Remark III. If in hypothesis (iii) only the set Ω_n (or only Ω_0) occurs, and in hypothesis (iv) we postulate only the possibility of solving the equation $F=0$ with respect to the variable y_n (or only with respect to y_0), then the solution of equation (1) can not exist in the whole interval (a, b). Anyhow, as is obvious from the proof, we can then infer the existence of continuous solutions of equation (1) in the interval $\langle a+\varepsilon, b)$ (or in the interval $(a, b-\varepsilon\rangle)$, where ε is an arbitrary positive number. In this case we must choose $x_0 \leqslant a+\varepsilon$ (or $x_0 \geqslant f_n^{-1}(b-\varepsilon)$) in the proof of the theorem.

References

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A simple proof of a certain result of Z. Opial

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1. Suppose that $x(t)$ is of class C^1 for $t \in \langle 0, h\rangle$ $(h>0)$, and that the following condition holds:

$$x(0) = x(h) = 0. \tag{1}$$

Under the assumptions given above Z. Opial [2] has proved the inequality

$$\int_0^h |x(t)x'(t)|\,dt \leqslant \tfrac{1}{4}h \int_0^h x'^2(t)\,dt. \tag{2}$$

The purpose of this note is to present a simple proof of this result of Opial.

In order to do so we shall use the following consequence of a well-known inequality of Buniakowski (see for example [1], p. 146):

$$\Big(\int_a^b |u(t)|\,dt\Big)^2 \leqslant (b-a)\int_a^b u^2(t)\,dt. \tag{3}$$

Let us observe that Opial, in his proof of (2), also used (3).

2. Denote by $y(t) = \int_0^t |x'(t)|\,dt$ and by $z(t) = \int_t^h |x'(t)|\,dt$. We have the following obvious relations:

$$y'(t) = |x'(t)| = -z'(t), \tag{4}$$

and

$$|x(t)| \leqslant y(t), \qquad |x(t)| \leqslant z(t), \qquad 0 \leqslant t \leqslant h. \tag{5}$$

By (4) and (5) we get

$$\int_0^{h/2} |x(t)x'(t)|\,dt \leqslant \int_0^{h/2} y(t)y'(t)\,dt = \tfrac{1}{2}y^2(\tfrac{1}{2}h),$$

$$\int_{h/2}^{h} |x(t)x'(t)|\,dt \leqslant -\int_{h/2}^{h} z(t)z'(t)\,dt = \tfrac{1}{2}z^2(\tfrac{1}{2}h).$$

Thus we have the inequality

$$\int_0^{h} |x(t)x'(t)|\,dt \leqslant \tfrac{1}{2}\big(y^2(\tfrac{1}{2}h)+z^2(\tfrac{1}{2}h)\big). \tag{6}$$

On the other hand using (3) we obtain the inequalities

$$y^2(\tfrac{1}{2}h) \leqslant \tfrac{1}{2}h\int_0^{h/2} x'^2(t)\,dt, \qquad z^2(\tfrac{1}{2}h) \leqslant \tfrac{1}{2}h\int_{h/2}^{h} x'^2(t)\,dt. \tag{7}$$

Inequalities (6) and (7) prove (2) immediately.

3. Since (3) holds for an arbitrary summable function $u(t)$, the above reasoning is valid for an absolutely continuous function $x(t)$. Therefore, *if $x(t)$ is absolutely continuous in the interval $\langle 0, h\rangle$ and satisfies assumption* (1), *then* (2) *holds.*

4. Z. Opial has also shown that if $x(t)$ satisfies (1) and the equality

$$\int_0^{h} |x(t)x'(t)|\,dt = \tfrac{1}{4}h\int_0^{h} x'^2(t)\,dt, \tag{8}$$

then $x(t)$ is of the form

$$x(t) = \begin{cases} At & \text{for} \quad 0 \leqslant t \leqslant \tfrac{1}{2}h, \\ A(h-t) & \text{for} \quad \tfrac{1}{2}h \leqslant t \leqslant h, \end{cases} \tag{9}$$

where A is constant.

This result may be obtained by the following arguments. By (8), (6) and (7) we get

$$\Big(\int_0^{h/2} |x'(t)|\,dt\Big)^2 = \tfrac{1}{2}h\int_0^{h/2} x'^2(t)\,dt, \tag{10}$$

$$\Big(\int_{h/2}^{h} |x'(t)|\,dt\Big)^2 = \tfrac{1}{2}h\int_{h/2}^{h} x'^2(t)\,dt. \tag{11}$$

It is easy to see that equalities (10) and (11) are possible if and only if $|x'(t)| = \text{const}$ almost everywhere in $\langle 0, \tfrac{1}{2}h\rangle$ and in $\langle \tfrac{1}{2}h, h\rangle$. Hence $y(t)$ and $z(t)$ are linear. Further, it follows from (8), (6), (10) and (11) that $|x(t)| = y(t)$ for $0 \leqslant t \leqslant \tfrac{1}{2}h$, and $|x(t)| = z(t)$ for $\tfrac{1}{2}h \leqslant t \leqslant h$. These facts imply (9).

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