

Definition. Ein nomographisches Polynom *ist von reduzierter Form*, wenn es von reduzierter Form in Bezug auf jede Veränderliche x_j $(j = 1, 2, \ldots, k)$ ist.

Hilfssatz 1. *Jedes nomographische Polynom kann man in eine reduzierte Form in Bezug auf eine beliebige Veränderliche bringen ohne die Anzahl der Polynomsummanden zu vergrößern.*