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On continuous solutions of a functional equation

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The object of the present paper is the functional equation

$$\varphi[f(x)] = G\big(x, \varphi(x)\big), \tag{1}$$

where $\varphi(x)$ denotes the required function, and $f(x)$ and $G(x, y)$ are known functions. We consider equation (1) in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of the equation

$$f(x) = x. \tag{2}$$

Equation (1) with the function $G(x, y)$ linear with respect to y,

$$\varphi[f(x)] = \lambda(x)\varphi(x) + F(x), \tag{3}$$

has been discussed in [2]. It has been proved there (under suitable assumptions) that

(∗) *If* $|\lambda(b)| > 1$, *then equation* (3) *possesses at most one solution that is continuous at the point* $x = b$.

(∗∗) *If* $|\lambda(b)| < 1$, *then every solution of equation* (3) *that is continuous in the interval* (a, b) *is also continuous at the point* $x = b$.

A property analogical to property (∗) has been proved for the equation of the general form (1) (under suitable assumptions) in [1]. The purpose of the present paper is to prove for equation (1) a property analogical to (∗∗).

We shall denote by $f^n(x)$ the n-th iteration of the function $f(x)$, i. e. we put

$$f^0(x) = x, \quad \begin{aligned} f^{n+1}(x) &= f[f^n(x)], \\ f^{n-1}(x) &= f^{-1}[f^n(x)], \end{aligned} \quad n = 0, \pm 1, \pm 2, \ldots$$

One can prove the following

LEMMA. *If the function* $f(x)$ *is continuous and strictly increasing in an interval* $\langle a, b\rangle$, *where* a *and* b *are two consecutive roots of equation* (2), *and moreover* $f(x) > x$ *in* (a, b), *then for every* $x \in (a, b)$ *the sequences* $\{f^n(x)\}$ *and* $\{f^{-n}(x)\}$ *are monotonic, and*

$$\lim_{n\to\infty} f^n(x) = b, \qquad \lim_{n\to\infty} f^{-n}(x) = a.$$

The proof of the above lemma is to be found in [3].

In the sequel we shall assume that

(i) The function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (2), and $f(x) > x$ in (a, b).

(ii) The function $G(x, y)$ is continuous and has a continuous derivative $\partial G/\partial y \neq 0$ in a region Ω, normal with respect to the x-axis.

For an arbitrary x we shall denote by Ω_x the x-section of the set Ω, i. e.

$$\Omega_x = \mathop{E}_{y}\{(x, y) \in \Omega\}.$$

We shall denote by Γ_x the set of values assumed by the function $G(x, y)$ on the set $x \times \Omega_x$ (the sign $\times$ denotes here the Cartesian product):

$$\Gamma_x = \mathop{E}_{z}\{\sum_{y} y \in \Omega_x,\ z = G(x, y)\}.$$

We shall suppose also (compare [1]) that

$$\text{(iii)} \qquad \Omega_x \neq 0, \quad \Gamma_x = \Omega_{f(x)} \quad \text{for} \quad x \in \langle a, b\rangle.$$

Hypotheses (i)-(iii) guarantee the existence of continuous solutions of equation (1) in the interval (a, b). Namely, it has been proved in [1] (under the above hypotheses) that for every $x_0 \in (a, b)$ and for every function $\bar\varphi(x)$ continuous in the interval $\langle x_0, f(x_0))$ and fulfilling the conditions

$$\bar\varphi(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, f(x_0)),$$

$$\lim_{x \to f(x_0)-} \bar\varphi(x) = G[x_0, \bar\varphi(x_0)],$$

there exists a function $\varphi(x)$ continuous in (a, b), satisfying equation (1) and such that

$$\varphi(x) \equiv \bar\varphi(x) \quad \text{for} \quad x \in \langle x_0, f(x_0)).$$

Let numbers c and d be roots of the equations

$$c = G(a, c) \quad \text{and} \quad d = G(b, d)$$

respectively. We shall prove the following

THEOREM. *If hypotheses* (i)-(iii) *are fulfilled, and moreover*

$$\text{(4)} \qquad \left|\frac{\partial G}{\partial y}(b, d)\right| < 1,$$

then equation (1) *possesses infinitely many solutions that are continuous in the interval* $(a, b\rangle$. *More precisely, there exist numbers* $\varepsilon > 0$ *and* $\eta > 0$ *such that every solution* $\varphi(x)$ *of equation* (1) *that is continuous in the interval* (a, b) *and fulfils the condition*

$$\text{(5)} \qquad |\varphi(x) - d| < \varepsilon \quad \textit{for} \quad x \in \langle x_0, f(x_0)),$$

where x_0 *is an arbitrary number from the interval* $(b-\eta, b)$, *is also continuous for* $x = b$.

Proof. Let $\varphi(x)$ be a solution of equation (1) that is continuous in the interval (a, b) and fulfils condition (5). We put

$$\psi(x) \overset{\text{df}}{=} G(f^{-1}(x), d), \qquad \varrho(x) \overset{\text{df}}{=} \psi(x) - \varphi(x), \qquad \alpha(x) \overset{\text{df}}{=} d - \psi(x).$$

The function $\varrho(x)$ is continuous in the interval (a, b), and the functions $\psi(x)$ and $\alpha(x)$ are continuous in the interval $\langle a, b\rangle$, and

$$\psi(b) = d, \qquad \alpha(b) = 0.$$

From condition (4) it follows that there exist positive numbers β, δ and ϑ, $\frac{1}{2} < \vartheta < 1$, such that $|\partial G/\partial y| < \vartheta$ for $|x-b| < \beta$, $|y-d| < \delta$. Let us put $\varepsilon = \delta/3$ and let us choose η so that $\eta < \beta$ and

$$\text{(6)} \qquad |\alpha(x)| < \frac{1-\vartheta}{\vartheta}\varepsilon$$

for $x \in (b-\eta, b)$.

We shall show at first that for $x \in (b-\eta, b)$ from the inequality

$$\text{(7)} \qquad |\varrho(x)| < 2\varepsilon$$

follows the inequality

$$\text{(8)} \qquad |\varrho[f(x)]| < 2\varepsilon.$$

Let us take an arbitrary $x \in (b-\eta, b)$, and let us suppose that inequality (7) holds. We have

$$\begin{aligned}\varrho[f(x)] = \psi[f(x)] - \varphi[f(x)] &= G(x, d) - G(x, \varphi(x)) \\ &= G_y[x, \varphi(x) + \theta(x)(d-\varphi(x))](d-\varphi(x)) \\ &= G_y[x, \varphi(x) + \theta(x)(d-\varphi(x))]\,\varrho(x) + G_y[x, \varphi(x) + \theta(x)(d-\varphi(x))]\,\alpha(x).\end{aligned}$$

We put

$$\lambda(x) \overset{\text{df}}{=} G_y[x, \varphi(x) + \theta(x)(d-\varphi(x))].$$

Thus

$$\varrho[f(x)] = \lambda(x)\varrho(x)+\lambda(x)\alpha(x). \tag{9}$$

By (6) and (7) we have

$$\begin{aligned}\big|\varphi(x)+\theta(x)\big(d-\varphi(x)\big)-d\big| &= |1-\theta(x)|\,|\varphi(x)-d| < |\varphi(x)-d| \\ &= |\varphi(x)-\psi(x)+\psi(x)-d| \leqslant |\varrho(x)|+|\alpha(x)| \\ &\leqslant 2\varepsilon+\frac{1-\vartheta}{\vartheta}\varepsilon = \varepsilon+\frac{\varepsilon}{\vartheta} \leqslant 3\varepsilon = \delta.\end{aligned}$$

Consequently

$$|\lambda(x)| < \vartheta. \tag{10}$$

By (9), (10), (6) and (7) we have

$$|\varrho[f(x)]| \leqslant |\lambda(x)|\,|\varrho(x)|+|\lambda(x)|\,|\alpha(x)| \leqslant 2\vartheta\varepsilon+\vartheta\frac{1-\vartheta}{\vartheta}\varepsilon = \vartheta\varepsilon+\varepsilon < 2\varepsilon,$$

which proves relation (8).

Now we shall show that the function $\varrho(x)$ is continuous for $x = b$. From inequality (5) it follows that

$$\begin{aligned}|\varrho(x)| &= |\psi(x)-\varphi(x)| = |\psi(x)-d+d-\varphi(x)| \leqslant |\alpha(x)|+|\varphi(x)-d| \\ &\leqslant \frac{1-\vartheta}{\vartheta}\varepsilon+\varepsilon = \frac{\varepsilon}{\vartheta} < 2\varepsilon\end{aligned}$$

for $x\in\langle x_0, f(x_0))$. Thus, on account of the lemma and of the first part of this proof, the inequality

$$|\varrho(x)| < 2\varepsilon \tag{11}$$

holds throughout the whole interval $\langle x_0, b)$. Hence it follows that the inequality

$$\big|\varphi(x)+\theta(x)\big(d-\varphi(x)\big)-d\big| < \delta,$$

and then also inequality (10) hold throughout the whole interval $\langle x_0, b)$. The function $\varrho(x)$ is continuous in the interval (a, b) and satisfies equation (9).

It has been proved in [2] that if inequality (10) holds in a (left-hand) neighbourhood of the point b and the functions $\lambda(x)$ and $F(x)$ are continuous in the interval $(a, b\rangle$, then every solution of equation (3) that is continuous in the interval (a, b) is also continuous at the point $x = b$. In our case, however, the functions $\lambda(x)$ and $F(x) = \lambda(x)\alpha(x)$ are not necessarily continuous in $(a, b\rangle$[1]. Nevertheless, the proof of property

[1] As has been proved by S. Łojasiewicz [4], the function $\theta(x)$ is continuous only if the function $G(x, y)$ is convex with respect to y.

(**) may be modified so as to be applicable in our case. We shall give here an outline of that proof only.

Just as in [2] it can be proved that the inequality

$$|\varrho[f^n(x)]| \leqslant M(x)\frac{1-\vartheta^n}{1-\vartheta}+\vartheta^n|\varrho(x)|,$$

where

$$M(x) \overset{\text{df}}{=} \sup_{\langle x,b\rangle}|\lambda(t)\alpha(t)|, \tag{12}$$

holds for $x\in\langle x_0, b)$ and all n. According to (11) and to the inequality $\vartheta < 1$ we have

$$|\varrho[f^n(x)]| \leqslant M(x)\frac{1}{1-\vartheta}+2\varepsilon\vartheta^n \quad \text{for} \quad x\in\langle x_0, b). \tag{13}$$

Since the function $\lambda(x)$ is bounded in a neighbourhood of b, and the function $\alpha(x)$ is continuous and vanishes at the point $x = b$, we have by (12)

$$\lim_{x\to b-} M(x) = 0. \tag{14}$$

The relation

$$\lim_{x\to b-}\varrho(x) = 0$$

follows easily from (13) and (14), which proves that the function $\varrho(x)$ is continuous for $x = b$. Consequently, also the function

$$\varphi(x) = \psi(x)-\varrho(x)$$

is continuous for $x = b$, which was to be proved.

Similarly one can prove that

If hypotheses (i)-(iii) *are fulfilled, and moreover*

$$\left|\frac{\partial G}{\partial y}(a, c)\right| > 1,$$

then equation (1) *possesses infinitely many solutions that are continuous in the interval* $\langle a, b)$. *More precisely, there exist numbers* $\varepsilon > 0$ *and* $\eta > 0$ *such that every solution* $\varphi(x)$ *of equation* (1) *that is continuous in the interval* (a, b) *and fulfils the condition*

$$|\varphi(x)-c| < \varepsilon \quad \textit{for} \quad x\in\langle x_0, f(x_0)),$$

where x_0 *is an arbitrary number from the interval* $(a, a+\eta)$, *is also continuous for* $x = a$.

Remark. As W. Kraj has remarked, in the hypotheses of our theorem the existence of the derivative $\partial G/\partial y$ and relation (4) can be replaced by the following condition:

The function $G(x, y)$ is — with fixed x — monotonic with respect to y, and fulfils in a neighbourhood of the point (b, d) the Lipschitz condition with respect to y:

$$|G(x, \bar{y}) - G(x, \bar{\bar{y}})| < K(x)|\bar{y} - \bar{\bar{y}}|,$$

with

$$K(x) \leqslant \vartheta < 1.$$

References

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